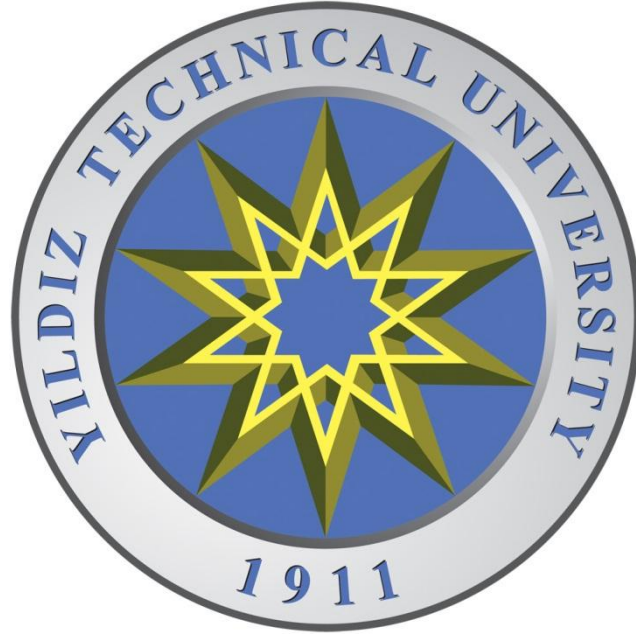




GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

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**REPUBLIC OF TURKEY
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**TYPES OF HOMOLOGY AND SOME APPLICATIONS
ON MANIFOLDS**

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**MSc. THESIS
DEPARTMENT OF MATHEMATICS
PROGRAM OF MATHEMATICS**

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Merve KAHRAMAN ARİMAN

TABLE OF CONTENTS

	Page
LIST OF SYMBOLS	vi
LIST OF ABBREVIATIONS.....	vii
LIST OF FIGURES	viii
ABSTRACT.....	ix
ÖZET	x
CHAPTER 1	
INTRODUCTION	1
1.1 Literature Review	1
1.2 Objective of the Thesis	1
1.3 Hypothesis	1
CHAPTER 2	
SIMPLICIAL HOMOLOGY.....	2
2.1 Simplices and Δ -complexes	2
2.2 Constructions	3
2.3 Computations.....	6
CHAPTER 3	
SINGULAR HOMOLOGY	12
3.1 Construction.....	12
3.2 Some Preliminaries About Exact Sequences	16
3.3 Relative Homology	16
3.4 The Excision Theorem.....	19
3.5 Isomorphism of Simplicial and Singular Homologies.....	21
3.6 Mayer Vietoris Sequence.....	22
CHAPTER 4	
CELLULAR HOMOLOGY	25

4.1 Degree	25
4.2 CW Complexes	25
4.3 Cellular Homology.	26
CHAPTER 5	
RESULTS AND DISCUSSION	29
REFERENCES	30
CURRICULUM VITAE	31

LIST OF SYMBOLS

$[v_0, \dots, v_n]$	n-simplex
$C_n(X)$	n-chains
∂_n	Boundary homomorphism
$H_n(X)$	n^{th} homology group of a topological space X
K	Klein bottle
B^2	Unit disc
$\widetilde{H}_n(X)$	Reduced homology group of a topological space X
$C_n(X, A)$	Relative chain group
$C_n^\Delta(X)$	Simplicial n-chains
X^k	k-skeleton of X
Σ_g	Orientable genus g surface
E^n	n-cell
$H_n^{\text{CW}}(X)$	n^{th} cellular homology group of a topological space X

LIST OF ABBREVIATIONS

CW Closure finite – Weak topology

LIST OF FIGURES

	Page
Figure 2.1 Δ -complex structure of torus and Klein bottle	3
Figure 2.2 Triangulation of S^1	6
Figure 2.3 Δ -complex structure of torus	7
Figure 2.4 Δ -complex structure of Klein bottle	8
Figure 2.5 Δ -complex structure of unit disc	10
Figure 3.1 Chain complex diagram.....	14
Figure 3.2 Chain complex diagram.....	15
Figure 3.3 Chain complex diagram.....	15
Figure 3.4 Chain complex diagram.....	17
Figure 3.5 Zigzag lemma diagram	18
Figure 3.6 Connecting homomorphism diagram	18
Figure 3.7 Homology group diagram	19
Figure 3.8 Five lemma diagram	21
Figure 3.9 Homology group diagram	21
Figure 3.10 Homology group diagram	22
Figure 3.11 Mayer-Vietoris diagram	23
Figure 3.12 Chain complex diagram.....	23
Figure 4.1 Homology group diagram	27

ABSTRACT

TYPES OF HOMOLOGY AND SOME APPLICATIONS ON MANIFOLDS

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MSc. Thesis

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Homology classes were first defined by Henri Poincare in his famous paper *Analysis Situs*. After Poincare, the development of combinatorial methods for the theory of simplicial complexes, simplicial homology, etc. was one of the most important expansions in topology. Continuous problems about spaces can be converted into combinatorial problems and can be solved by computers by aid of this theory.

In this thesis, simplicial, singular and cellular homology theories have been studied. Since it is more applicable and basic version of all, simplicial homology theory will be introduced at first.

Key Words: Homology classes, simplicial complexes, simplicial homology, cellular homology, singular homology

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**BENZEŞİM TÜRLERİ VE KATMANLI UZAYLARDA
BAZI UYGULAMALARI**

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Homoloji sınıfları ilk olarak Henri Poincare'in ünlü makalesi Analysis Situs içerisinde tanımlanmıştır. Poincare'den sonra Simpleksler Kompleksi ve Simpleksler Benzeşimi için geliştirilen işlemler metodlar topoloji alanında atılan en önemli adımlardan olmuştur. Bu gelişmenin yadımı ile çözüm uzayı sonsuz büyüklükte olan problemler, bilgisayarlar tarafından da kolayca çözülebilen işlemsel metodlara dayalı problemlere dönüştürülebilir.

Bu tez içerisinde; simpleks benzeşimi, tekil benzeşim ve hücrel benzeşim teorileri incelenmiş ve bu teorilerin faydaları ve birbirine üstünlükleri detaylı bir şekilde irdelenmiştir. Okuyucunun kolay takip edebilmesi amacı ile de tez, bu teorilerden en ilkeli olan basitçil benzeşim ile başlayacaktır.

Anahtar Kelimeler: Homoloji sınıfları, simpleksler kompleksi, simpleksler benzeşimi, hücrel benzeşim, tekil benzeşim

CHAPTER 1

INTRODUCTION

1.1 Literature Review

Homology theory is a general procedure to associate a sequence of abelian groups to a topological space. For a topological space, it is much easier to calculate the homology groups instead of some other invariants.

The motivation while defining homology theory is to distinguish shapes by their holes.

In a topological space, the holes always can not be defined obviously or the different kinds of holes can not be distinguished. Homology theory gives a rigorous method for defining and categorizing holes in a topological space.

1.2 Objective of the Thesis

Different types of homology theories have been introduced for calculating homology groups of topological spaces. The objective of this thesis to study the types of homology theories and and apply this theory to some topological spaces.

1.3 Hypothesis

Although there are different types of homology theories, these theories coincide for a topological space. It will be shown that the homology groups calculated with different homology theories give the same result for a topological space.

SIMPLICIAL HOMOLOGY

The first aim of this chapter is to define Δ -complex structure. Although such spaces may seem special, most of the spaces that we studied in topology admit this structure.

2.1 Simplices and Δ -complexes

Definition 2.1 (Affine Independence) A set $\{v_0, \dots, v_n\}$ of vectors in $\mathbb{R}^{n+1}$ is affinely independent if one of the followings satisfied:

- The set of vectors $\{v_1 - v_0, \dots, v_n - v_0\}$ is linearly independent.
- If $\exists a_i \in \mathbb{R}$ such that $\sum_{i=0}^n a_i v_i = 0$ and then $a_j = 0 \forall 0 \leq j \leq n$. ■

Definition 2.2 (Simplex) Let $\{v_0, \dots, v_n\}$, $v_i \in \mathbb{R}^n$ be an ordered $(n+1)$ -tuple of affinely independent vectors in $\mathbb{R}^{n+1}$. The n -simplex spanned by $\{v_0, \dots, v_n\}$ is defined to be the topological space $[v_0, \dots, v_n] := \{\sum_{i=0}^n a_i v_i : \sum_{i=0}^n a_i = 1, a_i \geq 0\}$. ■

The coefficients a_i are called the barycentric coordinates on $[v_0, \dots, v_n]$. If we delete one of the $n+1$ vertices of an n -simplex $[v_0, \dots, v_n]$, then remaining n vertices span an $(n-1)$ -simplex called a face of $[v_0, \dots, v_n]$.

Thus a zero simplex $[v_0]$ is a point, a 1-simplex $[v_0, v_1]$ is an edge, a 2-simplex $[v_0, v_1, v_2]$ is a triangle... Note that it is important to follow the ordering of a simplex, ‘ n -simplex’ will mean ‘ n -simplex with an ordering of its vertices’.

Definition 2.3 (Subsimplex) Any subset $\{v_{i_1}, \dots, v_{i_k}\}$ of $\{v_i\}$, with ordering given by the restriction of the ordering $(v_0, \dots, v_n)$ is a $(k - 1)$ -dimensional simplex $[v_{i_1}, \dots, v_{i_k}]$ called subsimplex of $[v_0, \dots, v_n]$. ■

We would like to build a space X by taking simplices and gluing them. The problem is to determine which simplices glue to which others. The proper way to construct this is

to think of each n -simplex of X as the image of a continuous map $f: \Delta^n \rightarrow X$ of the standard n -simplex Δ^n to X .

Definition 2.4 Let X be a topological space. A Δ -complex structure on X consists of the followings:

- **Decomposition into simplices:** A finite collection $S = \{\Delta_i\}$ of simplices with continuous maps $\sigma: \Delta_i \rightarrow X$, injective on the interior of σ such that:
 - $\bigcup_i \sigma(\Delta_i) = X$
 - Each $x \in X$ lies in the image of the interior of one simplex.
- **Closure under taking faces:** If $\sigma: \Delta \rightarrow X$ is an element of S , then the restriction of σ to any subsimplex τ of Δ is also an element of S . ■

The Δ -complex structure on X is seem to be a triangulation of X . The following figures are the Δ -complex structure of torus and Klein bottle, respectively:

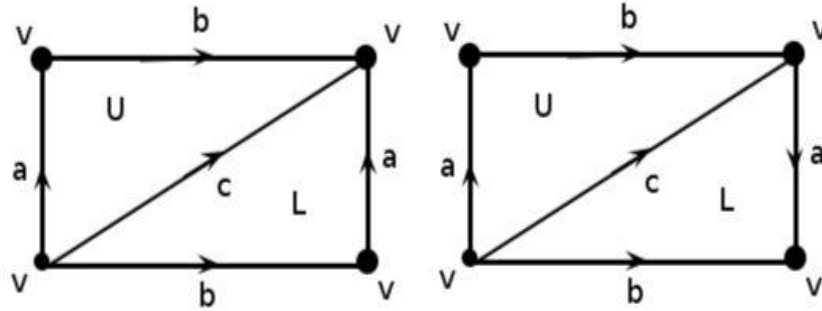


Figure 2.1 Δ -complex structure of torus and Klein bottle

2.2 Constructions

The aim of this section is to attach to any Δ -complex X and any integer $i \geq 0$ a finitely generated abelian groups $H_i(X)$, called the i^{th} homology group of X .

Definition 2.5 (Simplicial n -chain) Fix a Δ -complex X . $\forall n \geq 0$, the group of (simplicial) n -chains of X , denoted $C_n(X)$ is the free abelian group on the set of n -simplices of X . ■

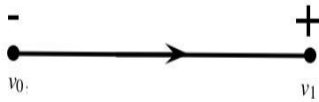
Since X has finitely many simplices, each $C_n(X)$ is finitely generated. The group operation in $C_n(X)$ is written as addition. Thus if $\{\sigma_i\}$ denotes the set of all n -simplices of X , then any element $c \in C_n(X)$ can be written uniquely as a formal finite sum $c = \sum_i a_i \sigma_i$ with $a_i \in \mathbb{Z}$.

Definition 2.6 For a general Δ -complex X , we define the boundary homomorphism $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ on any n -simplex $\sigma : [v_0, \dots, v_n] \rightarrow X$ via

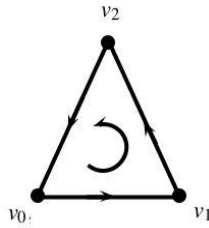
$$\partial_n \sigma := \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_{n+1}]} \quad (2.1)$$

where the hat symbol $\widehat{}$ over v_i indicates that this vertex is deleted from the sequence $v_0, \dots, v_n$. ■

Note that the signs are inserted to take orientations into account, so that all the faces of a simplex are oriented as shown below;



$$\partial[v_0, v_1] = [v_1] - [v_0] \quad (2.2)$$



$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \quad (2.3)$$

Thus the idea of the equation (2.1) is that ‘the boundary of a simplex $[v_0, \dots, v_n]$ is the sum of all of its faces, oriented properly’.

Proposition 2.7 Let X be any Δ -complex. For any $n \geq 1$ the homomorphism $\partial_n \circ \partial_{n+1} : C_{n+1}(X) \rightarrow C_{n-1}(X)$ is the zero homomorphism. ■

Proof We have $\partial_{n+1}(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_{n+1}]}$

$$\partial_n \circ \partial_{n+1}(\sigma) = \sum_{j < i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_{n+1}]} + \sum_{i < j} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_{n+1}]}$$

where in the second term it is $(-1)^{j-1}$ because there are only $j-1$ vertices before v_j as $j > i$ and v_i was removed. Switching i and j in the second sum, we have that the two

sums cancel as on one side we have $(-1)^{i+j}$ and on the other $(-1)^{i+j-1} = -(-1)^{i+j}$. Therefore $\partial_n \circ \partial_{n+1} = 0$. ■

At this point, given any finite Δ -complex X , we have a sequence of homomorphisms of finitely generated abelian groups:

$$\dots \rightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

where $n = \max \dim(\sigma)$ for σ a simplex in X and $\partial_{n-1} \circ \partial_n = 0 \forall n$.

Definition 2.8 A chain complex (of abelian groups) is a collection of abelian groups C_n and homomorphisms $\partial_n: C_n(X) \rightarrow C_{n-1}(X)$ satisfying $\partial_n \circ \partial_{n+1} = 0$. ■

Since a chain complex consists of abelian groups and homomorphisms between them, it is natural to consider kernels and images. Define $Z_n := \text{Ker}(\partial_n: C_n \rightarrow C_{n-1})$ and $B_n := \text{Im}(\partial_{n+1}: C_{n+1} \rightarrow C_n)$. Elements of Z_n are called ‘cycles’ and elements of B_n are called ‘boundaries’. The equation $\partial_n \circ \partial_{n+1} = 0$ means that $\text{Im} \partial_{n+1} \subset \text{Ker} \partial_n$.

Definition 2.9 (Homologous) Two cycles representing the same homology class are said to be homologous. ■

Definition 2.10 (Homology of Chain Complex) $\forall i \geq 0$, the i^{th} homology group of the chain complex C is defined by $H_i(C) := Z_n(C) / B_n(C)$. ■

Then we define the simplicial homology $H_i(X)$ of a Δ -complex X to be the homology of the simplicial chain complex of X :

$$H_i(X) := H_i(C(X))$$

Thus the route while seeking the homology group is:

$$\boxed{\text{Space } X} \rightsquigarrow \boxed{\text{Triangulation of } X} \rightsquigarrow \boxed{\text{Chain Complex}} \rightsquigarrow \boxed{H_i(X)}$$

Note that; since elements of Z_n are called n -cycles, and elements of B_n are called boundaries, then the homology group $H_n = Z_n / B_n$ measures how many n -cycles are not boundaries. In other words, H_n measures the number of nontrivial n -dimensional holes in X .

The following proposition shows that $H_0(X)$ in fact gives the number of path components of X .

Proposition 2.11 Let X be a Δ -complex with path components $X_1, \dots, X_k$. Then $H_0(X) \simeq \mathbb{Z}^k$.

Proof Omitted. [1]

2.3 Computations

Example 2.12 The simplicial homology group of S^1 .

Firstly, consider the triangulation of S^1 :

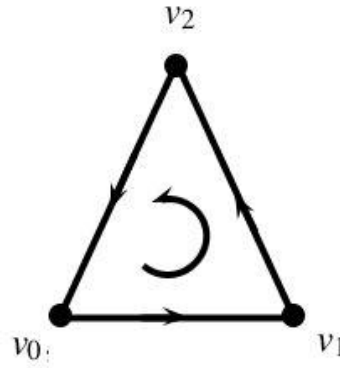


Figure 2.2 Triangulation of S^1

The simplicial complex contains three 0-simplices v_1, v_2, v_3 and three 1-simplices $[v_0, v_1], [v_1, v_2], [v_2, v_0]$. Then

$$C_0(S^1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$C_1(S^1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$C_n(S^1) \simeq 0 \text{ for } n > 1$$

$$Z_1(S^1) = \{c \in C_1(S^1) : \partial_1 c = 0\}$$

$$c \in C_1(S^1) \Rightarrow c = m_1 \langle v_0, v_1 \rangle + m_2 \langle v_1, v_2 \rangle + m_3 \langle v_2, v_0 \rangle \\ \text{where } m_1, m_2, m_3 \in \mathbb{Z}$$

$$\partial_1 c = m_1(\langle v_1 \rangle - \langle v_0 \rangle) + m_2(\langle v_2 \rangle - \langle v_1 \rangle) + m_3(\langle v_0 \rangle - \langle v_2 \rangle) \\ = (m_3 - m_1)\langle v_0 \rangle + (m_1 - m_2)\langle v_1 \rangle + (m_2 - m_3)\langle v_2 \rangle$$

$$\partial_1 c = 0 \Rightarrow m_1 = m_2 = m_3$$

$$Z_1(S^1) = \{n(\langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle) : n \in \mathbb{Z}\} \simeq \mathbb{Z}$$

$$B_1(S^1) = \text{Im}(\partial_2 : C_2 \rightarrow C_1)$$

$$\text{Since } C_2(S^1) = 0; B_1(S^1) = 0.$$

$$\text{Thus } H_1(S^1) = Z_1(S^1)/B_1(S^1) \simeq \mathbb{Z}/\{0\} = \mathbb{Z}.$$

$$Z_0(S^1) = \{c \in C_0(S^1) : \partial_0 c = 0\}$$

$$c \in C_0(S^1) \Rightarrow c = n_1 \langle v_0 \rangle + n_2 \langle v_1 \rangle + n_3 \langle v_2 \rangle \text{ where } n_1, n_2, n_3 \in \mathbb{Z}$$

$$Z_0(S^1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$B_0(S^1) = \text{Im}(\partial_1: C_1 \rightarrow C_0)$$

$$c \in C_1(S^1) \Rightarrow c = m_1 \langle v_0, v_1 \rangle + m_2 \langle v_1, v_2 \rangle + m_3 \langle v_2, v_0 \rangle \\ \text{where } m_1, m_2, m_3 \in \mathbb{Z}$$

$$\partial_1 c = m_1(\langle v_1 \rangle - \langle v_0 \rangle) + m_2(\langle v_2 \rangle - \langle v_1 \rangle) + m_3(\langle v_0 \rangle - \langle v_2 \rangle) \\ = (m_3 - m_1)\langle v_0 \rangle + (m_1 - m_2)\langle v_1 \rangle + (m_2 - m_3)\langle v_2 \rangle$$

$$n_1 = m_3 - m_1$$

$$n_2 = m_1 - m_2$$

$$n_3 = m_2 - m_3$$

These equations imply $n_1 + n_2 + n_3 = 0$.

$$B_0(S^1) = \{n_1 \langle v_0 \rangle + n_2 \langle v_1 \rangle + n_3 \langle v_2 \rangle \text{ where } n_1 + n_2 + n_3 = 0\} = \mathbb{Z} \oplus \mathbb{Z}$$

$$\text{Thus } H_0(S^1) = Z_0(S^1)/B_0(S^1) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} / \mathbb{Z} \oplus \mathbb{Z} \simeq \mathbb{Z}.$$

Also $H_n(S^1) = 0$ for $n \geq 2$.

In this case, it is important to notice $H_1(S^1) \simeq \mathbb{Z}$ as it shows the 1-dimensional hole in the middle of the triangle. This is precisely what makes cycles not boundaries there is a hole in the middle so we cannot write the cycles as the boundary of something. ■

Example 2.13 The simplicial homology of torus.

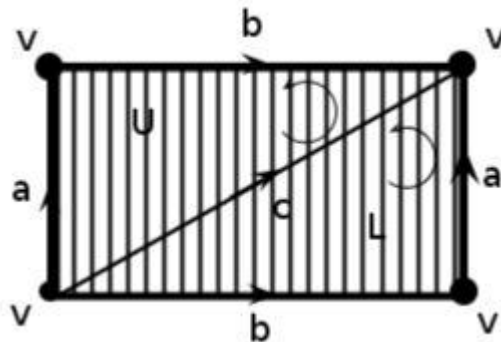


Figure 2.3 Δ -complex structure of torus

The simplicial complex of torus contains one 0-simplex v , three 1-simplices a, b, c and two 2-simplices U, L . Then;

$$C_0(T) = \mathbb{Z}[v]$$

$$C_1(T) = \mathbb{Z}^3[a, b, c]$$

$$C_2(T) = \mathbb{Z}^2[U, L]$$

$$\partial_1 a = \partial_1 b = \partial_1 c = v - v = 0. \text{ This implies } B_0(T) = \text{Im } \partial_1 = 0.$$

$$\partial_2 U = c - b - a, \quad \partial_2 L = b + a - c = -\partial_2 \Rightarrow B_1(T) = \mathbb{Z}[c - b - a]$$

$$\text{Also } B_n(T) = \{0\} \text{ for all } n \geq 2.$$

$$Z_0(T) = \text{Ker } \partial_0 = C_0(T) = \mathbb{Z}[v]$$

$$Z_1(T) = \text{Ker } \partial_1 = C_1(T) = \mathbb{Z}^3[a, b, c]$$

For finding $Z_2(T)$;

$$m_1 \partial_2 U + m_2 \partial_2 L = 0 \Leftrightarrow m_1(c - b - a) + m_2(b + a - c) = 0$$

$$\Leftrightarrow m_1 = m_2$$

$$Z_2(T) = \{m_1 U + m_1 L : m_1 \in \mathbb{Z}\} = \mathbb{Z}[U + L]$$

Also $Z_n(T) = 0$ for all $n \geq 3$. Thus;

$$H_0(T) = \mathbb{Z}[v] / \{0\} \simeq \mathbb{Z}[v]$$

$$H_1(T) = \mathbb{Z}^3[a, b, c] / \mathbb{Z}[c - b - a] \simeq \mathbb{Z}^2[a, b]$$

$$H_2(T) = \mathbb{Z}[U + L] / \{0\} \simeq \mathbb{Z}[U + L] \blacksquare$$

Example 2.14 The simplicial homology of Klein bottle.

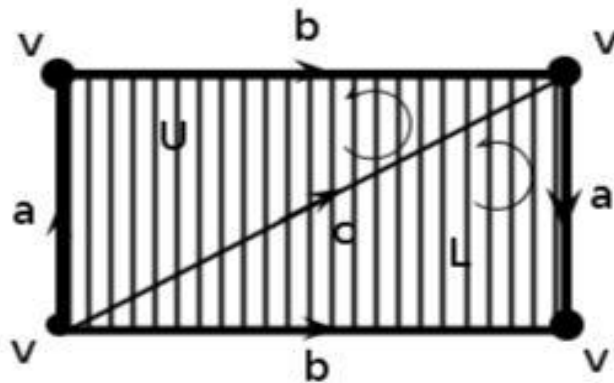


Figure 2.4 Δ -complex structure of Klein bottle

The simplicial complex of Klein bottle contains one 0-complex v , three 1-simplices a, b, c and two 2-simplices U, L . Then;

$$C_0(K) = \mathbb{Z} [v]$$

$$C_1(K) = \mathbb{Z}^3 [a, b, c]$$

$$C_2(K) = \mathbb{Z}^2 [U, L]$$

$$\partial_1 a = \partial_1 b = \partial_1 c = v - v = 0. \text{ This implies } B_0(K) = \text{Im } \partial_1 = 0.$$

$$\partial_2 U = c - b - a, \quad \partial_2 L = b - a - c = -\partial_2 \Rightarrow B_1(K) = \mathbb{Z}^2 [c - b - a, b - a - c]$$

$$\text{Also } B_n(K) = \{0\} \text{ for all } n \geq 2.$$

$$Z_0(K) = \text{Ker } \partial_0 = C_0(K) = \mathbb{Z}[v]$$

$$Z_1(K) = \text{Ker } \partial_1 = C_1(K) = \mathbb{Z}^3[a, b, c]$$

For finding $Z_2(K)$;

$$m_1 \partial_2 U + m_2 \partial_2 L = 0 \Leftrightarrow m_1(c - b - a) + m_2(b - a - c) = 0$$

$$\Leftrightarrow c(m_1 + m_2) + b(m_2 - m_1) + a(-m_1 - m_2) = 0$$

$$\Leftrightarrow m_1 + m_2 = m_1 - m_2 = 0$$

$$\Leftrightarrow m_1 = m_2 = 0$$

$$Z_2(K) = \{0\}. \text{ Thus;}$$

$$H_0(K) = \mathbb{Z}[v] / \{0\} \simeq \mathbb{Z}[v]$$

$$H_1(K) = \mathbb{Z}^3 [a, b, c] / \mathbb{Z}[c - b - a, b - a - c] \simeq \mathbb{Z} [a] / \mathbb{Z}[2a] \times \mathbb{Z}[b] \simeq \mathbb{Z} / 2\mathbb{Z} \times \mathbb{Z}$$

$$\text{And } H_n(K) = \{0\} \text{ for } n \geq 2.$$

Example 2.15 The simplicial homology of the unit disc B^2 .

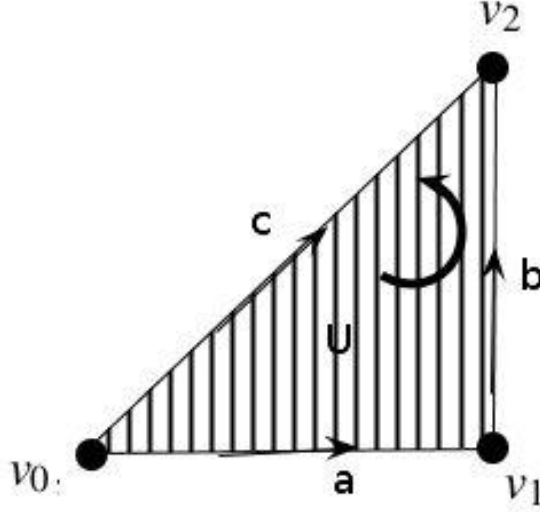


Figure 2.5 Δ -complex structure of unit disc

The simplicial complex of B^2 contains three 0-simplices, three 1-simplices and one 2-simplex.

$$C_0(B^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$C_1(B^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$C_2(B^2) = \mathbb{Z}$$

$$\partial_1 a = v_1 - v_0, \quad \partial_1 b = v_2 - v_1, \quad \partial_1 c = v_2 - v_0 = \partial_1 a + \partial_1 b$$

$$\text{This implies } B_0(B^2) = \mathbb{Z}^2 [v_1 - v_0, v_2 - v_1]$$

$$\partial_2 U = a + b - c \Rightarrow B_1(B^2) = \mathbb{Z} [a + b - c]$$

$$B_n(B^2) = \{0\} \quad \forall n \geq 2.$$

$$Z_0(B^2) = C_0(B^2) = \mathbb{Z}^3 [v_0, v_1, v_2]$$

To find $Z_1(B^2)$;

$$m_1 \partial_1 a + m_2 \partial_1 b + m_3 \partial_1 c = 0 \Leftrightarrow m_1(v_1 - v_0) + m_2(v_2 - v_1) + m_3(v_2 - v_0) = 0$$

$$\Leftrightarrow m_1 + m_3 = m_1 - m_2 = m_2 + m_3 = 0$$

$$\Leftrightarrow m_1 = m_2, \quad m_3 = -m_1$$

$$Z_1(B^2) = \{m_1 a + m_1 b - m_1 c\} = \mathbb{Z} [a + b - c]. \text{ Also } Z_n(B^2) = \{0\} \quad \forall n \geq 2. \text{ Thus}$$

$$H_0(B^2) = \mathbb{Z}^3 [v_0, v_1, v_2] / \mathbb{Z}^2 [v_1 - v_0, v_2 - v_1] \simeq \mathbb{Z}^2 [v_1, v_0] / \mathbb{Z} [v_1 - v_0] \simeq \mathbb{Z} [v_0]$$

$$H_1(B^2) = \mathbb{Z} [a + b - c] / \mathbb{Z} [a + b - c] = \{0\} \quad \blacksquare$$

Example 2.16 The simplicial homology of S^n for $n > 2$.

To find Δ -complex structure for S^n , let's take two n -simplices $[v_0, \dots, v_n]$ and $[u_0, \dots, u_n]$ and identify them along their boundaries, i.e.;

$$[v_0, \dots, \hat{v}_i, \dots, v_n] \simeq [u_0, \dots, \hat{u}_i, \dots, u_n] \quad 0 \leq i \leq n$$

But for a large n , this process will get complicated. To solve this problem, some other homology theories have been introduced. ■

SINGULAR HOMOLOGY

The problem with the simplicial homology is that it is defined only for Δ -complexes. Because of this, a new theory called ‘singular homology’ has been introduced and we are able to define same calculations for any topological space.

After defining singular homology, a fancy question arises: Are simplicial and singular homology groups coincide?

At the end of the chapter we will prove that simplicial and singular homology groups for Δ -complexes coincide.

3.1 Construction

The idea of singular homology is to consider maps from simplices into the topological space instead of simplices.

Definition 3.1 A singular n -simplex on a topological space X is a continuous map $\sigma: \Delta^n \rightarrow X$ from the standard n -simplex to X . ■

In this definition singular means that map σ need not to be injective, i.e., there may be singularities in the image of σ .

Let $C_n(X)$ be the free abelian group on the set of singular n -simplices of X . Elements of $C_n(X)$ are called n -chains. We can then define boundary map $\partial_n: C_n(X) \rightarrow C_{n-1}(X)$ in a similar fashion:

$$\partial_n \sigma := \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_{n+1}]}$$

The equation $\partial_n \circ \partial_{n+1} = 0$ holds with the same proof in simplicial homology so we can define singular homology group $H_n(X) = \text{Ker} \partial_n / \text{Im} \partial_{n+1}$

Proposition 3.2 Let X be a non-empty and path-connected topological space. Then $H_0(X) \simeq \mathbb{Z}$. ■

Proof Since $\partial_0 = 0$; $H_0(X) = C_0(X) / \text{Im}\partial_1$. Let f be a homomorphism s.t.

$$f: C_0(X) \rightarrow \mathbb{Z}$$

$$\sum_i n_i \sigma_i \mapsto \sum_i n_i$$

Since X is non-empty, f is surjective. Also since X is path-connected $\text{Ker} f = \text{Im}\partial_1$. Thus $C_0(X) / \text{Im}\partial_1 \simeq \mathbb{Z}$, i.e. $H_0(X) \simeq \mathbb{Z}$. ■

Proposition 3.3 Let X be a point. Then $H_0(X) \simeq \mathbb{Z}$ and $H_n(X) = 0$ for $n > 0$. ■

Proof For a singular n -simplex σ_n ;

$$\partial(\sigma_n) = \sum_i (-1)^i \sigma_{n-1} \quad (3.1)$$

The equation (3.1) is equal to 0 for n even and σ_{n-1} for n odd ($n \neq 0$). Then we have the chain complex

$$\dots \rightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0$$

where the boundary maps are isomorphisms and trivial maps except at the last $\mathbb{Z}$. Thus $H_0(X) \simeq \mathbb{Z}$ and $H_n(X) = 0$ for $n > 0$. ■

Definition 3.4 Reduced homology groups $\widetilde{H}_n(X)$ is defined to be the homology groups of the chain complex

$$\dots \rightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{f} \mathbb{Z} \rightarrow 0$$

where $f(\sum_i n_i \sigma_i) = \sum_i n_i$. ■

Since $f\partial_1 = 0$; f vanishes on $\text{Im}\partial_1$ and induces a map $H_0(X) \rightarrow \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$. This implies $H_0(X) \simeq \widetilde{H}_0(X) \oplus \mathbb{Z}$. Furthermore, $H_n(X) \simeq \widetilde{H}_n(X)$ for $n > 0$. ■

Definition 3.5 A chain map between two chain complexes $C_n(X)$ and $C_n(Y)$ is a collection of homomorphisms $\{f_i\}$ s.t. the following diagram commutes:

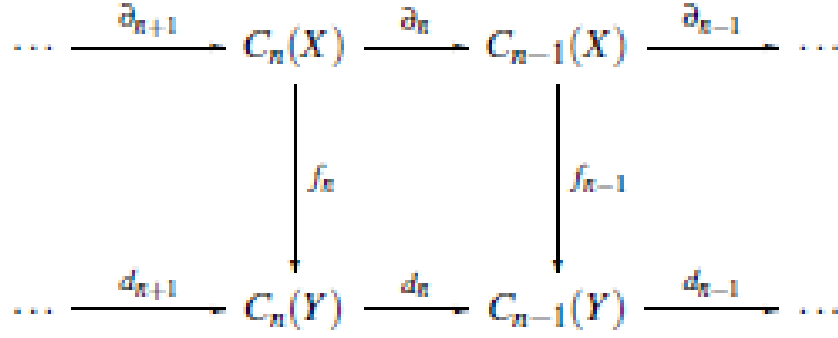


Figure 3.1 Chain complex diagram

Proposition 3.6 A chain map between chain complexes induces homomorphism between the homology groups of the two complexes. ■

Proof We have to show that f_n take cycles to cycles and boundaries to boundaries. Since $f_n: C_n(X) \rightarrow C_n(Y)$ implies $\partial_n(f_n \alpha) = f_n(\partial_n \alpha) = 0$. Thus f_n takes cycles to cycles. Also, since $f_n(\partial_n \beta) = \partial_n(f_n \beta)$; f_n takes boundaries to boundaries. Hence f_n induces a homomorphism $f_*: H_n(X) \rightarrow H_n(Y)$. ■

Proposition 3.7 A map $f: X \rightarrow Y$ induces a chain map from the singular chain complex of X to that of Y . ■

Proof If we have a map $f: X \rightarrow Y$, a singular simplex $\sigma: \Delta^n \rightarrow X$ induces a singular simplex $f_*(\sigma) = f \circ \sigma: \Delta^n \rightarrow Y$

We have to show that $f_*(\partial_n(\sigma)) = \partial_n(f_*(\sigma))$. Since $f_*(\sum_i a_i \sigma_i) = \sum_i a_i f_*(\sigma_i)$, we have;

$$f_*(\partial_n(\sigma)) = f_*(\sum_i (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_{n+1}]}) = \sum_i (-1)^i f_*(\sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_{n+1}]}) = \partial_n(f_*(\sigma)). \quad \blacksquare$$

It is useful to remember some important properties of induced homomorphism:

- If we have $X \xrightarrow{f} Y \xrightarrow{g} Z$ then,
 $(g \circ f)_*(\sigma) = (g \circ f) \circ \sigma = g \circ (f \circ \sigma) = (f_* \circ g_*)(\sigma)$
Thus $(g \circ f)_*(\sigma) = (f_* \circ g_*)(\sigma)$
- It is also clear that $\text{id}_* = \text{id}$ where id denotes the identity map of a space or a group.

Theorem 3.8 If two maps $f, g: X \rightarrow Y$ are homotopic, the homomorphism induced in homology by f and g are the same, i.e., $f_* = g_*: H_n(X) \rightarrow H_n(Y)$. ■

Proof Omitted. [1] ■

Corollary 3.9 If X and Y are homotopy equivalent, then all their homology groups coincide. ■

Proof Let X and Y be homotopy equivalent. Then $\exists f: X \rightarrow Y$ and $g: Y \rightarrow X$ s.t. $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. This implies $(g \circ f)_* = g_* \circ f_* = \text{id}_* = f_* \circ g_* = (f \circ g)_*$ where id is identity upon the homology groups. Thus g_* and f_* are inverses of each other, i.e. isomorphism. ■

Definition 3.10 Two chain maps $f_n: C_n(X) \rightarrow C_n(Y)$ and $g_n: C_n(X) \rightarrow C_n(Y)$ are chain homotopic if there exist maps $h_n: C_n(X) \rightarrow C_n(Y)$ s.t. $f - g = \partial h + h \partial$. ■

This definition says that, given the diagram,

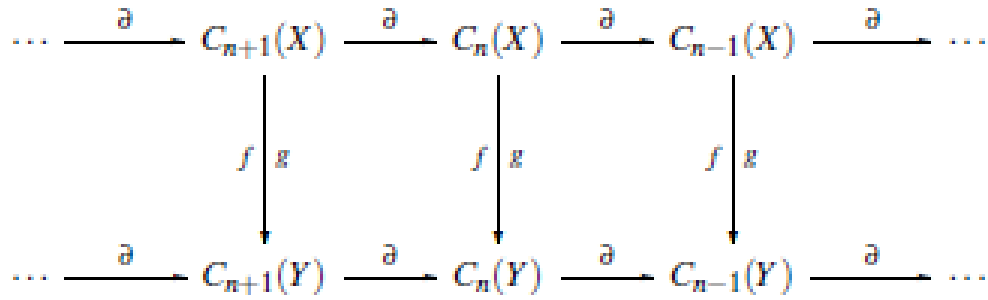


Figure 3.2 Chain complex diagram

h_n makes the following diagram commutative

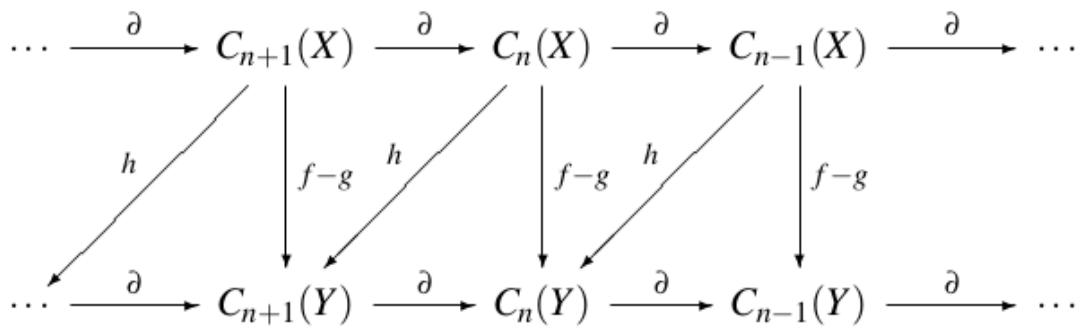


Figure 3.3 Chain complex diagram

It is shown in [1] in the proof of theorem (3.8) that $\partial P + P\partial = g - f$, i.e., P is a chain homotopy between the chain maps f and g . This implies the following:

Proposition 3.11 Two chain-homotopic chain maps induce the same homomorphisms in homology. ■

3.2 Some Preliminaries About Exact Sequences

The sequence of homomorphisms

$$\dots \rightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \rightarrow \dots$$

is said to be exact if $\text{Ker} f_n = \text{Im} f_{n+1} \forall n$. The inclusion $\text{Im} f_{n+1} \subset \text{Ker} f_n$ implies $f_n \circ f_{n+1} = 0$, so the sequence is a chain complex. Because of these facts, homology theory measures how far is a sequence from being exact.

Some properties proven in [1] are listed below:

- $0 \rightarrow A \xrightarrow{f} B$ is exact iff $\text{Ker} f = 0$.
- $A \xrightarrow{f} B \rightarrow 0$ exact iff $\text{Im} f = 0$.
- $0 \rightarrow A \xrightarrow{f} B \rightarrow 0$ is exact iff f is an isomorphism.
- $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is exact iff f is injective, g is surjective and $\text{Ker} g = \text{Im} f$, so g induces an isomorphism $C \simeq B / \text{Im} f$. This exact sequence is called short exact sequence.

3.3 Relative Homology

Let X be any topological space and $A \subseteq X$ be any subspace. The ‘relative chain group’ of X relative to A is defined to be $C_n(X, A) := C_n(X) / C_n(A)$.

Since the boundary map $\partial_n: C_n(X) \rightarrow C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$; it induces a boundary homomorphism $\partial'_n: C_n(X, A) \rightarrow C_{n-1}(X, A)$.

The equation $\partial^2 = 0$ implies that $(\partial')^2 = 0$. So we have a chain complex. Then the relative homology group is defined to be $H_n(X, A)$.

In this case relative cycles are the elements $\alpha \in C_n(X)$ s.t. $\partial\alpha \in C_{n-1}(A)$, and relative boundaries are elements $\alpha \in C_n(X)$ s.t. $\alpha = \partial\beta + \gamma$ where $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

With these properties, we can think of $H_n(X, A)$ as ‘homology of X modulo A ’.

Now consider the exact sequence

$$0 \rightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \rightarrow 0$$

where i is the inclusion and j is the quotient map.

Using this construction, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & C_{n+1}(A) & \xrightarrow{i} & C_{n+1}(X) & \xrightarrow{j} & C_{n+1}(X)/C_{n+1}(A) \longrightarrow 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
 0 & \longrightarrow & C_{n+1}(A) & \xrightarrow{i} & C_{n+1}(X) & \xrightarrow{j} & C_{n+1}(X)/C_{n+1}(A) \longrightarrow 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
 0 & \longrightarrow & C_{n+1}(A) & \xrightarrow{i} & C_{n+1}(X) & \xrightarrow{j} & C_{n+1}(X)/C_{n+1}(A) \longrightarrow 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Figure3.4 Chain complex diagram

Note that this diagram is commutative by the definition of the boundary maps.

Lemma 3.12 (Zigzag Lemma) Suppose we have the following ‘short exact sequence of chain complexes’:

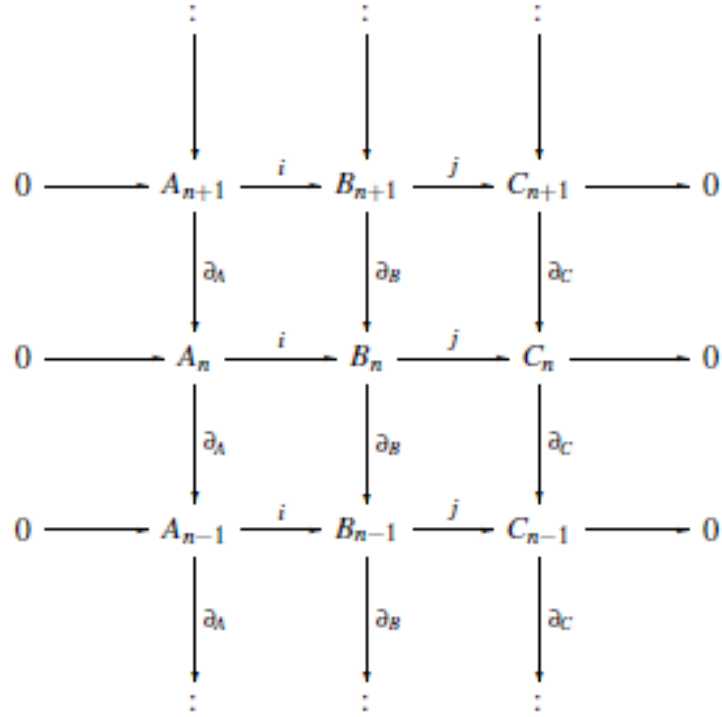


Figure 3.5 Zigzag lemma diagram

where the rows are exact sequences and each column is a complex. Then there is a family of ‘connecting homomorphisms’ $\partial_*: H_n(C) \rightarrow H_{n-1}(A)$ s.t

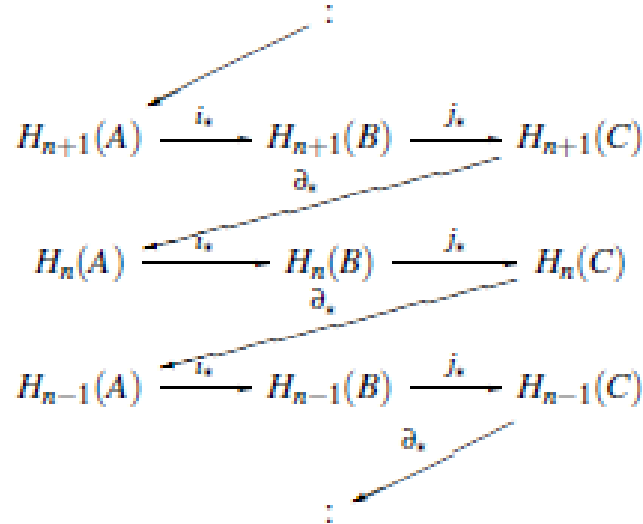


Figure 3.6 Connecting homomorphism diagram

So we have the following long exact sequence:

$$\dots \rightarrow H_{n+1}(C) \xrightarrow{\partial_*} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial_*} \dots \rightarrow H_0(C) \rightarrow 0 \blacksquare$$

Proof Ommited. [1]

This theorem implies a long exact sequence of homology groups:

$$\dots \rightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} \dots \rightarrow H_0(X, A) \rightarrow 0$$

3.4 The Excision Theorem

Theorem 3.13 (The Excision Theorem) Let X be a topological space with subspaces A , U with $U \subseteq A \subseteq X$. Assume that the closure of U is contained in the interior of A . Then the inclusion map $i: (X - U, A - U) \rightarrow (X, A)$ induces an isomorphism $i_*: H_n(X - U, A - U) \rightarrow H_n(X, A) \forall n \geq 0$. ■

This theorem basically says that excising away U from X and A does not change the relative homology.

Proof Ommited. [1]

The excision theorem has many applications. The following two theorems will be proven with the aid of the excision theorem.

Theorem 3.14 Let A be an open subset of a topological space X . Then the quotient map $p: (X, A) \rightarrow (X/A, A/A)$ induces an isomorphism

$$p_*: H_i(X, A) \rightarrow H_i(X/A, A/A) \simeq \widetilde{H}_i(X/A)$$

Proof Let A be an open subset of X . Since A is open, we can take $U = A$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{\simeq} & H_n(X - A, A - A) = H_n(X - A) \\ \downarrow p_* & & \downarrow (p|_{X-A})_* \\ H_n(X/A, A/A) & \xrightarrow{\simeq} & H_n(X/A - A/A, A/A) \end{array}$$

Figure 3.7 Homology group diagram

The horizontal arrows are isomorphism by the excision theorem since $(p|_{X-A})_*$ is the map induced by the restriction of p to $X - A$, which is a homomorphism, $(p|_{X-A})_*$ is

an isomorphism. Furthermore, since the diagram is commutative then p_* is also an isomorphism. ■

Theorem 3.15 Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. If $U \simeq V$ then $n = m$. ■

Proof Let $n, m > 1$. Take $x \in U$ then by excision theorem we have;

$$H_i(U, U - x) \simeq H_i(\mathbb{R}^n, \mathbb{R}^n - x).$$

Since $\mathbb{R}^n - x$ deformation retracts to S^{n-1} , we have the following long exact sequence

$$\dots \rightarrow H_i(\mathbb{R}^n) \rightarrow H_i(\mathbb{R}^n, \mathbb{R}^n - x) \rightarrow H_{i-1}(S^{n-1}) \rightarrow H_{i-1}(\mathbb{R}^n) \rightarrow \dots$$

So $H_i(\mathbb{R}^n, \mathbb{R}^n - x) \simeq H_{i-1}(S^{n-1}) \forall i > 1$ except $i = n$.

But any homeomorphism $f: U \rightarrow V$ induces a homeomorphism $h: (U, x) \rightarrow (V, f(x))$ which induces an isomorphism

$$H_i(\mathbb{R}^n, \mathbb{R}^n - x) \simeq H_i(\mathbb{R}^n, \mathbb{R}^n - f(x))$$

Thus $n = m$. ■

Now the knowledge that will be useful while finding the homology groups of S^n has been completed.

Example 3.16 Compute the homology groups of S^n ($n > 2$).

We can think S^n as the quotient of an n -disk by its boundary, i.e., $S^n \simeq D^n / \partial D^n$. Since $\partial D^n \simeq S^{n-1}$, we have the following exact sequence;

$$\dots \rightarrow H_i(D^n) \rightarrow H_i(D^n, S^{n-1}) \rightarrow H_{i-1}(S^{n-1}) \rightarrow H_{i-1}(D^n) \rightarrow \dots$$

Since the disk is contractible, all its homology groups vanish. So the above exact sequence becomes

$$0 \rightarrow H_i(D^n, S^{n-1}) \rightarrow H_{i-1}(S^{n-1}) \rightarrow 0$$

This implies $H_i(D^n, S^{n-1}) \simeq H_{i-1}(S^{n-1})$. We know that $\widetilde{H}_i(S^0) \simeq \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & \text{otherwise} \end{cases}$

It follows that $\widetilde{H}_i(S^n) \simeq \begin{cases} \mathbb{Z}, & i = n \\ 0, & \text{otherwise} \end{cases}$

Thus $H_i(D^n, S^{n-1}) \simeq H_i(S^n) \simeq \begin{cases} \mathbb{Z}, & i = n \\ 0, & \text{otherwise} \end{cases}$ ■

3.5 Isomorphism of The Simplicial and Singular Homology

Lemma 3.17 (Five Lemma) Suppose the following diagram of abelian groups commutes where the two rows are exact sequences:

$$\begin{array}{ccccccccc}
 A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
 \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\
 A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
 \end{array}$$

Figure 3.8 Five lemma diagram

If α, β, δ and ϵ are all isomorphisms then γ is also an isomorphism. ■

Proof Omitted. [1]

Let $f_i: C_n^\Delta(X, A) \rightarrow C_n(X, A)$ be a homomorphisms defined by sending each n -simplex $\Delta^n \in C_n^\Delta(X)$ of X to its corresponding singular n -simplex given by its characteristic map $\sigma: \Delta^n \rightarrow X$. Then f_i is a chain map so induces homomorphisms $f_*: H_n^\Delta(X, A) \rightarrow H_n(X, A)$.

Theorem 3.18 The homomorphisms $f_*: H_n^\Delta(X, A) \rightarrow H_n(X, A)$ are isomorphisms.

Proof • Case 1: X is finite dimensional

Firstly suppose $A = \emptyset$. For X^k the k -skeleton of X , we have the following exact sequence

$$H_{n+1}(X^k, X^{k-1}) \rightarrow H_n(X^{k-1}) \rightarrow H_n(X^k) \rightarrow H_n(X^k, X^{k-1}) \rightarrow H_{n-1}(X^{k-1})$$

Then by Zigzag lemma, we have the following commutative diagram;

$$\begin{array}{ccccccccc}
 H_{n+1}^\Delta(X^k, X^{k-1}) & \rightarrow & H_n^\Delta(X^{k-1}) & \longrightarrow & H_n^\Delta(X^k) & \longrightarrow & H_n^\Delta(X^k, X^{k-1}) & \rightarrow & H_{n-1}^\Delta(X^{k-1}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_{n+1}(X^k, X^{k-1}) & \rightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X^k) & \longrightarrow & H_n(X^k, X^{k-1}) & \rightarrow & H_{n-1}(X^{k-1})
 \end{array}$$

Figure 3.9 Homology group diagram

$C_n^\Delta(X^k, X^{k-1})$ is zero for $n \neq k$ and is free abelian with basis in bijection with the k -simplices of X when $n = k$, so we can say the same of $H_n^\Delta(X^k, X^{k-1})$. For the corresponding singular homology group $H_n(X^k, X^{k-1})$, let us consider the map

$\coprod_i (\Delta_i^k, \partial \Delta_i^k) \rightarrow (X^k, X^{k-1})$ consisting of all characteristic maps $\Delta^k \rightarrow X$. This induces $\coprod_i \Delta_i^k / \coprod_i \partial \Delta_i^k \simeq X^k / X^{k-1}$.

Then it induces isomorphisms on the singular homology groups. Thus $H_n(X^k, X^{k-1})$ is 0 for $n \neq k$ and free abelian with basis represented by the relative cycles given by the characteristic maps of the k -simplices of X when $n = k$. Hence the first and the fourth vertical maps in the diagram are isomorphisms. By induction on k , we may assume that the second and the fifth vertical maps are also isomorphisms. Five Lemma implies that the third vertical map is also an isomorphism. Hence $f_*: H_n^\Delta(X, A) \rightarrow H_n(X, A)$ are isomorphisms.

Now suppose $A \neq \emptyset$. Then by zigzag lemma, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 H_n^\Delta(A) & \longrightarrow & H_n^\Delta(X) & \longrightarrow & H_n^\Delta(X, A) & \longrightarrow & H_{n-1}^\Delta(A) & \longrightarrow & H_{n-1}^\Delta(X) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X)
 \end{array}$$

Figure 3.10 Homology group diagram

which has exact rows and the first, second, fourth and fifth vertical maps are isomorphisms. Again Five Lemma implies that the third vertical map is also an isomorphism.

• Case 2: Omitted. [1] ■

3.6 Mayer-Vietoris Sequences

Suppose A and B are two subspaces of X such that $X = \text{int}(A) \cup \text{int}(B)$. We then have the following commutative diagram:

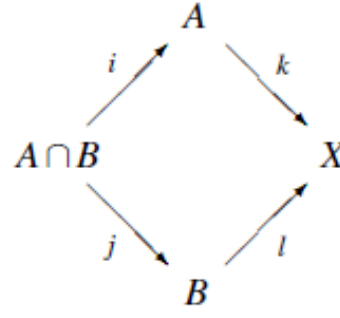


Figure 3.11 Mayer-Vietoris sequence

where i, j, k, l are inclusions. Then we have the following short exact sequence

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{(i_*, -j_*)} C_n(A) \oplus C_n(B) \xrightarrow{k_* + l_*} C_n(A + B) \longrightarrow 0$$

Then the following diagram is commutative

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_{n+1}(A \cap B) & \longrightarrow & C_{n+1}(A) \oplus C_{n+1}(B) & \longrightarrow & C_{n+1}(A + B) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_n(A \cap B) & \longrightarrow & C_n(A) \oplus C_n(B) & \longrightarrow & C_n(A + B) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_{n-1}(A \cap B) & \longrightarrow & C_{n-1}(A) \oplus C_{n-1}(B) & \longrightarrow & C_{n-1}(A + B) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Figure 3.12 Chain complex diagram

Note that this diagram is commutative since the horizontal arrows are inclusions. By Zigzag Lemma, there exists a long exact sequence:

$$H_{n+1}(X) \xrightarrow{\partial_*} H_n(A \cap B) \xrightarrow{(i_*, -j_*)} H_n(A) \oplus H_n(B) \xrightarrow{k_* + l_*} H_n(X) \xrightarrow{\partial_*} \dots \xrightarrow{k_* + l_*} H_0(X) \longrightarrow 0 \blacksquare$$

Mayer-Vietoris sequence can also be considered for reduced homology:

$$\tilde{H}_{n+1}(X) \xrightarrow{\partial_*} \tilde{H}_n(A \cap B) \xrightarrow{(i_*, -j_*)} \tilde{H}_n(A) \oplus \tilde{H}_n(B) \xrightarrow{k_* + l_*} \tilde{H}_n(X) \xrightarrow{\partial_*} \dots \xrightarrow{k_* + l_*} \tilde{H}_0(X) \rightarrow 0$$

Mayer-Vietoris sequence can be used for computing the homology groups of Klein bottle:

The Klein bottle can be decomposed as the union of two Möbius bands A and B glued along their boundary circles. A , B and $A \cap B$ are homotopy equivalent to circles. Then the exact sequence can be written as follows:

$$\begin{aligned} H_2(K) \xrightarrow{\partial_*} H_1(A \cap B) \xrightarrow{(i_*, -j_*)} H_1(A) \oplus H_1(B) \xrightarrow{k_* + l_*} H_1(K) \xrightarrow{\partial_*} H_0(A \cap B) \xrightarrow{(i_*, -j_*)} H_0(A) \\ \oplus H_0(B) \rightarrow H_0(K) \rightarrow 0 \end{aligned}$$

$$0 \rightarrow H_0(A \cap B) \xrightarrow{(i_*, -j_*)} H_0(A) \oplus H_0(B) \rightarrow H_0(K) \rightarrow 0$$

Then;

$$0 \rightarrow \mathbb{Z} \xrightarrow{(i_*, -j_*)} \mathbb{Z} \oplus \mathbb{Z} \rightarrow H_0(K) \rightarrow 0 \quad \Rightarrow \quad H_0(K) \simeq \mathbb{Z} \oplus \mathbb{Z} / \mathbb{Z} \simeq \mathbb{Z}$$

Since $H_1(A \cap B) \rightarrow H_1(A) \oplus H_1(B)$ is injective, we have;

$$0 \rightarrow H_2(K) \rightarrow 0 \text{ i.e, } H_2(K) \simeq 0.$$

Now consider the exact sequence

$$0 \rightarrow H_1(A \cap B) \xrightarrow{(i_*, -j_*)} H_1(A) \oplus H_1(B) \xrightarrow{k_* + l_*} H_1(K) \rightarrow 0$$

The map $(i_*, -j_*): \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ sends 1 to $(2, -2)$ since the boundary circle of a möbius band wraps twice around the central circle. We can write every element in $\mathbb{Z} \oplus \mathbb{Z}$ as $a(1, -1) + b(1, 0)$ for $a, b \in \mathbb{Z}$, with exactly the elements of the form $a \in 2\mathbb{Z}$ and $b = 0$. Then

$$\text{Im}(i_*, -j_*) = \text{Ker}(k_* + l_*) = 2\mathbb{Z}(1, -1)$$

$$\text{Thus } H_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z}(1, -1) \simeq \mathbb{Z}(1, -1) \oplus \mathbb{Z}(1, 0) / 2\mathbb{Z}(1, -1) \simeq \mathbb{Z} \oplus \mathbb{Z}_2.$$

CELLULAR HOMOLOGY

Cellular homology is a very efficient tool for computing the homology groups of topological spaces which are CW-complexes based on cell complex decompositions and degree calculations.

4.1 Degree

Let $f: S^n \rightarrow S^n$ be a continuous map. Then it induces a homomorphism $f_*: H_n(S^n) \rightarrow H_n(S^n)$. Since $\tilde{H}_n(S^n) \simeq \mathbb{Z}$, for any homomorphism $\phi: \mathbb{Z} \rightarrow \mathbb{Z}$ a unique $d \in \mathbb{Z}$ such that $\phi(z) = dz \forall z \in \mathbb{Z}$.

Definition 4.1 The degree of f is defined to be the unique integer such that $f_*(z) = (\deg f)z \forall z \in H_n(S^n)$. ■

Some properties of degree are listed below:

- Since $\mathbf{1}_* = \mathbf{1}$; $\deg \mathbf{1} = 1$.
- If f is not surjective then $\deg f = 0$.
- If $f \simeq g$ then $f_* = g_*$. This implies $\deg f = \deg g$.
- Since $(fg)_* = f_* g_*$; $\deg fg = \deg f \cdot \deg g$

4.2 CW Complexes

An orientable surface $\sum_g$ of genus g can be constructed from a polygon with $4g$ sides by identifying the edges. The $4g$ edges of the polygon is the union of $2g$ circles. These circles intersect in a single point. The interior of a polygon can be thought as an open disk attached to the circles. This open disk is called a 2-cell. The union of the circles can be obtained by attaching open arcs. These arcs are called 1-cells.

Thus if we want to build up a surface, take a point attach 1-cells to this point, then attach a 2-cell.

This process can be generalized as follows:

- Take a discrete set X^0 . X^0 is called the 0-skeleton of X . Points of X^0 are called 0-cells.
- Inductively, suppose the $(n-1)$ -skeleton X^{n-1} is defined for $n > 1$. Suppose we have a collection of n -disks $\{D_\alpha^n : \alpha \in I\}$ and $\phi_\alpha : \partial D_\alpha^n \rightarrow X^{n-1}$ is a continuous map $\forall \alpha$. X^n , the n -skeleton of X is then defined as follows:

$$X^n := [X^{n-1} \amalg_{\alpha \in I} D_\alpha^n] / \sim$$

where $\sim$ is defined by $x \sim \phi_\alpha(x) \forall \alpha \in I$.

Let $X = \amalg_{n \geq 0} X^n$. The space X is called a ‘CW complex’ and $\phi_\alpha(\partial D_\alpha^n)$ are called ‘ n -cells’. This process says that X^n is obtained from X^{n-1} by adding n -cells. The maps ϕ_α are called ‘attaching maps’.

Some examples of CW complexes are given below:

- A 1-dimensional CW complex is called a graph.
- The CW-complex structure of S^n for $n > 0$ consists of one 0-cell e^0 and one n -cell e^n where the attaching map is the constant map $\phi : S^{n-1} \rightarrow \{v\}$
- The CW complex structure of the genus g surface Σ_g consists of one vertex, $2g$ -edges $a_1, b_1, \dots, a_g, b_g$ with the obvious attaching maps and one 2-cell D^2 with attaching map $\phi : \partial D^2 \rightarrow (\cup a_i, \cup b_i)$ defined by dividing the circle ∂D^2 in $4g$ segments and mapping them onto the edges $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$.

4.3 Cellular Homology

Lemma 4.2 Let X be a CW complex. Then;

- $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is a free abelian group when $n = k$, with basis in bijection with the n -cells of X .
- $H_k(X^n) = 0$ for $k > n$. If X is finite dimensional then $H_k(X) = 0$ for $k > \dim X$.
- The inclusion $i : X^n \rightarrow X$ induces isomorphism $i_* : H_k(X^n) \rightarrow H_k(X)$ if $k < n$. ■

Proof Omitted. [1]

Let X be a CW-complex. By above lemma, there is a long exact sequence corresponding to the pairs (X^{k+1}, X^k) , (X^k, X^{k-1}) , (X^{k-1}, X^{k-2}) , ... which form the following commutative diagram:

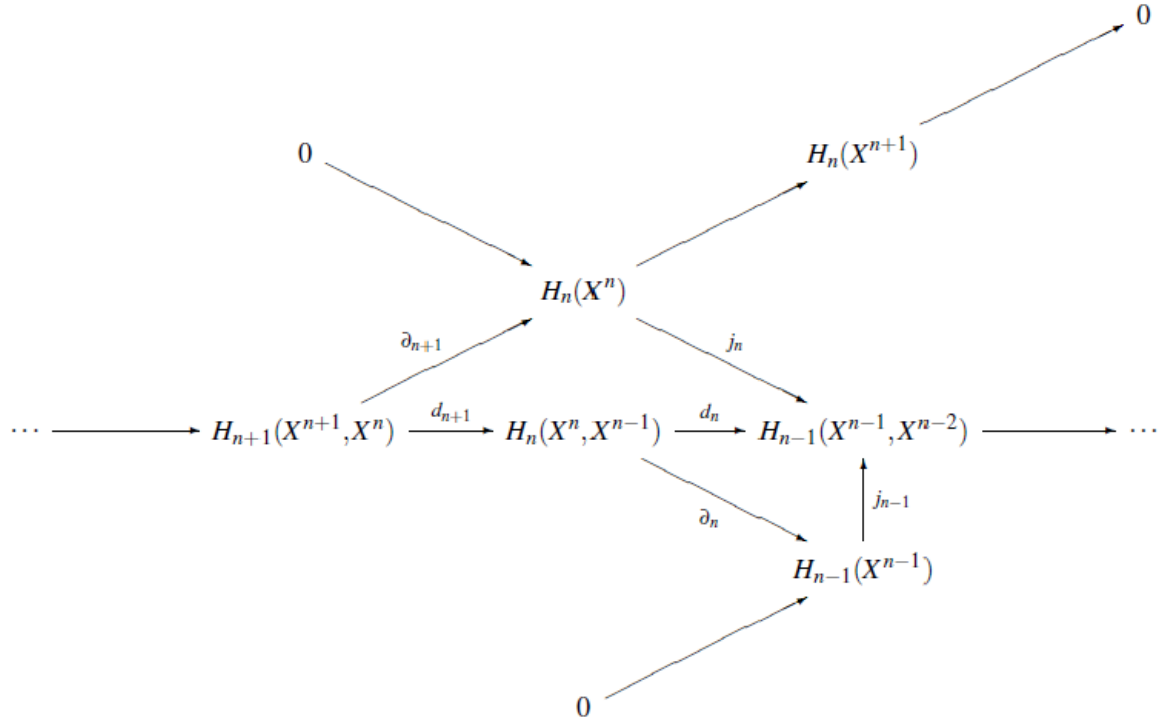


Figure 4.1 Homology group diagram

where $d_i = j_{i-1}\partial_i$, they allow us to consider the horizontal sequence. Thus the composition $d_i d_{i-1} = 0$ which implies that the horizontal sequence infact a chain complex.

$$\dots \rightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \rightarrow \dots$$

is called the ‘cellular chain complex’ of X . The elements of $H_n(X^n, X^{n-1})$ can be thought as linear combinations of n -cells of X . The homology groups of this cellular chain complex is called ‘cellular homology groups’ of X , and denoted by $H_n^{CW}(X)$.

Theorem 4.3 $H_n^{CW}(X) \simeq H_n(X) \quad \forall n. \blacksquare$

Proof By the above diagram $H_n(X) \simeq H_n(X^n) / \text{Im} \partial_{n+1}$. Since the sequence is exact, j_n is injective. Therefore $\text{Im}(\partial_{n+1}) = \text{Im}(j_n \partial_{n+1}) = \text{Im}(d_{n+1})$ and $H_n(X^n) \simeq \text{Im}(j_n) = \text{Ker}(\partial_n)$. Since j_{n-1} is injective; $\text{Ker}(\partial_n) = \text{Ker}(d_n)$. Thus

$$H_n(X^n) / \text{Im}(\partial_{n+1}) \simeq \text{Ker}(d_n) / \text{Im}(d_{n+1}).$$

The right hand side is the cellular homology and $H_n(X) \simeq H_n(X^n) / \text{Im}(\partial_{n+1})$. Thus $H_n(X) \simeq H_n^{\text{CW}}(X) \forall n$. ■

Some important facts given in [1] are listed below:

- If X is a CW complex with no n -cells then $H_n(X) = 0$.
- If X is a CW complex with k n -cells, then $H_n(X)$ is generated by at most k elements.
- If X is a CW complex having no two of its cells in adjacent dimensions, then $H_n(X)$ is free abelian with basis in one-to-one correspondence with the n -cells of X . This is because the cellular boundary maps d_n are automatically zero in this case.

Example 4.4 The closed orientable genus g surface Σ_g has CW structure consisting of one 0-cell, $2g$ 1-cells and one 2-cell attached by $[a_1, b_1], \dots, [a_g, b_g]$. Then the cellular chain complex can be constructed as follows:

$$0 \rightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \rightarrow 0$$

Since there is one 0-cell, d_1 must be 0. Also since a_i or b_i appears with its inverses, d_2 is 0. Since d_1 and d_2 are zero, the homology groups are in one-to-one correspondence with the cellular chain groups.

Thus $H_0(\Sigma_g) \simeq \mathbb{Z}$, $H_1(\Sigma_g) \simeq \mathbb{Z}^{2g}$, $H_2(\Sigma_g) \simeq \mathbb{Z}$.

CHAPTER 5

RESULTS AND DISCUSSION

Through out this thesis, simplicial, singular and cellular homology theories have been studied. It is seen that classification of the topological spaces can be easily made according to their holes with the aid of this theory. The homology groups of some manifolds have been calculated. As a result, it is shown that all these theories agree.

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