T.C.
YILDIZ TECHNICAL UNIVERSITY
INSTITUTE OF SCIENCE

THE CONNECTION BETWEEN BRAIDS AND THE FUNDAMENTAL GROUP OF CONFIGURATION SPACE

MAHMUT KUDEYT

MSc THESIS
DEPARTMENT OF MATHEMATICS
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ADVISOR
PROF. DR. AYŞE KARA

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May, 2011

Mahmut KUDEYT

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NOTATION

D	Unit Cube
d_i	i th string of n-braid in unit cube
B_n	n-braid group
I	Closed interval $[0,1]$
b_i	i th string of n-braid
D	Braid Diagrams
$\Omega_2^{\pm 1}$	Reidemeister move 2
$\Omega_3^{\pm 1}$	Reidemeister move 3
Δ	Delta move
ρ	Euclidean metric
B_n	n-braid group under equivalence relation
σ_i	Generator for braid group (i th)
$\mathbf{F}$	Free group
$\mathfrak{S}_n$	Symmetric group
P_n	Pure braid group
T_1	Tietze transformation (inserting)
T_2	Tietze transformation (deleting)
$\mathbf{A}_n$	Free subgroup of n-braid group
$\mathbf{H}_n$	The special subgroup of B_n
M_i	The right coset presentation
$F_{m,n}M$	The configuration space
R^2	Two dimensional Euclidean space
$\pi_1 B_{0,n} R^2$	Artin braid group with fundamental group
$\pi_1 F_{0,n} R^2$	Artin pure braid group with fundamental group
$\sum_n$	Symmetric group

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ABSTRACT

THE CONNECTION BETWEEN BRAIDS AND THE FUNDAMENTAL GROUP OF CONFIGURATION SPACE

Mahmut KUDEYT

Department of Mathematics

M.Sc. Thesis

Advisor: Prof. Dr. Ayşe KARA

The notion of braid is refined in algebraic and geometric settings. And the group structure of braid is given by generators and relations. In addition, the special subgroups of braid groups are introduced by their generators and relations.

The fundamental group of configuration space is presented by its fiber structure with some theorems. And we show that the fundamental group of configuration space has a structure of braid group. Finally, an isomorphism between a structure of braid group and braid group is indicated.

YILDIZ TECHNICAL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCE

ÖZET

ÖRGÜLER İLE DÜZENLENMİŞ UZAYIN TEMEL GRUPLARI ARASINDAKİ İLİŞKİ

Mahmut KUDEYT

Matematik Anabilim Dalı
Yüksek Lisans Tezi

Tez Danışmanı: Prof. Dr. Ayşe KARA

Örgü kavramı cebirsel ve geometriksel açıdan incelenmiş olup örgünün grup yapısı, üreteçleri ve bağıntıları ile beraber verilmiştir. Ayrıca bu grubun özel alt grupları, üreteçleri ve bağıntıları ile verilmiştir.

Düzenlenmiş (Configuration) uzayın temel grubu, fiber yapısı ile beraber teoremlerle açıklanmış ve örgü grup yapısına sahip olduğu gösterilip, örgü grup ile bu yapının arasında bir izomorfizma bulunmuştur.

CHAPTER 1

INTRODUCTION

1.1 Summary of the Literature

In the early part of the 20th century, using ordinary braids or plaits, found everywhere around us, as models, Emil Artin, a mathematician born in Germany, began a study [1],[2] that eventually developed into what is now known as braid theory.

As might be expected, the original ruminations were to some extent intuitive, based on the physical, tractable nature of braids and plaits.

But, over the course of the 20th century, braid theory has gradually been prospected, refined and polished, to use a goldmining analogy.

Braid theory is, now, recognized as one of the basic theories in mathematics and is of benefit in such branches as topology and algebraic geometry. Also, it is of profound use in other areas of the sciences - physics, statistical mechanics, chemistry and biology.

1.2 The Aim of the Thesis

The purpose of this master thesis to provide a connection between braids and fundamental groups.

The notion of braid is explained with its algebraic and geometrical properties. Especially, with the help of Reidemeister-Schreier method, algebraic properties of braid are refined for special subgroups.

The word problem is introduced to obtain the invariant subgroup of braid groups which is provided the connection between braid groups and fundamental groups.

Again, for our aim, configuration space is given with its fundamental group and structure of fiber bundle.

1.3 Hypothesis

Braid groups and fundamental group of configuration spaces are refined in algebraic and geometrical settings.

CHAPTER 2

BRAIDS

We start, in chapter 2, with the fundamental concepts, most notably the braid groups and pure braids.

2.1 Introduction

We introduce the definition of braid which follows directly from Artin's work (Artin [1]).

2.1.1 Basic Definition

Definition 2.1 Let D be a unit cube, so $\mathrm{D}=\{(x,y,z)|0\le x,y,z\le 1\}$. On the top face of cube place n points, $A_1,A_2,\ldots,A_n$, and, similarly, place n points on the bottom face, $B_1,B_2,\ldots,B_n$. In Figure 2.1, we have drawn such a configuration, but the cube has been placed in perspective.

For convenience, let us set $A_1=\left(\frac{1}{2},\frac{1}{n+1},1\right)$, $A_2=\left(\frac{1}{2},\frac{2}{n+1},1\right)$, … , $A_n=\left(\frac{1}{2},\frac{n}{n+1},1\right)$ and also $B_1=\left(\frac{1}{2},\frac{1}{n+1},0\right)$, $B_2=\left(\frac{1}{2},\frac{2}{n+1},0\right)$,…, $B_n=\left(\frac{1}{2},\frac{n}{n+1},0\right)$.

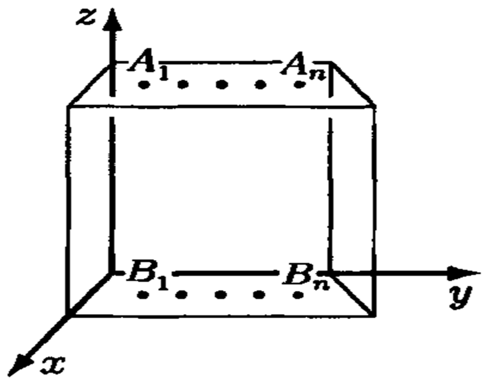


Figure 2.1 The unit cube

Now, join n points $A_1, A_2, ..., A_n$ with $B_1, B_2, ..., B_n$ by means of n polygonal segments/arcs $d_1, d_2, ..., d_n$ (Strictly speaking, the segments should be polygonal, but, in order to make diagrams that we will draw to easier to view, we shall draw these arcs as smooth curves.). However, the arcs can be attached in such a way that the following three conditions hold:

1) $d_1, d_2, ..., d_n$ are mutually disjoint.
2) Each d_i connects some A_j to some B_k, where j and k may or may not to be equal, d_i is not permitted to connect A_j and A_k (or B_j to B_k) .
3) Each plane E_s, such that $z = s$ and $0 \le s \le 1$ (in other words parallel to the xy-plane), intersects each arc d_i at one and only one point, Figure 2.2 (a) (In Figure 2.2(b), we give an example in which this condition does not hold).

Figure 2.2 An example of a 3-braid

Such a configuration of n arcs $d_1, d_2, ..., d_n$ is called n-braid, or braid with n strings. As might be expected, d_i is called a string.

Example 2.2 The braid in Figure 2.2(a) is a 3-braid. Other examples of braids are given in Figure 2.3.

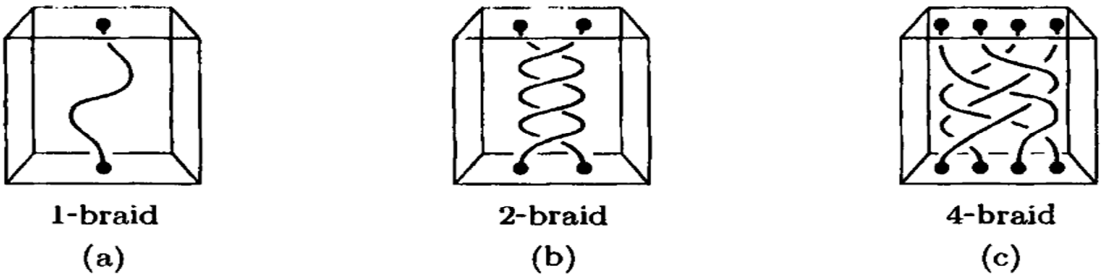


Figure 2.3 Some example of braids

Let us denote the set of all n-braid by B_n. With example 2.2, we can see that 1-braid, 2-braid, … , n-braid are mutually different braids. But we may refine the question, how many different braids are there for each n? In order to answer this question; we have to know when two braids are equivalent or not. So we present braids in geometric settings to understand the notion of equivalence.

Now, we shall extend D to $\mathbb{R}^2 \times I$ for the next sections.

2.2 Braids and Braid Diagrams

In this section we explain the braid groups in geometric terms. From now on, we denote by I the closed interval $[0,1]$ in the set of real numbers $\mathbb{R}$. By a topological interval, we mean a topological space homeomorphic to $I = [0,1]$. (For this section (Kassel and Turaev [3]))

2.2.1 Geometric Braids

Definition 2.3 A geometric braid on $n \geq 1$ strings $(n-braid)$ is a set $b \subset \mathbb{R}^2 \times I$ formed by n disjoint topological intervals called the strings of such that the projection $\mathbb{R}^2 \times I \to I$ maps each string homeomorphically onto I and

$$b \cap \left(\mathbb{R}^2 \times \{0\}\right) = \left\{\left(A_1,0,0\right),\left(A_2,0,0\right),...,\left(A_n,0,0\right)\right\} \tag{2.1}$$

$$b \cap \left(\mathbb{R}^2 \times \{1\}\right) = \left\{\left(B_1,0,1\right),\left(B_2,0,1\right),...,\left(B_n,0,1\right)\right\} \tag{2.2}$$

It is obvious that every string of b meets each plane $\mathbb{R}^2 \times \{t\}$ with $t \in I$ in exactly one point and connects a point $\left(A_i,0,0\right)$ to a point $\left(s(A_i),0,1\right)$, where $s\left(A_i\right)$ $\in \left\{B_1,B_2,...,B_n\right\}$ called the underlying permutation of b.

The Figure 2.4 is an example of geometric braid. Here x and y are the coordinates in $\mathbb{R}^2$ and $t \in I$. The underlying of permutation of this braid $\left(B_1,B_3,B_2,B_4\right)$

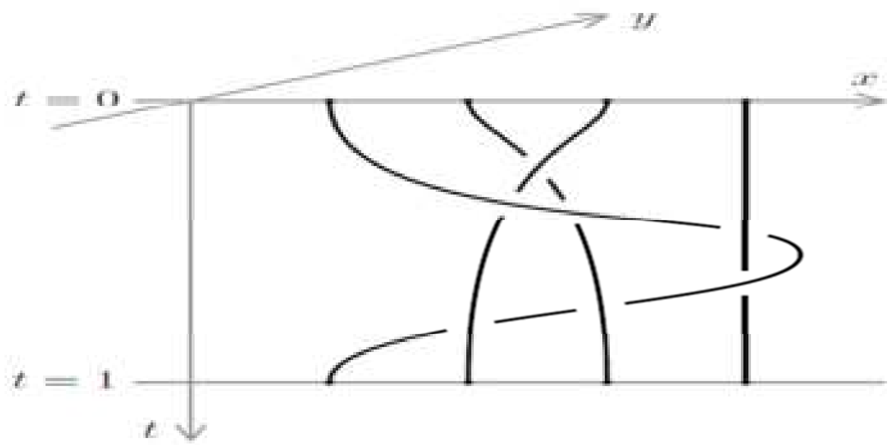


Figure 2.4 An example of a 4-braid

We introduce the concept of isotopy belongs to geometric braids.

Two geometric braids b and b' on n strings are isotopic, if b can be continuously deformed into b' in the class of braids. More formally, b and b' are isotopic if there is a continuous map $F : b \times I \rightarrow \mathbb{R}^2 \times I$ such that for each $s \in I$ the map $F_s : b \rightarrow \mathbb{R}^2 \times I$ sending $x \in b$ to $F(x,s)$ is an embedding whose image is a geometric braid on n strings, $F_0 = id_b : b \rightarrow b$, and $F_1(b) = b'$. Each F_s automatically maps every endpoint b to itself. Both the map F and the family of geometric braids $\left\{F_s(b)\right\}_{s \in I}$ are called an isotopy of $b = F_0(b)$ into $b' = F_1(b)$.

It is obvious that the relation of isotopy is an equivalence relation on the class of geometric braids on n strings. The corresponding equivalence classes are called braids on n strings.

2.2.2 Braid Diagrams

To specify a geometric braid, one can draw its projection to $\mathbb{R}\times\{0\}\times I$ along the second coordinate and indicate with string goes "under" the other one at each crossingpoint. To avoid local complications, we shall apply this procedure exclusively to those geometric braids whose projections to $\mathbb{R}\times\{0\}\times I$ have only double transversal crossings. These considerations lead to a notion of a braid diagram.

A braid diagram on n strands is a set $D\subset\mathbb{R}\times I$ split as a union of n topological intervals called the strands of D such that the following three conditions are met:

(i) The projection $\mathbb{R}\times I\to I$ maps each strand homeomorphically onto I.

(ii) Every point of $\{A_1, A_2, ..., A_n\}\times\{0\}$ or $\{B_1, B_2, ..., B_n\}\times\{1\}$ is the endpoint of a unique strand.

(iii) Every point of $\mathbb{R}\times I$ belongs to at most two strands. At each intersection point of two strands, these strands meet transversely, and one of them is distinguished and said to be undergoing, the other strand being overgoing.

Note that three strands of a braid diagram D never meet in one point. An intersection point of two strands of D is called a double point or a crossing of D. The transversality condition in (iii) means that in a neighborhood of a crossing, D looks, up to homoemorphism, like the set $\{(x,y)|xy=0\}$ in $\mathbb{R}^2$. Condition (iii) and the compactness of the strands easily imply that the number of crossings of D is finite.

In the figures, the strand going under a crossing is graphically represented by a line broken near the crossing; the strand going over a crossing is represented by a continued line. An example of a braid diagram is given in Figure 2.5. Here the top horizontal line represents $\mathbb{R}\times\{0\}$, the bottom horizontal line represents $\mathbb{R}\times\{1\}$. In the sequal we shall sometimes draw and sometimes omit these lines in the figures.

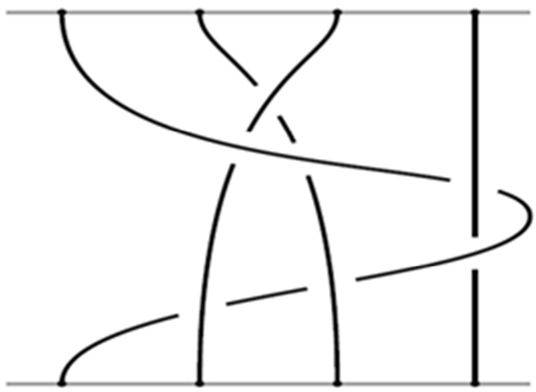


Figure 2.5 An example of a braid diagram on four strands

We now decribe the relationship between braids and braid diagrams. Each braid diagram D presents an isotopy class of geometric braids as follows. Using the obvious identification $\mathbb{R}\times I=\mathbb{R}\times\{0\}\times I$, we can assume that D lies on $\mathbb{R}\times\{0\}\times I\subset\mathbb{R}^2\times I$. In a small neighborhood of every crossing of D we slightly push the undergoing strand into $\mathbb{R}\times(0,\infty)\times I$ by increasing the secondcoordinate while keeping the first and third coordinates. This transforms D into a geometric braid on n strings. Its isotopy class is a well-defined braid presented by D. This braid is denoted by $\beta(D)$. For instance, the braid diagram in Figure 2.5 presents the braid drawn in Figure 2.4.

It is easy to see that any braid β can be presented by a braid diagram. To obtain a diagram of β, pick a geometric braid b that represents β and is generic with respect to the projection along the second coordinate. This means thatthe projection of b to $\mathbb{R}\times\{0\}\times I$ may have only double transversal crossings. At each crossing point of this projection choose the undergoing strand to be the one that comes from a subarc of *b* with larger second coordinate. The projection of b to $\mathbb{R}\times\{0\}\times I=\mathbb{R}\times I$ thus yields a braid diagram, D, and it is clear that $\beta(D)=\beta$.

Two braid diagrams D and D' on n strands are said to be *isotopic* if thereis a continuous map $F:D\times I\to\mathbb{R}\times I$ such that for each $s\in I$ the set

$D_s = F(D \times s) \subset \mathbb{R} \times I$ is a braid diagram on n strands, $D_0 = D$, and $D_1 = D^{'}$. It is understood that *F* maps the crossings of D to the crossingsof D_s for all $s \in I$ preserving the under/overgoing data. The family of braid diagrams $\{D_s\}_{s \in I}$ is called an *isotopy* of $D_0 = D$into $D_1 = D^{'}$. An example of an isotopy is given in Figure 2.6. It is obvious that if D is isotopic to $D^{'}$,then $\beta(D) = \beta(D^{'})$.

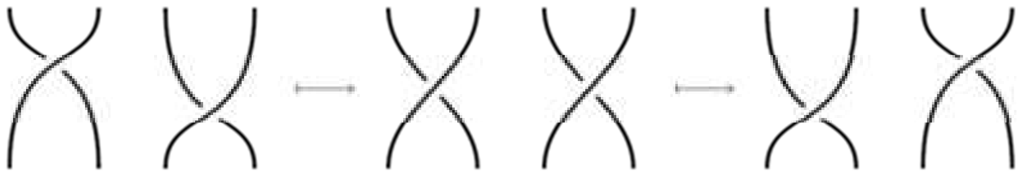


Figure 2.6 An example of isotopy of braid diagrams

2.2.3 Reidemeister Moves on Braid Diagrams

The transformations of braid diagrams Ω_2, Ω_3 shown in Figures 2.7 and 2.8, as well as the inverse transformations $\Omega_2^{-1}, \Omega_3^{-1}$ (obtained by reversing the arrows in Figures 2.7 and 2.8), are called Reidemeister moves. The moves affect only the position of a diagram in a disc inside $\mathbb{R} \times I$ and leave the remaining part of the diagram unchanged. The move Ω_2 involves two strands and creates two additional crossings (there are two types of Ω_2 -moves, as shown in Figure 2.7).

The move Ω_3 involves three strands and preserves the number of crossings. All these transformations of braid diagrams preserve the corresponding braidsup to isotopy.

Figure 2.7 Reidemeister moves (2)

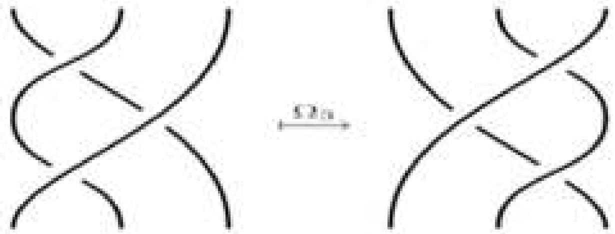


Figure2.8 Reidemeister moves (3)

We say that two braid diagrams $D, D^{'}$ are $\mathbb{R}-equivalent$ if D can be transformed into $D^{'}$ by a finite sequence of isotopies and Reidemeister moves $\Omega_2^{\pm 1}$, $\Omega_3^{\pm 1}$. It is obvious that if $D, D^{'}$ are $\mathbb{R}-equivalent$, then $\beta(D) = \beta(D^{'})$.

The following theorem asserts the converse.

Theorem 2.4 Two braid diagrams present isotopic geometric braids if and only if these diagrams $\mathbb{R}-equivalent$.

Proof The keypoint of Theorem 2.4 is that the diagrams of isotopic geometric braids are $\mathbb{R}-equivalent$. The proof of the theorem goes in four steps.

Step 1 We introduce some notation used in the next steps. Consider ageometric braid $b \subset \mathbb{R}^2 \times I$ on n strings. For $i = 1, ..., n$, denote the i th string of b, that is, the string adjacent to the point $(A_i, 0, 0)$, by b_i. Each plane $\mathbb{R}^2 \times \{t\}$ with $t \in I$ meets b_i in one point, denoted by $b_i(t)$. In particular, we have $b_i(0) = (A_i, 0, 0)$.

Let ρ be the Euclidean metric on $\mathbb{R}^3$. Given a real number $\varepsilon > 0$, the *cylinder* $\varepsilon - neighborhood$ of b_i consists of all points $(x, t) \in \mathbb{R}^2 \times I$ such that $\rho((x,t), b_i(t)) < \varepsilon$. This neighborhood meets each plane $\mathbb{R}^2 \times \{t\} \subset \mathbb{R}^2 \times I$ along the open disc of radius ε centered at $b_i(t)$.

For distinct $i, j \in \{1, ..., n\}$, the function $t \to \rho(b_i(t), b_j(t))$ is a continuous function on I with positive values. Since I is compact, this function has a minimum value.

Set

$$|b| = \frac{1}{2} \min_{1 \le i < j \le n} \min_{t \in I} \rho\left(b_i(t), b_j(t)\right) > 0 \tag{2.3}$$

It is clear that the cylinder $|b| - neighborhoods$ of the strings of b are pairwise disjoint. (In fact, $|b|$ is the maximal real number with this property.)

For any pair of geometric braids b, $b^{'}$ on n strings and any $i = 1,\ldots,n$ the function $t \to \rho\left(b_i(t), b_i^{'}(t)\right)$ is a continuous function on I with nonnegative values. Since I is compact, this function has a maximum value. Set

$$\tilde{\rho}\left(b, b^{'}\right) = \max_{1 \le i \le n} \max_{t \in I} \rho\left(b_i(t), b_j(t)\right) \ge 0 \tag{2.4}$$

The function $\tilde{\rho}$ satisfies the axioms of a metric: $\tilde{\rho}\left(b, b^{'}\right) = \tilde{\rho}\left(b^{'}, b\right)$; $\tilde{\rho}\left(b, b^{'}\right) = 0$ if and only if $b = b^{'}$; for any geometric braids, $b, b^{'}, b^{"}$ on n strings, we have $\tilde{\rho}\left(b, b^{"}\right) < \tilde{\rho}\left(b, b^{'}\right) + \tilde{\rho}\left(b^{'}, b^{"}\right)$. The latter follows from the fact that for some $i = 1,\ldots,n$ and $t \in I$,

$$\begin{aligned} \tilde{\rho}\left(b, b^{"}\right) &= \tilde{\rho}\left(b_i(t), b^{"}{}_i(t)\right) \\ &\le \tilde{\rho}\left(b_i(t), b^{'}{}_i(t)\right) + \tilde{\rho}\left(b^{'}{}_i(t), b^{"}{}_i(t)\right) \\ &\le \tilde{\rho}\left(b, b^{'}\right) + \tilde{\rho}\left(b^{'}, b^{"}\right) \end{aligned} \tag{2.5}$$

Note also that

$$|b| \le \left|b^{'}\right| + \tilde{\rho}\left(b, b^{'}\right) \tag{2.6}$$

Indeed, for some $t \in I$ and certain distinct $i, j \in 1,\ldots,n$

$$\begin{aligned}|b| &= \frac{1}{2}\rho\left(b_i(t), b_j(t)\right) \\ &\le \frac{1}{2}\left(\tilde{\rho}\left(b_i(t), b'_i(t)\right) + \tilde{\rho}\left(b'_i(t), b'_j(t)\right) + \tilde{\rho}\left(b'_j(t), b_j(t)\right)\right) \\ &\le \frac{1}{2}\left(\tilde{\rho}\left(b, b'\right) + 2\left|b'\right| + \tilde{\rho}\left(b', b\right)\right) \\ &= \left|b'\right| + \tilde{\rho}\left(b, b'\right)\end{aligned} \tag{2.7}$$

Step 2 A geometric braid is polygonal if all its strings are formed by consecutive (linear) segments; see Figure 2.9. Any geometric braid *b* on *n* strings can be approximated by polygonal braids as follows. Pick an integer $N \ge 2$ and an index $i = 1,\ldots,n$. For $k = 1,\ldots,N$, consider the segment in $\mathbb{R}^2 \times I$ with endpoints $b_i\left(\frac{k-1}{N}\right)$ and $b_i\left(\frac{k}{N}\right)$. The union of these N segments is a brokenline, b_i^N, with endpoints $b_i^N(0) = b_i(0) = (A_i, 0, 0)$ and $b_i^N(1) = b_i(1)$. For sufficiently large N, this broken line lies in the cylinder $|b| - neighborhoods$ of b_i. Therefore for sufficiently large N, the broken lines $b_1^N,\ldots,b_n^N$ are disjoint and form a polygonal braid, b^N, *approximating* b. Moreover, for any real number $\varepsilon > 0$ and all sufficiently large N, we have $\tilde{\rho}\left(b, b^N\right) < \varepsilon$. For instance, Figure 2.6 shows a polygonal approximation of the braid in Figure 2.4.

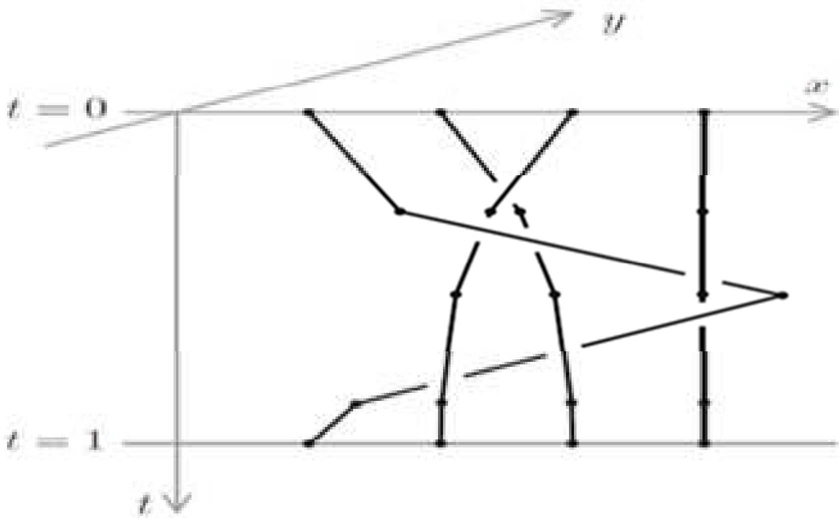


Figure 2.9 A polygonal braid

We now reformulate the notion of isotopy of braids in the polygonal setting. To this end, we introduce so-called $\Delta - moves$ on polygonal braids. Let A, B , C be three points in $\mathbb{R}^2 \times I$ such that the third coordinate of A is strictly smaller than the third coordinate of B and the latter is strictly smaller than the third coordinate of C. The move $\Delta(ABC)$ applies to a polygonal braid $b \subset \mathbb{R}^2 \times I$ whenever this braid meets the triangle ABC precisely along the segment AC. (By the triangle ABC, we mean the linear 2-simplex with vertices A, B , C.) Under this assumption, the move $\Delta(ABC)$ on b replaces $AC \subset b$ by $AB \cup BC$, keeping the rest of b intact; see Figure 2.10, where the triangle ABC is shaded. The inverse move $(\Delta(ABC))^{-1}$ applies to a polygonal braid meeting the triangle ABC precisely along $AB \cup BC$. This move replaces $AB \cup BC$ by AC. The moves $\Delta(ABC)$ and $(\Delta(ABC))^{-1}$ are called $\Delta - moves$

.

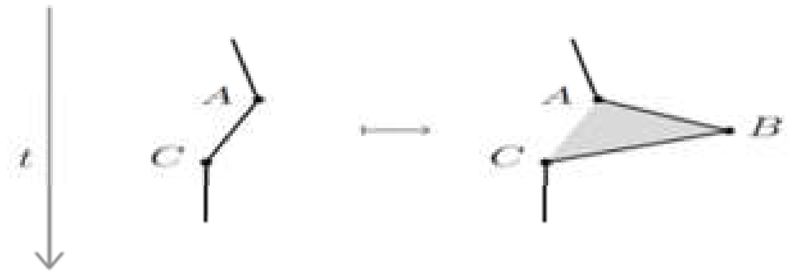


Figure 2.10 Delta move

It is obvious that polygonal braids related by a $\Delta - moves$ are isotopic. We establish a converse assertion.

Claim 2.5 If polygonal braids b, $b^{'}$ are isotopic, then b can be transformed into $b^{'}$ by a finite sequence of $\Delta - moves$.

Proof We first verify this claim under the assumption $\tilde{\rho}(b,b^{'}) < |b|/10$. Assumethat the i th string b_i is formed by $K \geq 1$ consecutive segments with vertices $A_0 = (i,0,0), A_1,..., A_K \in \mathbb{R}^2 \times I$. We write $b_i = A_0 A_1 ... A_K$. Similarly, assume that $b_i^{'} = B_0 B_1 ... B_L$ with $L \geq 1$ and $B_0, B_1,..., B_L \in \mathbb{R}^2 \times I$. Note that $A_0 = B_0$ and

$A_K = B_L \in \mathbb{R}^2 \times \{1\}$. Subdividing b_i, $b_i^{'}$ into smaller segments, we can ensure that $K = L$, the points A_j, B_j have the same third coordinate for all $j = 0,1,...,K$, and the Euclidean length of the segments A_jA_{j+1}, B_jB_{j+1} is smaller than $|b|/10$ for j = 0, 1, . . .,K −1. The assumption $\tilde{\rho}\left(b,b^{'}\right) < |b|/10$ implies that each horizontal segment A_jB_j has length $< |b|/10$. The move $\left(\Delta\left(A_0A_1A_2\right)\right)^{-1}$ transforms $b_i = A_0A_1...A_K$ into the string $A_0A_2...A_K = B_0A_2...A_K$. The move $\Delta\left(B_0B_1A_2\right)$ transforms the latter in the string $B_0B_1A_2...A_K$. Continuing by induction and applying the moves $\left(\Delta\left(B_jA_{j+1}A_{j+2}\right)\right)^{-1}$, $\Delta\left(B_jB_{j+1}A_{j+2}\right)$ for $j = 0,1,...,K-2$, we transform b_i into $b_i^{'}$. The conditions on the lengths imply that all the intermediate strings as well as the triangles $B_jA_{j+1}A_{j+2}$, $B_jB_{j+1}A_{j+2}$ determiningthese moves lie in the cylinder $|b| - neighborhoods$ of b_i; they are therefore disjoint from the cylinder $|b| - neighborhoods$ of the other strings of *b*. We apply these transformations for $i = 1,...,n$ and obtain thus a sequence of $\Delta - moves$ transforming b into $b^{'}$.

Consider now an arbitrary pair of isotopic polygonal braids b, $b^{'}$. Let $F : b \times I \to \mathbb{R}^2 \times I$ be an isotopy transforming $b = F_0(b)$ into $b^{'} = F_1(b)$ (the braids $F_s(b)$ with 0 < s <1 may be nonpolygonal). The continuity of F implies that the function *I* $\left(s,s^{'}\right) \to \tilde{\rho}\left(F_s(b), F_{s^{'}}(b)\right)$ is continuous. This function is equal to 0 on the diagonal $s = s^{'}$ of $I \times I$. These facts and the inequality (2.6) imply that the function $I \to \mathbb{R}$, $s \to |F_s(b)|$ *i*s continuous. Since $|F_s(b)| > 0$ for all s, there is a real number $\varepsilon > 0$ such that $|F_s(b)| > \varepsilon$ for all $s \in I$. The continuity of the function $\left(s,s^{'}\right) \to \tilde{\rho}\left(F_s(b), F_{s^{'}}(b)\right)$ now implies that for a sufficiently large integer N and all $k = 1,2,...,N$,

$$\tilde{\rho}\left(F_{(k-1)/N}(b), F_{k/N}(b)\right) < \varepsilon/10 \qquad (2.8)$$

Let us approximate each braid $F_{k/N}(b)$ by a polygonal braid p_k such that $\tilde{\rho}\left(F_{k/N}(b), p_k\right) < \varepsilon/10$. For p_0, p_N, we take b, b', respectively. By (2.6),

$$\left|p_k\right| \geq \left|F_{k/N}(b)\right| - \tilde{\rho}\left(F_{k/N}(b), p_k\right) > 9\varepsilon/10 \tag{2.9}$$

At the same time,

$$\begin{aligned}\tilde{\rho}\left(p_{k-1}, p_k\right) \leq \tilde{\rho}\left(p_{k-1}, F_{(k-1)/N}(b)\right) \\ + \tilde{\rho}\left(F_{(k-1)/N}(b), F_{k/N}(b)\right) + \tilde{\rho}\left(F_{k/N}(b), p_k\right) < 3\varepsilon/10\end{aligned} \tag{2.10}$$

Therefore $\tilde{\rho}\left(p_{k-1}, p_k\right) \leq \left|p_k\right|/2$ for $k = 1,...,N$. By the previous paragraph, p_{k-1} can be transformed into p_k by a sequence of $\Delta - moves$. Composing these transformations $b = p_0 \to p_1 \to ... \to p_N = b'$, we obtain a required transformation $b \to b'$. This completes the proof of Claim 2.5.

□

Step 3 A polygonal braid is *generic* if its projection to $\mathbb{R} \times I = \mathbb{R} \times \{0\} \times I$ along the second coordinate has only double transversal crossings. Slightly deforming the vertices of a polygonal braid b (keeping ∂b), we can approximate this braid by a generic polygonal braid. Moreover, if b, b' are generic polygonal braids related by a sequence of $\Delta - moves$, then slightly deforming the vertices of the intermediate polygonal braids, we can ensure that these polygonal braids are also generic. Note the following corollary of this argument and Claim 2.5 .

Claim 2.6 *If generic polygonal braids b, b' are isotopic, then b can be transformed into b' by a finite sequence of $\Delta - moves$ such that all the intermediate polygonal braids are generic.*

To present generic polygonal braids, we can apply the technique of braid dagrams. The diagrams of generic polygonal braids are the braid diagrams, whose strands are formed by consecutive straight segments. Without loss of generality, we can always assume that the vertices of these segments do notcoincide with the crossing points of the diagrams.

Claim 2.7 *The diagrams of two generic polygonal braids related by a* $\Delta - move$ *are R-equivalent.*

Proof Consider a $\Delta - move$ $\Delta(ABC)$ on a generic polygonal braid b producing a generic polygonal braid b'. Pick points A', C' inside the segments AB, BC, respectively. Pick a point D inside the segment AC such that the third coordinate of D lies strictly between the third coordinates of A' and C'. Applying to b the moves $\Delta(AA'D)$, $\Delta(DC'C)$, we transform the segment AC into the broken line $AA'DC'C$. Further applying the moves $\left(\Delta(A'DC')\right)^{-1}$ and $\Delta(A'BC')$, we obtain b'. This shows that the move $\Delta(ABC)$ can be replaced by a sequence of four $\Delta - moves$ along smaller triangles (one should choose the points A', C', D so that the intermediate polygonal braids are generic). This expansion of the move $\Delta(ABC)$ can be iterated. In this way, subdividing the triangle ABC into smaller triangles and expanding $\Delta - moves$ as compositions of $\Delta - moves$ along the smaller triangles, we can reduce ourselves to the case in which the projection of ABC to $\mathbb{R} \times I$ meets the rest of the diagram of b either along a segment or along two segments with one crossing point.

Consider the first case. If both endpoints of the segment in question lie on $AB \cup BC$, then the diagram of b is transformed under $\Delta(ABC)$ by Ω_2. If one endpoint of the segment lies on AC and the other one lies on $AB \cup BC$, then the diagram is transformed by an isotopy.

If the projection of ABC to $\mathbb{R} \times I$ meets the rest of the diagram along two segments having one crossing, then we can similarly distinguish several subcases. Subdividing if necessary the triangle ABC into smaller triangles and expanding our $\Delta - move$ as a composition of $\Delta - moves$ along the smaller triangles, we can reduce ourselves to the case in which the move preserves the part of the diagram lying outside a small disk in $\mathbb{R} \times I$ and changes the diagram inside this disk via one of the following six formulas:

$$d_1^+ d_2^+ d_1^+ \leftrightarrow d_2^+ d_1^+ d_2^+ \, , d_1^+ d_2^+ d_1^- \leftrightarrow d_2^- d_1^+ d_2^+ \, , \quad d_1^- d_2^- d_1^+ \leftrightarrow d_2^+ d_1^- d_2^- \qquad (2.11)$$

$$d_1^- d_2^- d_1^- \leftrightarrow d_2^- d_1^- d_2^- \, , d_1^+ d_2^- d_1^- \leftrightarrow d_2^- d_1^- d_2^+ \, , \quad d_1^- d_2^+ d_1^+ \leftrightarrow d_2^+ d_1^+ d_2^- \qquad (2.12)$$

Here $d_1^{\pm}$ and $d_2^{\pm}$ are the braid diagrams on three strands shown in Figure 2.11; for the definition of the product of braid diagrams, see Figure 2.4. It remains to prove that for each of them, the diagrams on the left-hand and right-hand sides are R-equivalent. The transformation $d_1^+ d_2^+ d_1^+ \leftrightarrow d_2^+ d_1^+ d_2^+$ is just Ω_3. For the other five transformations, the R-equivalence is established by the following sequences of moves:

$$w = \left(d_1^+ d_2^+ d_1^- \xrightarrow{\Omega_2} d_2^- d_2^+ d_1^+ d_2^+ d_1^- \xrightarrow{\Omega_3^{-1}} d_2^- d_1^+ d_2^+ d_1^+ d_1^- \xrightarrow{\Omega_2^{-1}} d_2^- d_1^+ d_2^+ \right),$$

$$\gamma = \left(d_1^- d_2^- d_1^+ \xrightarrow{\Omega_2} d_1^- d_2^- d_1^+ d_2^+ d_2^- \xrightarrow{w^{-1}} d_1^- d_1^+ d_2^+ d_1^- d_2^- \xrightarrow{\Omega_2^{-1}} d_2^+ d_1^- d_2^- \right),$$

$$\mu = \left(d_1^- d_2^- d_1^- \xrightarrow{\Omega_2} d_2^- d_2^+ d_1^- d_2^- d_1^- \xrightarrow{\gamma^{-1}} d_2^- d_1^- d_2^- d_1^+ d_1^- \xrightarrow{\Omega_2^{-1}} d_2^- d_1^- d_2^- \right),$$

$$d_1^+ d_2^- d_1^- \xrightarrow{\Omega_2} d_1^+ d_2^- d_1^- d_2^- d_2^+ \xrightarrow{\mu^{-1}} d_1^+ d_1^- d_2^- d_1^- d_2^+ \xrightarrow{\Omega_2^{-1}} d_2^- d_1^- d_2^+ ,$$

$$d_1^- d_2^+ d_1^+ \xrightarrow{\Omega_2} d_1^- d_2^+ d_1^+ d_2^+ d_2^- \xrightarrow{\Omega_3^{-1}} d_1^- d_1^+ d_2^+ d_1^+ d_2^- \xrightarrow{\Omega_2^{-1}} d_2^+ d_1^+ d_2^- .$$

This completes the proof of Claim 2.7.

□

Figure 2.11 Presentations of some braids (for generators)

Step 4 We can now complete the proof of Theorem 2.4. It is obvious that $R-equivalent$ braid diagrams present isotopic braids. To prove the converse, consider two braid diagrams D_1, D_2 presenting isotopic braids. For $i = 1, 2,$ straightening D_i near its crossing points and approximating the rest of D_i by broken lines as at Step 2, we obtain a diagram, $D_i^{'}$, of a generic polygonal braid, b_i. If the approximation is close enough, then $D_i^{'}$ is isotopic to D_i.

Then the braids b_1, b_2 are isotopic. Claim 2.6 implies that b_1 can be transformed into b_2 by a finite sequence of $\Delta - moves$ in the class of generic polygonal braids. Claim 2.7 implies that the diagrams $D_1^{'}, D_2^{'}$ are R-equivalent. Therefore the diagrams D_1, D_2 are $R - equivalent$.

□

CHAPTER 3

THE GROUP OF BRAIDS

3.1 The Group of Braids

As showing above, we understand the notion of isotopy of briads. Now, to try and figure out how we may attempt to show that B_n for each n has an infinite number of n-braids, we know $B_1 = \{1\}$ is trivial group and let us to look at the simplest non-trivial case, namely $n = 2$. Every 2-braid is equivalent to one of two types of braid, an example of the 2 types is shown in Figure 3.1 where the left-hand figure has 3 twists, while the right-hand figure has 4 twists. We will see later that if p and q are the number of twist of two braids, β_p and γ_q,respectively, then $p,q \geq 1$ and $p \neq q$ these braids are not eqivalent. So if we can prove this, then it follows that B_2 has an infinite number of distinct braids.

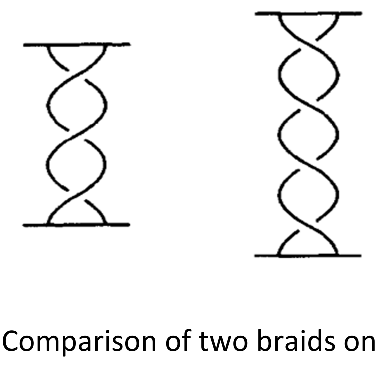


Figure 3.1 Comparison of two braids on two strings

We would like to introduce the concept of the braid groups. The braid group is also sometimes called Artin's braid group.

Now, we define the product of n-braid.

Definition3.1 Given two n-braid $b_1, b_2 \subset \mathbb{R}^2 \times I$, we define their product $b_1 b_2$ to be set of poins $(x, y, t) \in \mathbb{R}^2 \times I$ such that $(x, y, 2t) \in b_1$ if $0 \le t \le 1/2$ and $(x, y, 2t-1) \in b_2$ if $1/2 \le t \le 1$. If we think the product with figures, we shall see how it figures out in Figure 3.2.

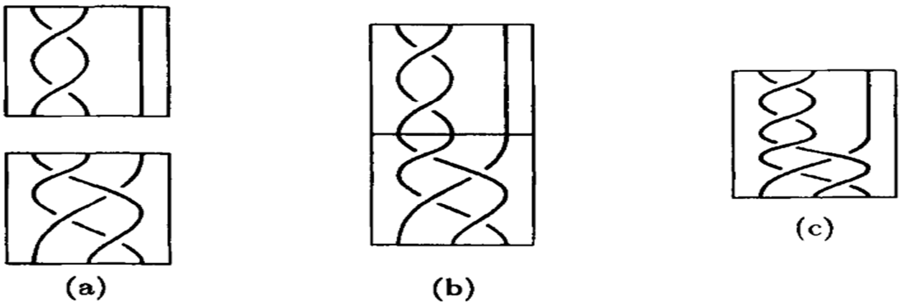


Figure 3.2 The product of two braids on three strings

Proposition 3.2 If b_1, b_2 are isotopic to n-braids b_1', b_2', respectively, then $b_1 b_2$ is isotopic to $b_1' b_2'$.

Proof $b_1 \sim b_1'$ and $b_2 \sim b_2'$. From the definition of isotopy, b_1 can be transformed into b_1' by finite sequence of Reidemeister moves or $\Delta - moves$. Similarly, b_2 can be transformed into b_2' by finite sequence of Reidemeister moves or $\Delta - moves$. Let m,k≥ 0 be integers.

$$b_1 = (b_1)_0 \to (b_1)_1 \to .... \to (b_1)_m = b_1' \tag{3.1}$$

$b_1 b_2 = (b_1)_0 b_2 \to (b_1)_1 b_2 \to ... \to (b_1)_m b_2 = b_1' b_2$, so we obtain that $b_1 b_2 \sim b_1' b_2$.

$$b_2 = (b_2)_0 \to (b_2)_1 \to .... \to (b_2)_k = b_2' \tag{3.2}$$

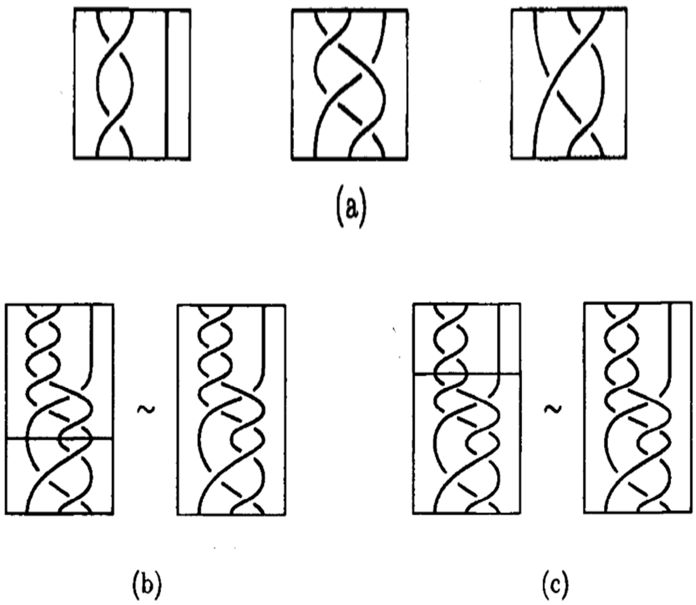
$b_1' b_2 = b_1' (b_2)_0 \to b_1' (b_2)_1 \to ... \to b_1' (b_2)_k = b_1' b_2'$, so we also obtain $b_1' b_2 \sim b_1' b_2'$. Therefore we combine two results; $b_1 b_2 \left(\sim b_1' b_2\right) \sim b_1' b_2'$.

□

Proposition 3.3 The product of braids is associative, that is,

$$\left(b_1 b_2\right) b_3 \sim b_1\left(b_2 b_3\right) \tag{3.3}$$

Proof In the Figure 3.3 (a)-(c), we give diagrams to prove above statement. Diagram (a) shows each n-braid, b_1, b_2, b_3, respectively. Diagram (b) indicates $\left(b_1 b_2\right) b_3$ and diagram (c) indicates $b_1\left(b_2 b_3\right)$. So, we can see that the product of braids is associative.



(a)

(b) (c)

Figure 3.3 The proof of associative of braids

□

Proposition 3.4 The product of braids has a natural element. We shall denote this element by 1_n.

Proof Let 1_n be the n-braid shown in figure. We see that this braid connects A_j to B_j.

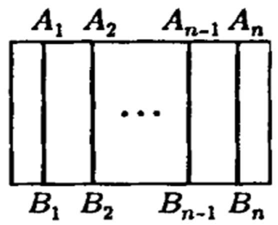


Figure 3.4 Trivial braid

And 1_n obviously satifies this relations:

$$b(1_n) \sim b \text{ and } (1_n)b \sim b \tag{3.4}$$

□

Proposition 3.5 For each n-braid b, there exists a n-braid $\bar{b}$ such that $b\bar{b} \sim 1_n$ and $\bar{b}b \sim 1_n$.

Such a n-braid is called the inverse of b and denoted by b^{-1}.

With the product operation and properties, we all have the necessary requirements for B_n to be a group.

If $\beta \in B_n$ be a n-braid, its equivalence classes is denoted by $[\beta]$.

Theorem3.6 The set of equivalence class of n-braids B_n, forms a group. This group is usually called n-braid group or Artin's n-braid group.

Proof The product is given by Definition 3.1; associavity as a consequence of Proposition 3.2, the identity element is 1_n (Proposition 3.3) and the inverse element of $[\beta]$, denoted by $[\beta^{-1}]$ (Proposition 3.4).

□

3.2A Presentation For Braid Group

In this section we define the generators of B_n and obtain a group presentation for B_n.

We begin with defining the generators of B_n and answer this question, " how do this generators generate B_n ?

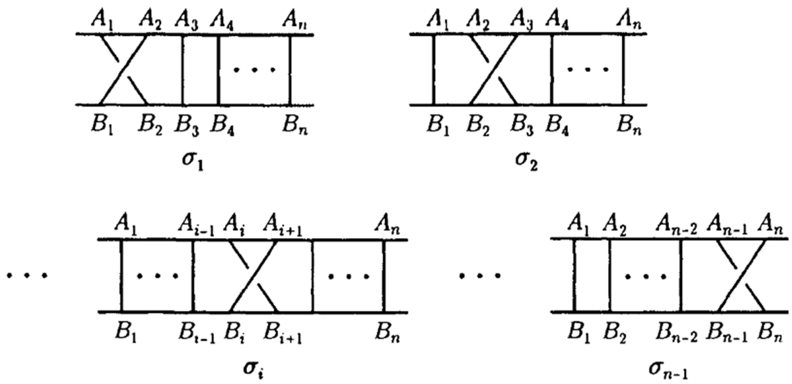


Figure 3.5 The generators of braid groups

As in Figure 3.5 we shall denote these (n-1) braids by $\sigma_1, \sigma_2, ..., \sigma_{n-1}$.

The second set of (n-1) braids may be formed by interchanging the overcrossing and undercrossing information for each of n-braids of Figure 3.5. Thus, this set of n-braids is exactly the set of the inverse of each element in the first set. Therefore, we shall denote these by $\sigma_1^{-1}, \sigma_2^{-1}, ..., \sigma_{n-1}^{-1}$, see also Figure 3.6.

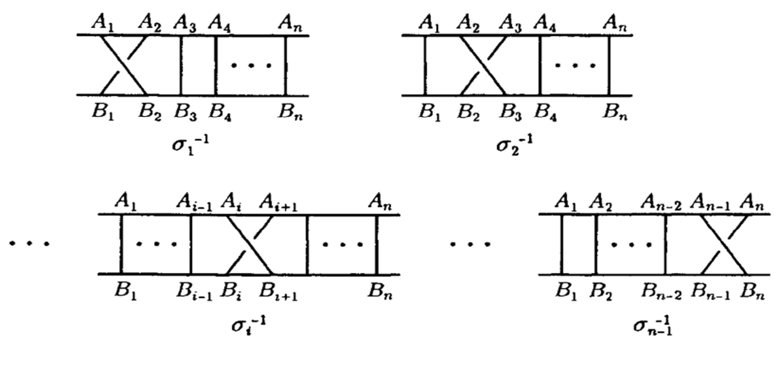


Figure 3.6 The inverse of generators of braid groups

Proposition3.7 Any n-braid β (in B_n) can be written as a product of elements from the set $\{\sigma_i^{\pm}\}$ with $i = 1, 2, ..., n-1$, e.i,

$$\beta = \sigma_{i_1}^{\varepsilon_1} \sigma_{i_2}^{\varepsilon_2} ... \sigma_{i_k}^{\varepsilon_k} \tag{3.5}$$

where each ε_j is either + or − and $i_1,...,i_k \in \{1,2,...,n-1\}$.

Proof Let the Figure 3.7(a) be the braid projection of β (in B_n) denoted by D. We may partition this braid diagram by means of level planes such that two consecutive level planes only two strings are braided with preserving overgoing and undergoing datas as an example.

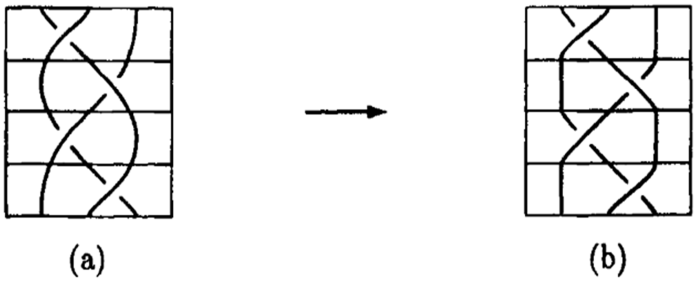


Figure3.7 Indication of generators on braids

This diagram can be approximated by polygonal braid and each section of Figure 3.7(b) shows an element of $\{\sigma_i^{\pm}\}$ with $i = 1,2,...,n-1$. So we obtain proof of proposition.

To find a presentation for B_n, we must also find a set of defining relations. For this reason, we use induction with begining B_3.

To obtain general relation, we must show that this equation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ is true. If we draw each equation's figure, we see that this eqution's figures are polygonal approximation of Reidemeister move Ω_3. Therefore they are in the same equivalence classes. So they present same element.

Actually the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ still holds in B_n for any $n \geq 4$.

Similarly, we have to show that

$$\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \tag{3.6}$$

Figure 3.8 Proof of second relation of braid groups on the example

We can see from their figures that first strings are constant, therefore we again use Ω_3 and we obtain the proof.

Continuing this process, we see that in B_n for n > 2 the following relations hold,

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } i = 1,...,n-2 \tag{3.7}$$

Consider, in B_4 the product $\sigma_1\sigma_3$, Figure 3.9(a) , we show using elementary moves that this product is equal to $\sigma_3\sigma_1$, see Figure 3.9 (a)-(c)

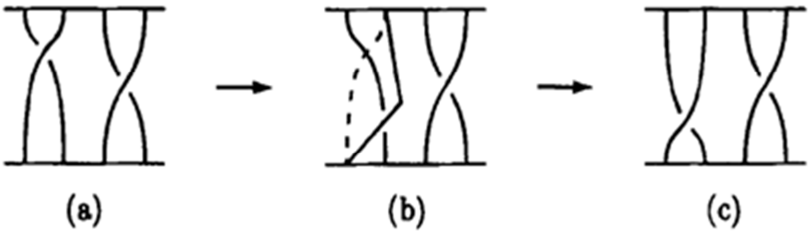


Figure3.9 Proof of first relation of braid groups on the example

If we generalize this, we can easily see the following relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \tag{3.8}$$

for all $i, j = 1,...,n-1$ with $|i-j| \geq 2$.

□

Theorem 3.8 For any $n \geq 1$ the n-braid group B_n has the following presentation;

$$\mathrm{B}_n = \left\langle \sigma_1, \sigma_2, ..., \sigma_{n-1} \middle| \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } i = 1,...,n-2 \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for all } i, j = 1,...,n-1 \text{with } |i-j| \geq 2. \end{array} \right\rangle.$$

Proof For proof, we have to find an isomorphism between a free group (G) and B_n. Let us begin with defining abstractly, in terms of a presentation, the following group G,

$$G = \left\langle x_1, x_2, \ldots, x_{n-1} \,\middle|\, \begin{array}{l} x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ for all } i = 1, \ldots, n-2 \\ x_i x_j = x_j x_i \text{ for all } i, j = 1, \ldots, n-1 \text{ with } |i-j| \geq 2. \end{array} \right\rangle$$

Now, we need to establish that G and B_n are isomorphic as groups. That is to say, the natural correspondence $x_i \to \sigma_i$, for $i = 1, 2, \ldots, n-1$, is a group isomorphism.

Firstly, let us define a mapping $\varphi : G \to \mathrm{B}_n$ as follows, let $W = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_k}^{\varepsilon_k}$ be an arbitrary element of G, then set

$$\varphi(W) = \varphi\left(x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_k}^{\varepsilon_k}\right) = \left[\sigma_{i_1}^{\varepsilon_1} \sigma_{i_2}^{\varepsilon_2} \ldots \sigma_{i_k}^{\varepsilon_k}\right] \in \mathrm{B}_n \tag{3.9}$$

Now, our aim is to show that the mapping φ is a group isomorphism between G and B_n. For this purpose, we use the following theorem.

Theorem3.9 Let

$$G = \left\langle x_1, x_2, \ldots, x_n \mid R_1 = 1, R_2 = 1, \ldots, R_m = 1 \right\rangle \tag{3.10}$$

where $R_j = x_{j_1}^{\varepsilon_1} x_{j_2}^{\varepsilon_2} \ldots x_{j_k}^{\varepsilon_k}$, with $1 \leq j_1, j_2, \ldots, j_k \leq n$ and $\varepsilon_i = \pm 1$. Further, let $\mathbf{H}$ be an arbitrary group and f a mapping from $\mathbf{F}\left\langle x_1, x_2, \ldots, x_n \right\rangle$ to $\mathbf{H}$ defined by

$$f(x_i) = W_i \tag{3.11}$$

for $i = 1, 2, \ldots, n$.

If, for $j = 1, 2, \ldots, m$,

$$\begin{aligned} f(R_j) &= f\left(x_{j_1}\right)^{\varepsilon_1} f\left(x_{j_2}\right)^{\varepsilon_2} \ldots f\left(x_{j_k}\right)^{\varepsilon_k} \\ &= W_{j_1}^{\varepsilon_1} W_{j_2}^{\varepsilon_2} \ldots W_{j_k}^{\varepsilon_k} = 1 \end{aligned} \tag{3.12}$$

in $\mathbf{H}$, then f defines a homomorphism $\hat{f} : G \to \mathbf{H}$ with

$$\hat{f}\left(x_{\lambda_1}^{\eta_1}x_{\lambda_2}^{\eta_2}...x_{\lambda_l}^{\eta_l}\right)=\hat{f}\left(x_{\lambda_1}\right)^{\eta_1}\hat{f}\left(x_{\lambda_2}\right)^{\eta_2}...\hat{f}\left(x_{\lambda_l}\right)^{\eta_l} \tag{3.13}$$

By theorem 3.9, it is sufficient to show that φmaps defining relations of G to the identity element in B_n . This is quite easy to show, for

$$\varphi\left(x_k x_l x_k^{-1} x_l^{-1}\right)=\left[\sigma_k\sigma_l\sigma_k^{-1}\sigma_l^{-1}\right]=\left[1_n\right] \text{ for } |k-l|\geq 2 \tag{3.14}$$

$$\varphi\left(x_i x_{i+1} x_i x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1}\right)=\left[\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}\right]=\left[1_n\right] \tag{3.15}$$

To complete proof, we need to show that the homomorphism φ is onto and one-to-one.

Firstly, let us show that φ is onto. Suppose β is an arbitrary element of B_n . By proposition 3.7, β can be written as

$$\beta=\sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}...\sigma_{i_k}^{\varepsilon_k} \tag{3.16}$$

where $1\leq i_1,i_2,...,i_k\leq n-1$ and $\varepsilon_i=\pm 1$. Now, consider the element $W=x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}...x_{i_k}^{\varepsilon_k}$ in G , by definition,

$$\varphi(W)=\varphi\left(x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}...x_{i_k}^{\varepsilon_k}\right)=\left[\sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}...\sigma_{i_k}^{\varepsilon_k}\right]=[\beta] \tag{3.17}$$

Hence, φ is onto.

Suppose, now, in G there exist elements g and g' such that $\varphi(g)=\varphi(g')$. We may write $g=x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}...x_{i_k}^{\varepsilon_k}$ and $g'=x_{j_1}^{\eta_1}x_{j_2}^{\eta_2}...x_{j_l}^{\eta_l}$, and let $\beta=\sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}...\sigma_{i_k}^{\varepsilon_k}$ and $\beta'=\sigma_{j_1}^{\eta_1}\sigma_{j_2}^{\eta_2}...\sigma_{j_k}^{\eta_k}$.

Since by assumption $\varphi(g)=\varphi(g')$, we must also have $\beta\sim\beta'$. Hence we need to show either that $g=g'$ as elements of G, or equivalently $W=x_{j_1}^{\varepsilon_1}x_{j_2}^{\varepsilon_2}...x_{j_k}^{\varepsilon_k}$ and $W'=x_{j_1}^{\eta_1}x_{j_2}^{\eta_2}...x_{j_l}^{\eta_l}$ thought of as words in free group $\mathbf{F}\left(x_1,x_2,....,x_{n-1}\right)$ can be connected by a finite sequence as follows,

$$W=W_1\to W_2\to...\to W_r=W' \tag{3.18}$$

where for $i=1,2,\ldots,r-1$, W_{i+1} is obtained from W_i by either insertion or deleting of a conjugate of one of the relation of G. If we denote the set of all relations of G by

R then we can say that $W \underset{R}{\sim} W'$.

We know from our previous work that since $\beta \sim \beta'$ we can construct a finite sequence

$$\beta = \beta_1 \to \beta_2 \to \ldots \to \beta_s = \beta' \tag{3.19}$$

We will show that it is possible to construct of type in (3.18) from (3.19).

With this mind, suppose that X_i, for $i=1,2,\ldots,s-1$, is a word in G whose image under φ in β_i in (3.19). In particular, let us set $X_1 = W$ and $X_s = W'$. Then, if we can show that X_i is equivalent to X_{i+1} relative to R, for $i=1,2,\ldots,s-1$, will be able to construct exactly the finite sequence we require, namely,

$$W = X_1 \to X_2 \to \ldots \to X_s = W' \tag{3.20}$$

and hence $W \underset{R}{\sim} W'$.

Clearly, it is sufficient to look at only part of the sequence in (3.20), $\beta_i \to \beta_{i+1}$ say. This, im turn, allows us to simply notation to $\beta \to \beta'$.

Now, β' is obtained from β by applying a solitary elementary move. By definition of an elementary move, we replace an edge AB by the edges $AC \cup CB$ in the triangle ΔACB see in Figure 3.10. Also, by definition, the image of ΔACB is also triangle.

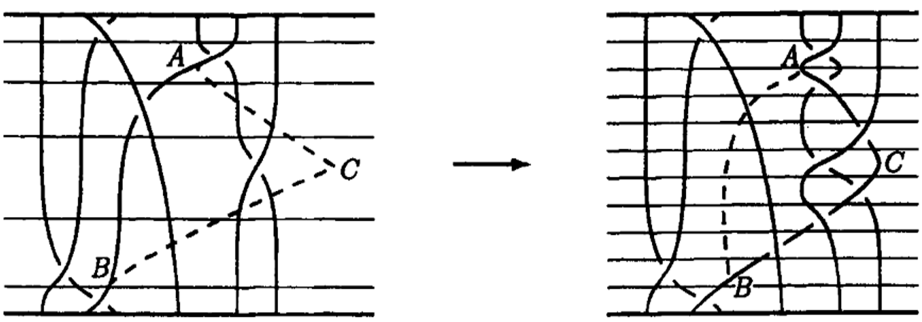
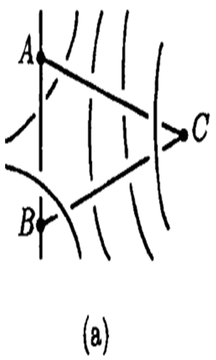


Figure 3.10 Delta move on 5-braid

It is easy to see from Figure 3.10 that the effect on β' of an elementery move is to introduce the straight lines (arcs) AC and CB. In general, AC and CB will produce new intersections with other strings in β'.

Since $\beta = \sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}...\sigma_{i_k}^{\varepsilon_k}$, we can partition β by k-1 level lines into k rectangles in each of which there is exactly one crossing of the form $\sigma_{i_s}^{\varepsilon_s}$. However, the introduction of ΔACB will require that we add several level lines to take into account the extra intersections caused by AC and CB.

An important point to note is that, even though we add these extra intersections and hence level lines, when a string enters, say, over (under) an edge of ΔABC then this string will always exit over (under) some edge of ΔABC, see Figure 3.11.



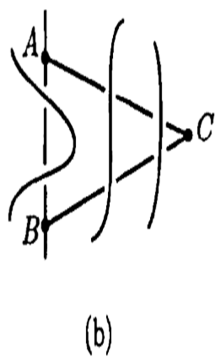


Figure 3.11 The position of strings in the delta move

Now, let $P_1, P_2, ..., P_m$ and $Q_1, Q_2, ..., Q_m$ be the points of intersection of the level lines with AB and $AC \cup CB$, respectively, see Figure 3.10. Next, for each $i = 1, 2, ..., m$ create a new point P_i' on AB just above P_i. Clearly, we can construct P_i' in such a way that the narrow triangle $\Delta P_i' Q_i P_i$ does not contain any of the crossing in β or β', see Figure 3.12.

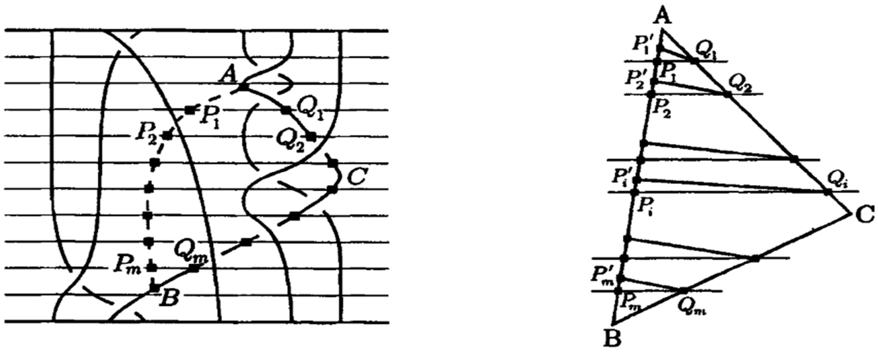


Figure 3.12 Subdiving for delta moves

The above construction, in fact, is nothing more than the creation of a series of elementary moves and what may be termed $\Delta - moves$.

We continue this process of appliying $\Delta - moves$ until we reach $P_m{}'Q_m \cup Q_i B$. We replace this last polygonal segment by $P_m{}'B$, this is just an $\Delta - move$, see Figure 3.12.

The above process yields the following sequence between β' and β,

$$\beta' = \gamma_0 \to \gamma_1 \to \ldots \to \gamma_m \to \gamma_{m+1} = \beta,$$

where $\gamma_i \to \gamma_{i+1}$, for $i = 1, 2, \ldots, m-1$, is the replacement of $P_i'Q_i \cup Q_i Q_{i+1}$ by $P_i'P_{i+1}' \cup P_{i+1}'Q_{i+1}$. While, $\gamma_0 \to \gamma_1$ and $\gamma_m \to \gamma_{m+1}$ are $\Delta - moves$.

To next step is to find words $X_1, X_2, \ldots, X_m, X_{m+1}$ in G such that $\varphi(X_i) = \gamma_i$. To find these words, we need to find an expression for each of the γ_i in the terms of product $\sigma_i^{\pm 1}$. Essentially, there are two cases to consider.

Case 1

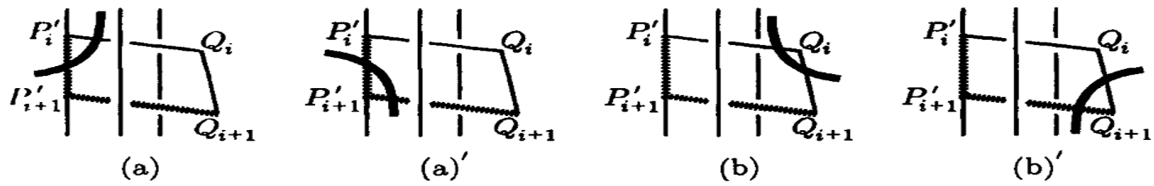


Figure 3.13 The positions of strings in delta move (case 1)

In Figure 3.13, if the bold face string enters quadrangle $P_i'Q_iQ_{i+1}P_{i+1}'$ under (or over) a given edge, then it exists the same quadrangle under (or over) some edge. It is clear for Figure 3.13 that γ_i and γ_{i+1} are of exactly the same form, namely

$$\gamma_i = \gamma_i' \sigma_j^{\pm1} \sigma_{j+1}^{\pm1} \ldots \sigma_l^{\pm1} \gamma_i'' = \gamma_{i+1} \tag{3.21}$$

So, in this case, X_i and X_{i+1} are the same word, i.e.,

$$X_i = X_{i+1} = X_i' x_j^{\pm1} x_{j+1}^{\pm1} \ldots x_l^{\pm1} X_i'' \tag{3.22}$$

where X_i' and X_i'' are words that correspond to γ_i' and γ_i'' respectively.

On the other hand, if the string enters $P_i'Q_iQ_{i+1}P_{i+1}'$ over (or under) $P_i'Q_i$ say, and it exits from the same edge $P_i'Q_i$, then it is easy to see that γ_i and γ_{i+1} are of exactly the same form

Case 2

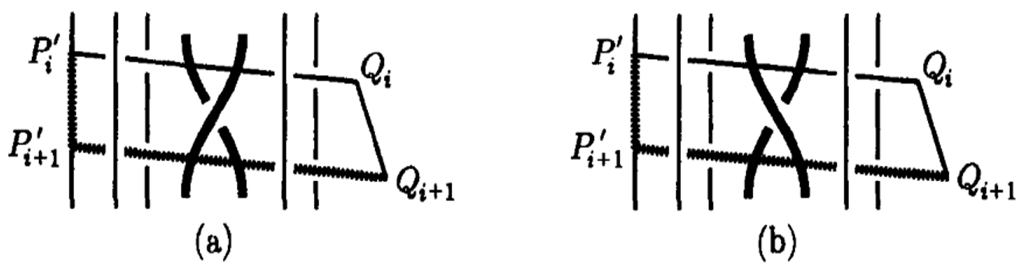


Figure 3.14 The positions of strings with one crossing in delta move (case 2)

The same remark as under Figure 3.13 applies to Figure 3.14.

$$\gamma_i = \gamma_i' \sigma_j^{\pm1} \sigma_{j+1}^{\pm1} \ldots \sigma_{p-2}^{\pm1} \sigma_{p-1}^{\lambda} \sigma_p^{\mu} \sigma_{p+1}^{\pm1} \ldots \sigma_q^{\pm1} \sigma_{p-1}^{\nu} \gamma_i'' \tag{3.23}$$

where λ, μ and ν are either +1 or -1, but if $\lambda \neq \mu$ then $\nu = \mu$.

On the other hand,

$$\gamma_{i+1} = \gamma_i' \sigma_p^{\nu} \sigma_j^{\pm1} \sigma_{j+1}^{\pm1} \ldots \sigma_{p-2}^{\pm1} \sigma_{p-1}^{\mu} \sigma_p^{\lambda} \sigma_{p+1}^{\pm1} \ldots \sigma_q^{\pm1} \gamma_i'' \tag{3.24}$$

It is important that the sign of λ, μ, ν is observed carefully.

Therefore,

$$X_i = X_i' x_j^{\pm 1} x_{j+1}^{\pm 1} \ldots x_{p-2}^{\pm 1} x_{p-1}^{\lambda} x_p^{\mu} x_{p+1}^{\pm 1} \ldots x_q^{\pm 1} x_{p-1}^{\nu} X_i'' \tag{3.25}$$

and

$$X_{i+1} = X_i' x_p^{\nu} x_j^{\pm 1} x_{j+1}^{\pm 1} \ldots x_{p-2}^{\pm 1} x_{p-1}^{\mu} x_p^{\lambda} x_{p+1}^{\pm 1} \ldots x_q^{\pm 1} X_i'' \tag{3.26}$$

But in *G,* since $x_p^{\nu} \rightleftharpoons x_j^{\pm 1}, \ldots, x_{p-2}^{\pm 1}$ and $x_{p-1}^{\nu} \rightleftharpoons x_{p+1}^{\pm 1}, \ldots, x_q^{\pm 1}$, we have that

$$X_i \underset{R}{\sim} \hat{X}_i = X_i' x_j^{\pm 1} x_{j+1}^{\pm 1} \ldots x_{p-2}^{\pm 1} x_{p-1}^{\lambda} x_p^{\mu} x_{p-1}^{\nu} x_{p+1}^{\pm 1} \ldots x_q^{\pm 1} X_i'' \tag{3.27}$$

Similarly,

$$X_{i+1} \underset{R}{\sim} \hat{X}_{i+1} = X_i' x_j^{\pm 1} x_{j+1}^{\pm 1} \ldots x_{p-2}^{\pm 1} x_p^{\nu} x_{p-1}^{\mu} x_p^{\lambda} x_{p+1}^{\pm 1} \ldots x_q^{\pm 1} X_i'' \tag{3.28}$$

From the relations of *G,* we know that $x_{p-1}^{\lambda} x_p^{\mu} x_{p-1}^{\nu} = x_p^{\nu} x_{p-1}^{\mu} x_p^{\lambda}$ where λ, μ and ν are either +1 or -1, but if $\lambda \neq \mu$ then $\nu = \mu$. Thus, $X_i \underset{R}{\sim} X_{i+1}$. Hence, we can say that $W = X_0 \sim X_{m+1} = W'$.

Therefore, φ is one-to-one, so the proof of theorem is complete.

□

Now we indicate homomorphism to extend our view with symmetric group (Kassel and Turaev [3]).

Given a homomorphism f from B_n to a group G, the elements $\{s_i = f(\sigma_i)\}_{i=1,\ldots,n-1}$ of G satisfy the braid relations

$$s_i s_j = s_j s_i \tag{3.29}$$

for all $i, j = 1,...,n-1$ with $|i-j| \geq 2$, and

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \tag{3.30}$$

for $i = 1,...,n-2$. Then, there is a converse relation which is given by the following lemma.

Lemma 3.10 If $s_1, s_2, ..., s_{n-1}$ are elements of a group satisfying the braid relations, then there is a unique group homomorphism $f : B_n \to G$ such that $s_i = f(\sigma_i)$ for all $i = 1,...,n-1$.

Proof Let F_n be a free group generated by the set $\{\sigma_1, \sigma_2, ..., \sigma_{n-1}\}$. There is a unique group homomorphism $\overline{f} : F_n \to G$ such that $\overline{f}(\sigma_i) = s_i$ for all $i = 1,...,n-1$. This homomorphism induces group homomorphism $f : B_n \to G$ provided $\overline{f}(r^{-1}.r') = 1$, equivalently, provided $\overline{f}(r^{-1}) = \overline{f}(r')$ for all braid relations $r^{-1} = r'$. To verify the first braid relation, we have

$$\overline{f}(\sigma_i \sigma_j) = \overline{f}(\sigma_i)\overline{f}(\sigma_j) = s_i s_j = s_j s_i = \overline{f}(\sigma_j)\overline{f}(\sigma_i) = \overline{f}(\sigma_j \sigma_i).$$

To verify the second braid relation, we similarly have

$$\overline{f}(\sigma_i \sigma_{i+1} \sigma_i) = s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} = \overline{f}(\sigma_{i+1})\overline{f}(\sigma_i)\overline{f}(\sigma_{i+1}) = \overline{f}(\sigma_{i+1} \sigma_i \sigma_{i+1}).$$

$\square$

We introduce new relation between $\mathfrak{S}_n$ which is symmetric group, and B_n . This relation provides us how to imagine the generators σ_i .

3.2.1 Projection to the symmetric group

We apply the previous lemma to the symmetric group $G = \mathfrak{S}_n$. An element of $\mathfrak{S}_n$ is a permutation of the set $\{1, 2, ..., n\}$. Consider the simple transpositionswhere s_i permutes i and $i+1$ and leaves allthe other elements of $\{1, 2, ..., n\}$ fixed. It is an easy exercise to verify that the simple transpositions satisfy the braid relations. By Lemma 2.9, there is a unique group homomorphism $\pi : B_n \to \mathfrak{S}_n$ such that $s_i = \pi(\sigma_i)$ for all $i = 1,...,n-1$. This homomorphism is surjective because, as is well known, the simple transpositions generate $\mathfrak{S}_n$.

3.2.2Definition of pure braids

The kernel of the natural projection $\pi : B_n \to \mathfrak{S}_n$ is called the pure braid group and is denoted by P_n:

$$P_n = Ker\left(\pi : B_n \to \mathfrak{S}_n\right).$$

Pure braid group has some generators and relations. Before we indicate these, in the next section, we try to find a practical method that allow us to determine if a n-braid is equivalent to another n-braid or not. This sort of determination problem is called the word problem for group, in this case the braid group. We use the pure braid to solve problem.

3.3 Word Problem

We mentioned the notion of equivalence of n-braids in first chapter. Now, we introduce practical method to determine that whether two n-braids are equivalent or not [4].

3.3.1 Word Problem For The Braid Group

Definition 3.11 (Word problem forthe braid group): Given any two braids, β_1 and β_2 say, find a method that will allow us to decide if or not $\beta_1 = \beta_2$.

It easy to see that we can modify this to say, find a method that will allow us to decide if or not $\beta = 1$(since if $\beta_1 = \beta_2$ then $\beta_1\beta_2^{-1} = 1$).

For this aim of this section is to introduce the various steps that make up the algeorithm.

(I) First Step- Is the braid in the question a pure braid or not?

The reason why we ask this question is because the trivial braid 1 is a pure braid. Therefore, if we can show that a given n-braid β is not pure braid then, immediately, we can say that β cannot be 1. Hence, the algorithm terminates.

So, how can we determine whether or not β is pure braid? The answer is quite simple, all we need to consider is its braid permutation, $\pi(\beta)(\in \mathfrak{S}_n)$, defined above. For if $\pi(\beta)=(1)$ then β is a pure braid, however if $\pi(\beta)\neq(1)$ then β is not a pure braid.

On determining the braid permutation, if β is a pure braid then we need to proceed to step (II).

(II) Second Step – The braid in question is a pure braid.

Since β is a pure n-braid, from the definition, we know each string of the braid, d_i for $i=1,2,...,n$, starts at the point A_i and terminates at B_i. So, let us remove last string, d_n, and replace it by a straight line joining A_n to B_n, Figure 3.21(b), and denote the resultant n-braid by γ.

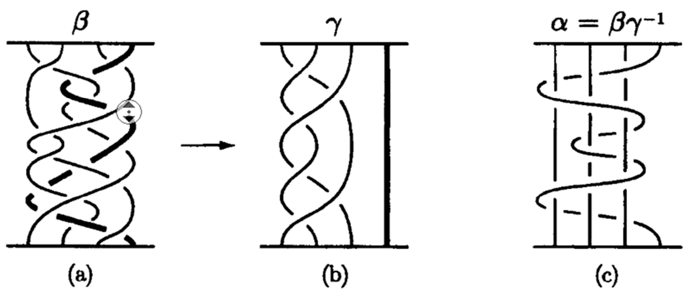


Figure 3.15 An example of construction of combed braid

Now, we may form another n-braid α by taking the product of β and γ^{-1}, i.e, $\alpha=\beta\gamma^{-1}$. By construction, if we remove the final string of α then the resultant (n-1)-braid, α' say, is equivalent to trivial braid. Hence, we may think of α' as just n-1 parallel lines. Therefore, α , itself, may be thought of as a n-braid in which the first n-1 strings are parallel and the last string links with these parallel lines. Such a braid is said to be a combed braid, Figure 3.15(c).

Let us set $\alpha = \alpha_1$ and $\gamma = \gamma_1$, so $\beta = \alpha_1\gamma_1$, and let us shift our attention to γ_1. Since the final string of γ_1 is a straight line, to apply the above process to γ_1 we must start with the (n-1)th string. Let us denote by γ_2 the n-braid obtained from γ_1 by removing the (n-1)th string and replacing it by straight line. Working through the above process, we shall obtain a combed braid α_2, but in this case the first (n-2) strings and the nth string are parallel and the (n-1)th string links with only the first (n-2) parallel strings. Thus, the process yields $\gamma_1 = \alpha_2\gamma_2$.

By repeating the above process, finally, we shall arrive at a decomposition of β in the form

$$\beta = \alpha_1\alpha_2 ... \alpha_{n-1} \tag{3.31}$$

where each n-braid α_i is a combined braid, and in α_{n-1} every string except the second string is a straight line and second string links only with the first string.

Proposition3.12 Let β be a pure n-braid. Then, β is the trivial braid if and only if each of the α_i in the decomposition given in (3.31) is trivial briad.

Proof If each α_i is the trivial braid then clearly β is also the trivial braid.

Conversely, let us suppose that β is the trivial braid. In addition, let ξ_i be the (n-i)-braid obtained from β by removing its last i strings. By construction, ξ_0 is β, ξ_{n-1} is the 1-braid and ξ_n is empty. Obviously, since β is the trivial braid, each of the ξ_i, for $i = 0,1,2,...,n-1$, is also trivial braid.

Similarly, let us define $\alpha_{j,i}$ as the (n-i)-braid obtained from α_j by removing the last i strings. Now, by construction,

$$\begin{aligned} \beta = \xi_0 &= \alpha_1\alpha_2\ldots\alpha_{n-1}, \\ \xi_1 &= \alpha_{1,1}\alpha_{2,1}\ldots\alpha_{n-1,1}, \\ \xi_2 &= \alpha_{1,2}\alpha_{2,2}\ldots\alpha_{n-1,2}, \\ &\vdots \\ \xi_{n-2} &= \alpha_{1,n-2}\alpha_{2,n-2}\ldots\alpha_{n-1,n-2}, \end{aligned} \tag{3.32}$$

where $\alpha_{1,1}$ is the trivial (n-1)-braid, $\alpha_{1,2}$, $\alpha_{2,2}$ are trivial (n-2)-braids, and, in general, $\alpha_{1,i}, \alpha_{2,i}, \ldots, \alpha_{i,i}$, for $i =, 1, 2, \ldots, n-2$, are all trivial (n-i)-braids.

Now , $\alpha_{1,n-2}, \alpha_{2,n-1}, \ldots, \alpha_{n-2,n-2}$ are all trivial 2-braids and α_{n-1} is n-braid obtained from 2-braid $\alpha_{n-1,n-2}$ by adding n-2 parallel straight line strings. Therefore, the triviality of $\xi_{n-2} = \alpha_{n-1,n-2}$ implies that α_{n-1} is the trivial n-braid. Hence,

$$\beta = \alpha_1\alpha_2\ldots\alpha_{n-2}.$$

By the very same reasoning as above, we can say that $\xi_{n-3} = \alpha_{n-2,n-3}$, and hence α_{n-2} is the trivial braid. So, continiing in this way, we will eventually show that each α_i is indeed the trivial braid.

□

(III) Third Step- Determine if or not each α_i is the trivial braid?

It is well known that the word problem is solvable for free group which is the following theorem. So, if we can show that each α_i is an element of a free group, we can determine if or not each α_i is trivial braid. However, the arguments to prove contains some terms and notations, and we postpone these to the next section.

Clearly, this third step is the final step of the algorithm. Hence, the steps allow us to completely solve the Word problem for B_n .

Theorem 3.13 The Word problem for a group is solvable.

Proof Let F be a free group of rank n generated by $x_1, x_2, \ldots, x_n$. An element

$$g = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_m}^{\varepsilon_m} \tag{3.33}$$

of F is equal empty word, 1, if and only if we can eleminate each $x_{i_j}^{\varepsilon_j}$ by a serious of T_1 (inserting) and T_2 (deleting) transformations. That is to say, we can only cancel products within g of the form $x_i x_i^{-1}$ and $x_i^{-1} x_i$. If we cannot such cancelations, then g is never equal to the empty word.

Therefore, to solve word problem for an arbitrary word, g, of a free group, F, we need only check if $x_i x_i^{-1}$ or $x_i^{-1} x_i$ exist within g. Such a straight-forward method can be deemed reasonably practical, so the word problem may be said to be solvable for a free group.

Now, we give an example to understand this steps.

Example 3.14 Let us consider the pure 4-braid

$$\beta = \sigma_3\sigma_1\sigma_2^{-2}\sigma_3\sigma_2\sigma_1\sigma_3\sigma_1\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_3 \tag{3.34}$$

Already shown in Figure 3.15(a). From Figure 3.15(a) and (b) , it is easy to see that $\gamma = \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1$. While, in Figure 3.16 we show by a sequence of diagrams that $\beta\gamma^{-1}$ is combined braid in Figure 3.15(c), and so

$$\alpha_1 = \beta\gamma^{-1} = \left(\sigma_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-1}\right)\left(\sigma_3^{-1}\sigma_2^{-2}\sigma_3\right)\left(\sigma_3\sigma_2\sigma_1^{2}\sigma_2\sigma_3\right) \tag{3.35}$$

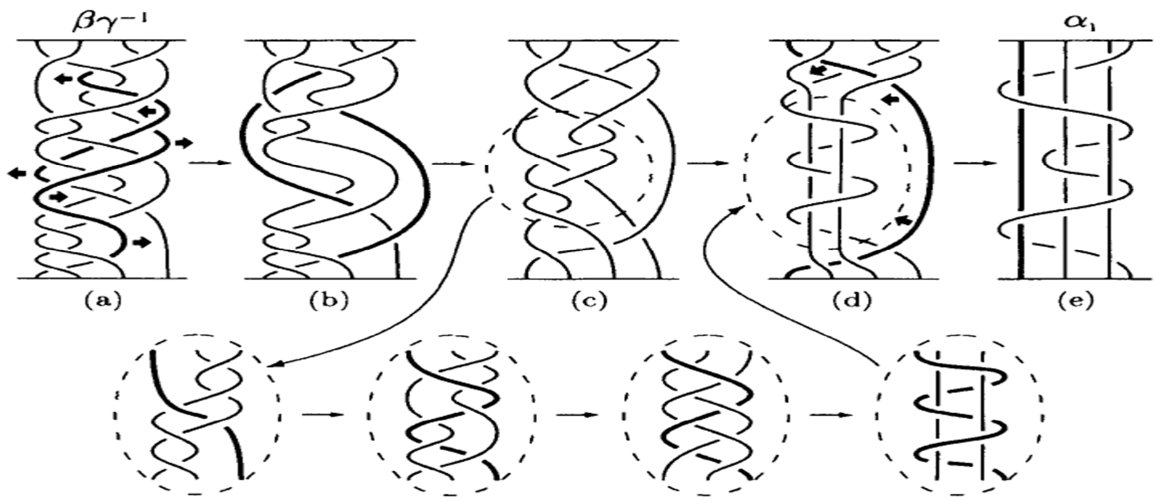


Figure 3.16 An example of process of word problem's solution

Now, let us turn our attention to γ. By replacing the third string of γ by a straight line, we obtain $\gamma_2 = \sigma_1^2$. Therefore,

$$\alpha_2 = \gamma\gamma_2^{-1} = \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1\sigma_1^{-2} = \overbrace{\sigma_1\sigma_2\sigma_1}^{\rightarrow}\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2\sigma_1\overbrace{\sigma_2\sigma_1\sigma_2}^{\rightarrow}\sigma_1^{-1}$$
$$= \sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1^{-1} = \sigma_2\sigma_1^2\sigma_2 \qquad (3.36)$$

and finally $\alpha_3 = \gamma_2$. Hence, wemay write β as, see also Figure 3.17,

$$\beta = \alpha_1\alpha_2\alpha_3 \qquad (3.37)$$

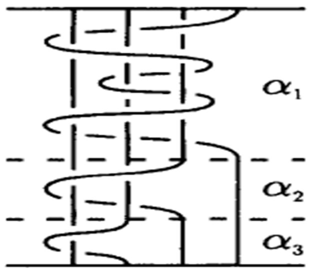


Figure 3.17 The conclusion of word problem for the example

3.3.2 A Solution of Word Problem

Our aim in this section to show that combed braids are elements of a free group. Then, since the word problem is solvable for a free group, this allow us to solve the word problem for B_n. To this end, let us define $\mathbb{A}_n$ to be set of those combed braids for which the removal of last string results in the trivial (n-1)-braid.

Now, we introduce some technical information about $\mathbb{A}_n$.

Proposition 3.15 The group $\mathbb{A}_n$ is generated by the following (n-1) elements, see also Figure 3.18,

$$a_1 = (\sigma_{n-1}\sigma_{n-2}...\sigma_2)\sigma_1^2(\sigma_2^{-1}...\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}),$$
$$a_2 = (\sigma_{n-1}\sigma_{n-2}...\sigma_3)\sigma_2^2(\sigma_3^{-1}...\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}),$$
$$\vdots$$
$$a_{n-2} = \sigma_{n-1}\sigma_{n-2}^2\sigma_{n-1}^{-1}$$
$$a_{n-1} = \sigma_{n-1}^2$$

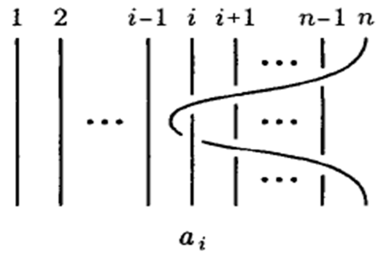


Figure 3.18 A combed braid

Proof Let β be an element of A_n. If we assume that β is not the trivial n-braid, then the nth string d_n of β has k, say, points $p_1, p_2, ..., p_k$ with a vertical tangent on the left-hand side of the curve d_n and k-1 points $q_1, q_2, ..., q_{k-1}$ with a vertical tangent on the right-side, see Figure 3.19(a).

Now keeping the points $p_1, p_2, ..., p_k$ fixed, pull the piece of the string at each of the points q_i to the right, so that it clears all the strings on the right-hanf side, see Figure 3.19(b). For clarity, we shall denote by $q_i^{'}$, respectively, the points with a vertical tangent on the right-hand side on the new string.

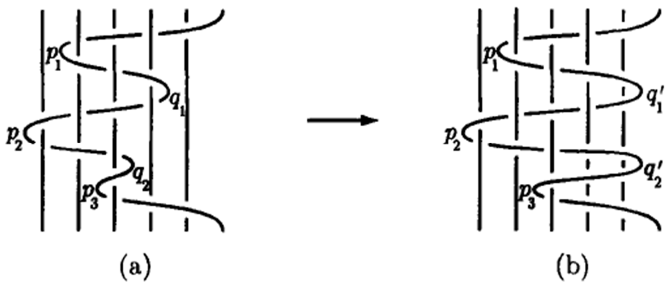


Figure 3.19 The process of delta moves to obtain new braid

□

What the above process allows us to do is to deform β into the following form

$$\beta = \beta_1 \beta_2 \ldots \beta_k \tag{3.38}$$

in which for some i each β_j is of the form

$$\beta_j = \left(\sigma_{n-1}^{\varepsilon_{n-1}} \sigma_{n-2}^{\varepsilon_{n-2}} \ldots \sigma_{i+1}^{\varepsilon_{i+1}}\right) \sigma_1^{2\varepsilon_i} \left(\sigma_{i+1}^{\varepsilon'_{i+1}} \ldots \sigma_{n-2}^{\varepsilon'_{n-2}} \sigma_{n-1}^{\varepsilon'_{n-1}}\right) \tag{3.39}$$

Where, for $1 \le i \le n-1$ and $l = i+1, \ldots, n-1,\ \varepsilon_l,\ \varepsilon'_l,\ \varepsilon_i$ are either +1 or -1.

Therefore, to prove the proposition it sufficies to show that each β_j is a product of a_i and their inverses. A diagrammatic proof is given in Figure 3.20(a) and (b). In Figure 3.20(a), firstly, we choose four points a, b, c, d that lie close to each undercrossing point from q to p and from p to q'. Having established these 4 points, we pull them to the right side, Figure 3.20(b). Then, it can be seen that β is the product of a_i and a_i^{-1}.

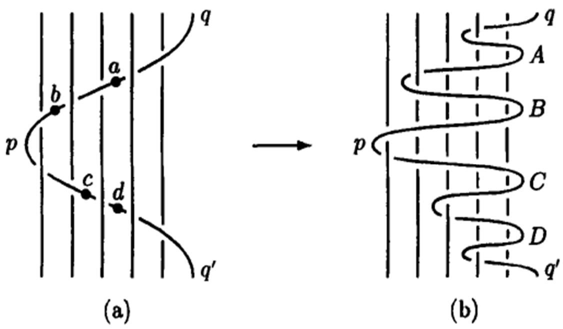


Figure 3.20 The process of delta moves to obtain new braid

Actually, we know that $a_1, a_2, \ldots, a_{n-1}$ generate $\mathbb{A}_n$ from proposition 3.15. As we shall see, the exact nature of these generators relies on the use of the Reidemeister-Schreier method [5].

Proposition 3.16 The elements $a_1, a_2, \ldots, a_{n-1}$ defined as in Proposition 3.15, freely generate the group $\mathbb{A}_n$. In other words, $\mathbb{A}_n$ is a free group freely generated by $a_1, a_2, \ldots, a_{n-1}$.

The remaining part of this section is devoted to the proof of this proposition.

Firstly, let us consider a subset, $\mathbb{H}_n$ say, of B_n that consists of all n-braids β with the property $\pi(\beta)(n)=n$, i.e, the braid permutation, $\pi(\beta)$ fixes n. This group obviously is a subgroup of B_n. Since β_1 and β_2 belong to the same right coset if and only if $\beta_1\beta_2^{-1}\in\mathbb{H}_n$, and so $\pi(\beta_1)(n)=\pi(\beta_2)(n)$, each right coset $\mathbb{H}_n\beta$ of $\mathbb{H}_n$ in B_n consists of n-braids β with $\pi(\beta)(n)=k$ for some $1\le k\le n.$ Therefore, there are exactly n distinct right cosets of $\mathbb{H}_n$ in B_n, and hence,

$$[\mathrm{B}_n:\mathbb{H}_n]=n \tag{3.40}$$

In particular, each right coset is represented by

$$M_i=\sigma_{n-1}\sigma_{n-2}...\sigma_{n-i+1} \tag{3.41}$$

for $i=1,2,...,n,$ and $M_1=1.$

Definition 3.17 Let G be a group given by $G=\langle x_1,x_2,...,x_n\,|P,R,Q,...\rangle$, where each x_i is generator of G and P,R,Q,... are relators for G. And let M_i be right coset representative for G, in addition, we may write

$$M_i=x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}...x_{i_k}^{\varepsilon_k} \tag{3.42}$$

where $1\le i_1,i_2,...,i_k\le n$ and $\varepsilon_l=\pm$ (for our case) and $M_1=1.$ The set $M=\{M_1,M_2,...\}$ is said to be Schreier system if for each M_i in (2.42) the following $k-1$ consecutive, initial parts of M_i,

$$x_{i_1}^{\varepsilon_1},x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2},x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}x_{i_3}^{\varepsilon_3},...,x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}...x_{i_{k-1}}^{\varepsilon_{k-1}} \tag{3.43}$$

also belong to M .

From definition 3.15, it is clear that $M=\{M_1,M_2,...,M_n\}$ forms a Schreier system of right coset representatives. Now, using this system, we can compute a presentation for

$\mathbb{H}_n$. Since this process is straight-forward application of the Reidemeister-Schreier method.

First, we shall determine the generators of $\mathbb{H}_n$.

Lemma 3.18 The group $\mathbb{H}_n$ is generated by

$$\sigma_1, \sigma_2, ..., \sigma_{n-2} \tag{3.44}$$

and for $j = 1, 2, ..., n-1$ by

$$a_j = \left(\sigma_{n-1}\sigma_{n-2}...\sigma_{j+1}\right)\sigma_j^2\left(\sigma_{j+1}^{-1}...\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\right) \tag{3.45}$$

Having established the nature of the generators for $\mathbb{H}_n$, the next step is to find the set of defining relations. Once we apply Reidemeister-Schreier method to obtain relations with M .

The first type of relations come from

$$\tau\left(M_i\sigma_k\sigma_l\sigma_k^{-1}\sigma_l^{-1}M_i^{-1}\right) = 1 \tag{3.46}$$

for $i = 1, 2, ..., n$ and $k, l = 1, 2, ..., n-1$ with $|k-l| \geq 2$ and τ is rewriting function (for the Reidemeister-Schreier method). But,

$$\begin{aligned}\tau\left(M_i\sigma_k\sigma_l\sigma_k^{-1}\sigma_l^{-1}M_i^{-1}\right) &= M_i\sigma_k\overline{M_i\sigma_k}^{-1} \times \overline{M_i\sigma_k}\sigma_l\overline{M_i\sigma_k\sigma_l}^{-1} \\ &\times \overline{M_i\sigma_k\sigma_l\sigma_k^{-1}}\,\overline{M_i\sigma_k\sigma_l\sigma_k^{-1}}^{-1} \times \overline{M_i\sigma_k\sigma_l\sigma_k^{-1}}\sigma_l^{-1}\overline{M_i\sigma_k\sigma_l\sigma_k^{-1}\sigma_l^{-1}}^{-1}\end{aligned} \tag{3.47}$$

Since M is a Schreier system, the remaining factors are all the identity. The latter re-arrangement allows us to determine relations.

Claim 3.19

$$\overline{M_i\sigma_k} = \begin{cases} M_i & \text{if } n-i>k \text{ or } n-i<k-1 \\ M_i\sigma_k & \text{if } n-i=k \\ M_i\sigma_{\mathrm{k}}^{-1} & \text{if } n-i=k-1 \end{cases} \tag{3.48}$$

where $M_i = \sigma_{n-1}\sigma_{n-2}...\sigma_{n-i+1}$ and $i,k=1,2,...,n-1$.

Proof: If we look the definition of M_i and use relation when $|k-i|\geq 2$, then we obtain the map.

From the relation (3.46) and Claim 3.17, we derived these relations of H_n:

(1) $\sigma_i\sigma_k = \sigma_k\sigma_i$, $\quad |k-i|\geq 2, \quad i,k=1,2,...,n\text{-}2$ (3.49)

(2) $\sigma_i a_k \sigma_{\mathrm{i}}^{-1} = a_k$, $k\neq i,\ i+1$ (3.50)

The second and final of relations can be determined from

$$\tau\left(M_i\sigma_j\sigma_{j+1}\sigma_j\sigma_{j+1}^{-1}\sigma_j^{-1}\sigma_{j+1}^{-1}M_i^{-1}\right)=1 \tag{3.51}$$

for $i=1,2,...,n$ and $j=1,2,...,n-2$.

From the relation (3.51) and claim 3.17, we derived these relations of H_n:

(3) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad i=1,2,...,n-3,$ (3.52)

(4) $\sigma_i a_i \sigma_{\mathrm{i}}^{-1} = a_{i+1}$, (3.53)

(5) $\sigma_i a_{i+1}\sigma_{\mathrm{i}}^{-1} = a_{\mathrm{i+1}}^{-1}\, a_i a_{i+1}$. (3.54)

Now, we found a presentation for H_n with help of Reidemeister-Schreier method. We shall indicate this presentation in the following proposition:

Proposition 3.20 The group $\mathbb{H}_n$ has a presentation of the form

$$\begin{aligned}\langle \sigma_1,\sigma_2,....,\sigma_{n-2},a_1,a_2,...,a_{n-1} \,|\, & \sigma_i\sigma_k=\sigma_k\sigma_i, \quad |k-i|\geq 2, \quad i,k=1,2,...,n\text{-}2 \\ & \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1} \qquad i=1,2,...,n-3 \\ & \sigma_i a_k \sigma_{\mathrm{i}}^{-1}=a_k \quad k\neq i,\ i+1 \\ & \sigma_i a_i \sigma_{\mathrm{i}}^{-1}=a_{i+1}, \\ & \sigma_i a_{i+1}\sigma_{\mathrm{i}}^{-1}=a_{\mathrm{i}+1}^{-1}\,a_i a_{i+1}\rangle.\end{aligned}$$

Now, let **F** be a free group generated by the $n-1$ elements $u_1,u_2,...,u_{n-1}$. The reason for the the introduction **F** is that we wish to establish a homomorphism from $\mathbb{H}_n$ to Aut(**F**) , the group of aumorphism of **F**. Once we have defined the appropriate mapping, we shall use it to show that indeed $\mathbb{A}_n$ is a free group generated by $a_1,a_2,...,a_{n-1}$.

So, let us define a mapping $\phi:\mathbb{H}_n\to Aut(\mathbf{F})$ by, $k=1,2,...,n-2$,

$$\phi(\sigma_k):\begin{cases}u_i\to u_i \quad \text{if } i\neq k,\ k+1,\ 1\leq i\leq n-1\\ u_k\to u_{k+1}\\ u_{k+1}\to u_{k+1}^{-1}u_k u_{k+1}\end{cases} \qquad 3.55)$$

and for $j=1,2,...,n-1$ by

$$\phi(a_j):u_i\to u_j^{-1}u_i u_j$$

(3.56)

where $i=1,2,...,n-1$.

Lemma 3.21 The mapping ϕ defined above is a homomorphism from $\mathbb{H}_n$ to Aut (**F**) .

Proof For the proof of lemma, we shall show the computation for the one of relations:

$$\sigma_i a_i \sigma_{\mathrm{i}}^{-1}=a_{i+1} \qquad (3.57)$$

$$\phi\left(\sigma_i a_i \sigma_{\mathrm{i}}^{-1}\right): u_j \xrightarrow{\sigma_{\mathrm{i}}^{-1}} u_j \xrightarrow{a_i} u_i^{-1} u_j u_i \xrightarrow{\sigma_i} u_{i+1}^{-1} u_j u_{i+1},$$
$$\phi\left(a_{i+1}\right): u_j \rightarrow u_{i+1}^{-1} u_j u_{i+1}.$$

$$\phi\left(\sigma_i a_i \sigma_{\mathrm{i}}^{-1}\right): u_i \xrightarrow{\sigma_{\mathrm{i}}^{-1}} u_i^{-1} u_{i+1} u_i \xrightarrow{a_i} u_i u_i^{-1} u_i^{-1} u_{i+1} u_i u_i^{-1} = u_{i+1} \xrightarrow{\sigma_i} u_{i+1}^{-1} u_i u_{i+1},$$
$$\phi\left(a_{i+1}\right): u_j \rightarrow u_{i+1}^{-1} u_i u_{i+1}.$$

$$\phi\left(\sigma_i a_i \sigma_{\mathrm{i}}^{-1}\right): u_{i+1} \xrightarrow{\sigma_{\mathrm{i}}^{-1}} u_i \xrightarrow{a_i} u_i \xrightarrow{\sigma_i} u_{i+1},$$
$$\phi\left(a_{i+1}\right): u_{i+1} \rightarrow u_{i+1}.$$

□

We are now at the final stage of the proof of Proposotion 3.14. All the information is at hand to show that $a_1, a_2, ..., a_{n-1}$ are free generators. If $n=2$, then a_1 generates a free group of rank 1, since $a_1^k \neq 1$ for any $k \neq 0$. So we assume that $n \geq 3$.

Suppose that $a_1, a_2, ..., a_{n-1}$ are not free generators for $\mathbb{H}_n$. Then, there exist a non-trivial relation in $\mathbb{H}_n$ in terms of $a_1, a_2, ..., a_{n-1}$, i.e. ,

$$W\left(a_1, a_2, ..., a_{n-1}\right) = 1 \tag{3.58}$$

If we apply the above homomorphism ϕ to W, we obtain that

$$\phi(W)\left(a_1, a_2, ..., a_{n-1}\right) = W\left(\phi\left(a_1\right), \phi\left(a_2\right), ..., \phi\left(a_{n-1}\right)\right) = id_{\mathrm{F}} \tag{3.59}$$

Suppose that

$$W\left(a_1, a_2, ..., a_{n-1}\right) = a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_{n-1}^{\varepsilon_{n-1}} \tag{3.60}$$

Then for $j = 1, 2, ..., n-1$,

$$\phi(W)\left(u_j\right) = \left(u_{i_1}^{-\varepsilon_1} ... u_{i_{k-1}}^{-\varepsilon_{k-1}} u_{i_k}^{-\varepsilon_k}\right) u_j \left(u_{i_1}^{\varepsilon_1} ... u_{i_{k-1}}^{\varepsilon_{k-1}} u_{i_k}^{\varepsilon_k}\right) \tag{3.61}$$

However, since $u_1, u_2, ..., u_{n-1}$ are free generators of **F**, $\phi(W) = id_{\mathrm{F}}$ implies

$$u_j = \left(u_{i_1}^{-\varepsilon_1} \dots u_{i_{k-1}}^{-\varepsilon_{k-1}} u_{i_k}^{-\varepsilon_k}\right) u_j \left(u_{i_1}^{\varepsilon_1} \dots u_{i_{k-1}}^{\varepsilon_{k-1}} u_{i_k}^{\varepsilon_k}\right) \tag{3.62}$$

for $j = 1, 2, \dots, n-1$.

Let us denote $u_{i_k}^{\varepsilon_k} u_{i_{k-1}}^{\varepsilon_{k-1}} \dots u_{i_1}^{\varepsilon_1}$ by g, then from the above relation, we see that g commutes with u_j for each $j = 1, 2, \dots, n-1$. Hence there exist an integer λ_j such that $g = u_j^{\lambda_j}$, for $j = 1, 2, \dots, n-1$.

However, the above implies that if $j \neq l$ then $u_j^{\lambda_j} = u_l^{\lambda_l}$. But $u_1, u_2, \dots, u_{n-1}$ are free generators of **F**. Hence, $\lambda_j = \lambda_l = 0$. Therefore, g, itself, must be 1. So, by a finite number of T1 and/or T2 operations, g collapses down to the empty word. Consequently, the relation $W = 1$ must be trivial relation. However, this contradicts our orginal assumption that there exists a non-trivial relation in $\mathbb{H}_n$ in terms of $a_1, a_2, \dots, a_{n-1}$. Hence, $a_1, a_2, \dots, a_{n-1}$ form a set of free generators. So, we have finally reached the conclusion of the proof of Proposition 3.16.

□

With the completion of the above proof, we are finally in a position to answer in a methodical fashion if or not a given n-braid β is a trivial n-braid.

3.3.3 A Presentation For The Pure n-braid Group

We know that the set of all pure n-braids forms a normal subgroup P_n of B_n . Since $(P_n) = (\ker \pi)$ is the normal subgroup of B_n .

Claim 3.22 The quotient group B_n / P_n is isomorfic to the symmetric group $\mathfrak{S}_n$.

Proof We know that there is a homomorhism $\pi : \mathrm{B}_n \to \mathfrak{S}_n$. And $(P_n) = (\ker \pi)$. So if we can use first isomorphism theorem, we can easily prove the claim. Because this claim is the application of this theorem. According to this theorem, there is a isomorphism such that $\pi' : \mathrm{B}_n / P_n \to \mathfrak{S}_n$.

□

Therefore, it is possible to find a Schreier system of right cosets [5] representatives of P_n in B_n. In this section, we use Reidemeister- Schreier method for P_n. But, we use induction which based on the presentation H_n to procedure a presentation of P_n.

For this purpose, we shall recall, H_n is the set of all n-braids β with the property $\pi(\beta)=n$, that is, those that fix n. Let H_{n-1} be subset of H_n consisting of all n-braids β with the property $\pi(\beta)(n-1)=n-1$. Hence every element of H_{n-1} fixes both n-1 and n. It is not too hard to see that H_{n-1} is a subgroup of H_n. We may define a subgroup H_k, for $k=1,2,...,n$, of H_{k+1} that consists of all n-braids that fix k. Therefore, H_k consists of all n-braids β with $\pi(\beta)(l)=l$ for $l=k,k+1,...,n.$ It is easy to see that we have the following sequence of subgroups,

$$\mathrm{B}_n \supset \mathrm{H}_n \supset \mathrm{H}_{n-1} \supset ... \supset \mathrm{H}_2 \supset \mathrm{H}_1 \qquad (3.63)$$

Furthermore, $[\mathrm{H}_{k+1}:\mathrm{H}_k]=k$ for $k=1,2,...,n-1$ and $\mathrm{H}_1(=\mathrm{H}_2)$ is the pure n-braid group P_n.

By choosing a suitable Schreier system, our aim is to use the presentation of H_n as the first step in an inductive process that yields a presentation of H_k, for $k=n-1,n-2,...,2$. In fact, a Schreier system of right coset representatives of H_k in H_{k+1} for $k=n-1,n-2,...,2$ is given by $\left\{N_i^{(k)}, i=1,2,...,k\right\}$, with

$$N_i^{(k)}=\sigma_{k-1}\sigma_{k-2}...\sigma_i \text{ and } N_k^{(k)}=1 \qquad (3.64)$$

We have shown that H_n is generated by $\sigma_1,\sigma_2,...,\sigma_{n-2}$ and the pure n-braids $a_1,a_2,...,a_{n-1}$. More generally, we define a pure n-braid $A_{i,j}$, for $1 \le i < j \le n$, as

$$A_{i,j}=\left(\sigma_{j-1}\sigma_{j-2}...\sigma_{i+1}\right)\sigma_i^2\left(\sigma_{i+1}^{-1}...\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\right) \qquad (3.65)$$

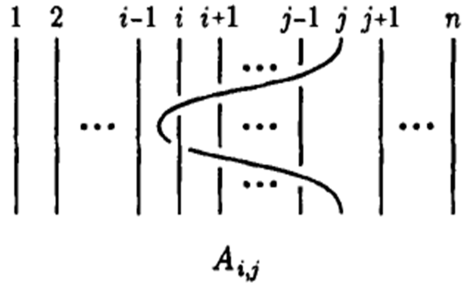


Figure 3.21 A generator of pure braid group

By convention, we assume that $A_{j,j} = 1$. In terms of notation, a_i is just $A_{i,n}$.

We can use $\left\{A_{i,n}\right\}$ to obtain a presentation for $\mathbb{H}_{n-1}$. In fact, $\mathbb{H}_{n-1}$ is generated by

$$g\left(N_i^{(n-1)}, \sigma_j\right) = N_i^{(n-1)} \sigma_j \overline{N_i^{(n-1)} \sigma_j}^{-1} \tag{3.66}$$

and

$$g\left(N_i^{(n-1)}, A_{k,n}\right) = N_i^{(n-1)} A_{k,n} \overline{N_i^{(n-1)} A_{k,n}}^{-1} \tag{3.67}$$

for $i,k = 1,2,\ldots,n-1$ and $j = 1,2,\ldots,n-2$.

To obtain more explicit forms of these generators of these generators, we need the following lemma.

Lemma 3.23 With the N_i as above,

(1)
$$\begin{aligned} \overline{N_i^{(n-1)} \sigma_j} &= \overline{\sigma_{n-2}\sigma_{n-3}\ldots\sigma_i\sigma_j} \\ &= \begin{cases} \sigma_{n-2}\sigma_{n-3}\ldots\sigma_i & \text{if } \ j < i-1 \text{ or } i < j \\ \sigma_{n-2}\sigma_{n-3}\ldots\sigma_i\sigma_{i-1} & \text{if } \ i-1 = j \\ \sigma_{n-2}\sigma_{n-3}\ldots\sigma_{i+1} & \text{if } \ i = j \end{cases} \end{aligned} \tag{3.68}$$

(2)
$$\overline{N_i^{(n-1)} A_{k,n}} = N_i^{(n-1)} \tag{3.69}$$

Proof Let $\beta = N_i^{(n-1)}\sigma_j$, and let us compute $\pi(\beta)(n-1)$. If $j < i-1 \text{ or } i < j$ then $\pi(\beta)(n-1) = i$. Hence,

$$\overline{N_i^{(n-1)}\sigma_j} = N_i^{(n-1)} \tag{3.70}$$

The other case in (1), if we use same method which is given for first case, we can easily obtain the other results.

Turning now to (2), since $A_{k,n}$ is a pure n-braid, we have

$$\pi\left(N_i^{(n-1)}A_{j,n}\right)(n-1) = \pi\left(N_i^{(n-1)}\right)(n-1) = i \tag{3.71}$$

Now, (2) is a direct consequence of above equation.

□

Therefore, for $j = 1,2,\ldots,n-2$ and $i = 1,2,\ldots,n-1$, we obtain that

$$g\left(N_i^{(n-1)},\sigma_j\right) = \begin{cases} \sigma_j & \text{if } j < i-1 \\ \sigma_{j-1} & \text{if } i < j \\ 1 & \text{if } i\text{-}1 = j \\ A_{j,n-1} & \text{if } i = j \end{cases} \tag{3.72}$$

Further, for $i, j = 1,2,\ldots,n-1$,

$$g\left(N_i^{(n-1)}A_{j,n}\right) = \begin{cases} A_{j,n} & \text{if j} < i \\ A_{n-1,n} & \text{if } i = j \\ A_{n-1,n}^{-1}A_{j-1,n}A_{n-1,n} & \text{if i} < j \end{cases} \tag{3.73}$$

Therefore, H_{n-1} is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-3}$ and $A_{1,n-1}, \ldots, A_{n-2,n-1}, A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}$.

The next step is to determine the defining relaitons for H_{n-1}. In fact, there are two types of defining relations for H_n.

$$(\mathrm{I})\begin{cases}(1)\ \sigma_i\sigma_k=\sigma_k\sigma_i & \text{for } i,k=1,2,\ldots,n\text{-}2\\(2)\ \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i=1,2,\ldots,n-3\end{cases} \tag{3.74}$$

$$(\mathrm{II})\begin{cases}(1)\ \sigma_i A_{k,n}\sigma_{\mathrm{i}}^{-1}=A_{k,n} & \text{for } i=1,2,\ldots,n-2,\ k=1,2,\ldots,n\text{-}1,\ \mathrm{k}\neq \mathrm{i,i+1}\\(2)\ \sigma_i A_{i,n}\sigma_{\mathrm{i}}^{-1}=A_{i+1,n} & \text{for } i=1,2,\ldots,n-2\\(3)\ \sigma_i A_{i+1,n}\sigma_{\mathrm{i}}^{-1}=A_{\mathrm{i+1,n}}^{-1}A_{i,n}A_{i+1,n} & \text{for } i=1,2,\ldots,n-2\end{cases} \tag{3.75}$$

Now, in order to find some information on the relations for $\mathbf{H}_{n-1}$, we need to compute $\tau\left(N_i^{(n-1)}R\left(N_i^{(n-1)}\right)^{-1}\right)$ with $1\le i\le n-1$ for various relations R in (I) and (II).

Proposition 3.24 For $1\le j<k\le n-2$ and $\left|j-k\right|\ge 2$, the relation $(\mathrm{I}_n)(1)$ $R=\sigma_j\sigma_k\sigma_j^{-1}\sigma_k^{-1}=1$ yields two types of relations.

(1) $\sigma_j\sigma_k=\sigma_k\sigma_j$ for $1\le j<k\le n-3$ and $\left|j-k\right|\ge 2$

$$(2)\ \ \sigma_p A_{q,n-1}\sigma_p^{-1}=A_{q,n-1}\ \text{ for } 1\le p\le n-3,\ 1\le q\le n-2 \text{ and } q\neq p,p+1 \tag{3.76}$$

Proof Let us compute $\tau\left(N_i^{(n-1)}R\left(N_i^{(n-1)}\right)^{-1}\right)=\tau\left(N_i^{(n-1)}\sigma_j\sigma_k\sigma_j^{-1}\sigma_k^{-1}\left(N_i^{(n-1)}\right)^{-1}\right)=1$ and find relations.

$$\begin{aligned}&\tau\left(N_i^{(n-1)}\sigma_j\sigma_k\sigma_j^{-1}\sigma_k^{-1}\left(N_i^{(n-1)}\right)^{-1}\right)=\\&=N_i^{(n-1)}\sigma_j\,\overline{N_i^{(n-1)}\sigma_j}^{-1}\times\overline{N_i^{(n-1)}\sigma_j\sigma_k}\ \overline{N_i^{(n-1)}\sigma_j\sigma_k}^{-1}\times\overline{N_i^{(n-1)}\sigma_j\sigma_k\sigma_j^{-1}}\ \overline{N_i^{(n-1)}\sigma_j\sigma_k\sigma_j^{-1}}^{-1}\\&\times\overline{N_i^{(n-1)}\sigma_j\sigma_k\sigma_j^{-1}}\sigma_k^{-1}\ \overline{N_i^{(n-1)}\sigma_j\sigma_k\sigma_j^{-1}\sigma_k^{-1}}^{-1}\end{aligned} \tag{3.77}$$

For $1\le i\le n-1$. From the above equation, we find this relations:

$$
\begin{array}{ll}
(1)\ \sigma_j\sigma_k = \sigma_k\sigma_j & \text{if } 1 \le j < k < i-1 \\
(2)\ \text{the trivial relation} & \text{if } 1 \le j < k = i-1 \\
(3)\ \sigma_j A_{k-1,n-1}\sigma_j^{-1} = A_{k-1,n-1} & \text{if } k = i \\
(4)\ \sigma_j\sigma_{k-1} = \sigma_{k-1}\sigma_j & \text{if } j < i-1,\ k = i+1 \\
(5)\ \text{the trivial relation} & \text{if } j = i-1,\ k > i \\
(6)\ \sigma_{k-1}A_{j,n-1}\sigma_{k-1}^{-1} = A_{j,n-1} & \text{if } j = i < k \le n-2 \\
(7)\ \sigma_{j-1}\sigma_{k-1} = \sigma_{k-1}\sigma_{j-1} & \text{if } i+1 \le j < k \le n-2
\end{array}
\tag{3.78}
$$

And if we generalize above relations, we find two relations are given in proposition 3.22.

Now, we introduced general relations with where they derived from.

$$(\mathrm{I}_n)(2)\ \sigma_j\sigma_{j+1}\sigma_j\sigma_{j+1}^{-1}\sigma_j^{-1}\sigma_{j+1}^{-1} = 1 \tag{3.79}$$

$$
\begin{array}{l}
(1)\ \sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1} \quad \text{for } 1 \le j < n-4 \\
(2)\ \sigma_j A_{j,n-1}\sigma_j^{-1} = A_{j+1,n-1} \quad \text{for } 1 \le j \le n-3 \\
(3)\ \sigma_j\left(A_{j+1,n-1}\right)\sigma_j^{-1} = A_{j+1,n-1}^{-1}A_{j,n-1}A_{j+1,n-1} \quad \text{for } 1 \le j \le n-3
\end{array}
\tag{3.80}
$$

$$(\mathrm{II}_n)(1)\quad \sigma_i A_{k,n}\sigma_{\mathrm{i}}^{-1}A_{k,n}^{-1} = 1 \tag{3.81}$$

$$
\begin{array}{l}
(1)\ \sigma_j A_{k,n}\sigma_j^{-1} = A_{k,n} \quad \text{if } 1 \le j \le n-3,\ k \ne j,\ j+1 \\
(2)\ A_{j,n-1}A_{k,n} = A_{k,n}A_{j,n-1} \quad \text{if } 1 \le k \le j \le n-2 \\
(3)\ A_{j,n-1}A_{n-1,n}^{-1}A_{k,n}A_{n-1,n}A_{j,n-1}^{-1} = A_{n-1,n}^{-1}A_{k,n}A_{n-1,n} \quad \text{if } 1 \le j \le n-2,\ 1 \le k \le n-1,\ j < k
\end{array}
\tag{3.82}
$$

$$(\mathrm{II}_n)(2)\ \ \sigma_i A_{i,n}\sigma_{\mathrm{i}}^{-1}A_{i+1,n}^{-1} = 1 \tag{3.83}$$

$$
\begin{array}{l}
(1)\ \sigma_j A_{j,n}\sigma_j^{-1} = A_{j+1,n} \quad \text{for } j = 1,2,\dots,n-3 \\
(2)\ A_{j,n-1}A_{j,n}A_{j,n-1}^{-1} = A_{n-1,n}^{-1}A_{j,n}A_{n-1,n} \quad \text{for } j = 1,2,\dots,n-2
\end{array}
\tag{3.84}
$$

$$(\mathrm{II}_n)(3)\ \ \sigma_i A_{i+1,n}\sigma_{\mathrm{i}}^{-1}A_{\mathrm{i+1,n}}\,A_{i,n}^{-1}A_{i+1,n}^{-1} = 1 \tag{3.85}$$

$$
\begin{aligned}
&(1)\ \sigma_j A_{j+1,n}\sigma_j^{-1} = A_{j+1,n}^{-1}A_{j,n}A_{j+1,n} \quad \text{for } 1 \le j \le n-3 \\
&(2)\ A_{j,n-1}A_{n-1,n}A_{j,n-1}^{-1} = A_{n-1,n}^{-1}A_{j,n}^{-1}A_{n-1,n}A_{j+1,n}A_{n-1,n} \quad \text{for } j = 1,2,...,n-2
\end{aligned} \tag{3.86}
$$

Now, we present all generators and relations for $\mathbb{H}_{n-1}$.

Proposition 3.25 The group $\mathbb{H}_{n-1}$ has the following presentation,

generators:

$$
\begin{aligned}
&\sigma_1, \sigma_2, ..., \sigma_{n-3}, \\
&A_{1,n-1}, A_{2,n-1}, ..., A_{n-2,n-1}, \\
&A_{1,n}, A_{2,n}, ..., A_{n-1,n}
\end{aligned} \tag{3.87}
$$

relations:

$(\mathrm{I})_{n-1}$

(1) $\sigma_j \rightleftharpoons \sigma_k$ if $1 \le j < k \le n-3$ and $|j-k| \ge 2$

(2) $\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}$ if $j = 1,2,...,n-4$

$(\mathrm{II})_{n-1}$

(1) $\sigma_j A_{k,n}\sigma_{\mathrm{j}}^{-1} = A_{k,n}$ if $1 \le j \le n-3,\ 1 \le k \le n-1$ and $k \ne j, j+1$

(2) $\sigma_j A_{k,n-1}\sigma_{\mathrm{j}}^{-1} = A_{k,n-1}$ if $1 \le j \le n-3,\ 1 \le k \le n-2$ and $k \ne j, j+1$

$(\mathrm{III})_{n-1}$

(1) $\sigma_j A_{j,n}\sigma_{\mathrm{j}}^{-1} = A_{j+1,n}$ if $1 \le j \le n-3$

(2) $\sigma_j A_{j,n-1}\sigma_{\mathrm{j}}^{-1} = A_{j+1,n-1}$ if $1 \le j \le n-3$

$(\mathrm{IV})_{n-1}$

(1) $\sigma_j A_{j+1,n}\sigma_{\mathrm{j}}^{-1} = A_{\mathrm{j+1,n}}^{-1} A_{j,n}A_{j+1,n}$ if $1 \le j \le n-3$

(2) $\sigma_j A_{j+1,n-1}\sigma_{\mathrm{j}}^{-1} = A_{\mathrm{j+1,n-1}}^{-1} A_{j,n-1}A_{j+1,n-1}$ if $1 \le j \le n-3$

$(\mathrm{V})_{n-1}$

(1) $A_{j,n-1} \rightleftharpoons A_{k,n}$ if $1 \le k < j \le n-2$

$(\mathrm{VI})_{n-1}$

(1) $A_{j,n-1}A_{j,n}A_{j,n-1}^{-1} = A_{n-1,n}^{-1}A_{j,n}A_{n-1,n}$ if $j = 1,2,...,n-2$

$(\mathrm{VII})_{n-1}$

(1) $A_{j,n-1}A_{j,n}A_{j,n-1}^{-1} = A_{n-1,n}^{-1}A_{j,n}^{-1}A_{n-1,n}A_{j,n}A_{n-1,n}$ if $j = 1, 2, ..., n-2$

$(\mathrm{VIII})_{n-1}$

(1) $A_{n-1,n}^{-1}A_{k,n}A_{n-1,n} \rightleftharpoons A_{j,n-1}$ if $1 \le j \le n-2,\ 1 \le k \le n-1$ and $j < k$

From proposition 3.24, we can see that there are two types of relations: one set involving σ_i and $A_{j,k}$, and the other set involving only $A_{j,k}$. Since P_n does not involve any σ_i, the first set of relations will eventually disappear. But, the second set of relations can be derived from the first set of relations. In addition, the second set of relations does not produce any new type of relations. With this in mind, we will give a presentation for P_n in following theorem.

Theorem 3.26 The pure n-braid group P_n has following presentation,

generators: $A_{j,k}\left(=\sigma_{k-1}\sigma_{k-2}...\sigma_{j+1}\sigma_j^2\sigma_{j+1}^{-1}...\sigma_{k-2}^{-1}\sigma_{k-1}^{-1}\right)$ for $1 \le j < k \le n$

relations:

(A) $A_{r,s} \rightleftharpoons A_{i,j}$ if $1 \le r < s < i < j \le n$ or $1 \le r < i < j < s \le n$

(B) $A_{r,s}A_{r,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}A_{s,j}$ if $1 \le r < s < j \le n$

(C) $A_{r,s}A_{s,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}A_{r,j}A_{s,j}$ if $1 \le r < s < j \le n$

(D) $A_{i,j}^{-1}A_{s,j}A_{i,j} \rightleftharpoons A_{r,i}$ if $1 \le r < s < i < j \le n$

(D) is equivalent to

(D') $A_{r,i}A_{s,j}A_{r,i}^{-1} = \left[A_{i,j}^{-1}, A_{r,j}^{-1}\right]A_{s,j}\left[A_{i,j}^{-1}, A_{r,j}^{-1}\right]^{-1}$ if $1 \le r < s < i < j \le n.$

Proof Assuming the other relations hold, we shall show, firstly, that relations (D) and (D') are equivalent.

$$
\begin{aligned}
(D) &\Leftrightarrow A_{r,i}A_{i,j}^{-1}A_{s,j}A_{i,j}A_{r,i}^{-1} = A_{i,j}^{-1}A_{s,j}A_{i,j} \\
&\Leftrightarrow \left(A_{r,i}A_{i,j}^{-1}A_{r,i}^{-1}\right)\left(A_{r,i}A_{s,j}A_{r,i}^{-1}\right)\left(A_{r,i}A_{i,j}A_{r,i}^{-1}\right) = A_{i,j}^{-1}A_{s,j}A_{i,j} \\
&\Leftrightarrow A_{r,i}A_{s,j}A_{r,i}^{-1} = \left(A_{r,i}A_{i,j}A_{r,i}^{-1}\right)A_{i,j}^{-1}A_{s,j}A_{i,j}\left(A_{r,i}A_{i,j}^{-1}A_{r,i}^{-1}\right) \\
&\overbrace{\Leftrightarrow}^{(C)} A_{r,i}A_{s,j}A_{r,i}^{-1} = \left(A_{i,j}^{-1}A_{r,j}^{-1}A_{i,j}A_{r,j}A_{i,j}\right)\left(A_{i,j}^{-1}A_{s,j}A_{i,j}\right)\left(A_{i,j}^{-1}A_{r,j}^{-1}A_{i,j}^{-1}A_{r,j}A_{i,j}\right) \\
&\Leftrightarrow A_{r,i}A_{s,j}A_{r,i}^{-1} = \left[A_{i,j}^{-1}, A_{r,j}^{-1}\right]A_{s,j}\left[A_{i,j}^{-1}, A_{r,j}^{-1}\right]^{-1} \\
&\Leftrightarrow (D^{'})
\end{aligned}
$$

Lemma 3.27 Let us write N_p for $N_p^{(k)}$, and set $p = 1, 2, ..., k.$

Case 1, $r < k,$

(1) $\tau\left(N_p A_{r,s} N_p^{-1}\right) = A_{r,s}$ if $r < p \le k < s$

(2) $\tau\left(N_p A_{r,s} N_p^{-1}\right) = A_{k,s}$ if $r = p \le k < s$

(3) $\tau\left(N_p A_{r,s} N_p^{-1}\right) = A_{k,s}^{-1}A_{r-1,s}A_{k,s}$ if $p < r < k < s$

Case 2, $r = k$,

(1) $\tau\left(N_p A_{r,s} N_p^{-1}\right) = A_{k,s}^{-1}A_{r-1,s}A_{k,s}$

Case 3, $r > k$,

(1) $\tau\left(N_p A_{r,s} N_p^{-1}\right) = A_{r,s}$

Lemma 3.28 The following commutative relations are consequences of relations $(A)-(D)$.

(1) $A_{k,s}^{-1}A_{l,s}A_{k,s} \rightleftharpoons A_{k,j}$ if $1 \le l < k < s < j \le n$

(2) $A_{k,j}^{-1}A_{l,j}A_{k,j} \rightleftharpoons A_{k,s}^{-1}A_{m,s}A_{k,s}$ if $1 \le l < m < k < s < j \le n$

(3) $A_{r,j}A_{k,j} \rightleftharpoons A_{r,k}$ if $1 \le r < k < j \le n$

Proof

$$(1)\ A_{k,j}A_{k,s}^{-1}A_{l,s}A_{k,s}=A_{k,s}^{-1}A_{l,s}A_{k,s}A_{k,j}$$

$$\Leftrightarrow A_{k,s}A_{k,j}A_{k,s}^{-1}=A_{l,s}\boxed{A_{k,s}A_{k,j}A_{k,s}^{-1}}A_{l,s}^{-1}$$

$$\underbrace{\Leftrightarrow}^{(B)} A_{s,j}^{-1}A_{k,j}A_{s,j}=A_{l,s}A_{s,j}^{-1}A_{k,j}A_{s,j}A_{l,s}^{-1}$$

$$\Leftrightarrow A_{s,j}^{-1}A_{k,j}A_{s,j}\rightleftharpoons A_{l,s}$$

$$\Leftrightarrow (D)$$

$$(2)\ A_{k,j}^{-1}A_{l,j}A_{k,j}A_{k,s}^{-1}A_{m,s}A_{k,s}=A_{k,s}^{-1}A_{m,s}A_{k,s}A_{k,j}^{-1}A_{l,j}A_{k,j}$$

$$\underbrace{\Leftrightarrow}^{(B)} A_{l,k}A_{l,j}A_{l,k}^{-1}\boxed{A_{k,s}^{-1}A_{m,s}A_{k,s}}=\boxed{A_{k,s}^{-1}A_{m,s}A_{k,s}}A_{l,k}A_{l,j}A_{l,k}^{-1}$$

$$\underbrace{\Leftrightarrow}^{(D)} A_{l,k}A_{l,j}A_{k,s}^{-1}A_{m,s}A_{k,s}A_{l,k}^{-1}=A_{l,k}A_{k,s}^{-1}A_{m,s}A_{k,s}A_{l,j}A_{l,k}^{-1}$$

$$\Leftrightarrow A_{l,j}A_{k,s}^{-1}A_{m,s}A_{k,s}=A_{k,s}^{-1}A_{m,s}A_{k,s}A_{l,j}$$

This holds since $A_{l,j}\rightleftharpoons A_{k,s}$ and $A_{l,j}\rightleftharpoons A_{m,s}$, relation (A).

$$(3)\ A_{r,k}A_{r,j}A_{k,j}A_{r,k}^{-1}=A_{r,j}A_{k,j}$$

$$\Leftrightarrow \boxed{A_{r,k}A_{r,j}A_{r,k}^{-1}}\boxed{A_{r,k}A_{k,j}A_{r,k}^{-1}}=A_{r,j}A_{k,j}$$

$$\underbrace{\Leftrightarrow}^{(B)\ (C)} A_{k,j}^{-1}A_{r,j}A_{k,j}A_{k,j}^{-1}A_{r,j}^{-1}\ A_{k,j}A_{r,j}A_{k,j}=A_{r,j}A_{k,j}$$

$$\Leftrightarrow A_{r,j}A_{k,j}=A_{r,j}A_{k,j}$$

□

We need to look at the relation $R_p=N_p\left(A_{r,s}A_{i,j}A_{r,s}^{-1}A_{i,j}^{-1}\right)N_p^{-1}=1$. Then, $\tau\left(R_p\right)=1$ yields the following relations.

Case 1, $r<k<s$

(1) If $p > r$ then $A_{r,s} \rightleftharpoons A_{i,j}$
(2) If $p = r$ then $A_{k,s} \rightleftharpoons A_{i,j}$
(3) If $i < p < r$ then $A_{i,j} \rightleftharpoons A_{k,s}^{-1} A_{r-1,s} A_{k,s}$
(4) If $i = p$ then $A_{k,j} \rightleftharpoons A_{k,s}^{-1} A_{r-1,s} A_{k,s}$
(5) If $p < i$ then $A_{k,j}^{-1} A_{i-1,j} A_{k,j} \rightleftharpoons A_{k,s}^{-1} A_{r-1,s} A_{k,s}$.

The relations in (1), (2) and (3) are consequences of the relation (A), while the relations (4) and (5) are same as (1) and (2) in Lemma 3.28.

Case 2, $r = k$

(1) If $i < p < r$ then $A_{k,s}^{-1} A_{k-1,s} A_{k,s} \rightleftharpoons A_{k,j}^{-1} A_{i-1,j} A_{k,j}$
(2) If $p = i$ then $A_{k,s}^{-1} A_{k-1,s} A_{k,s} \rightleftharpoons A_{k,j}$
(3) If $p < i$ then $A_{k,s}^{-1} A_{k-1,s} A_{k,s} \rightleftharpoons A_{i,j}$.

Relation (1) above is just Lemma 3.28(2), relation (2) is Lemma 3.28(1), and Lemma 3.25(2), and (3) is a consequence of (A).

Case 3, $i < k < r$

(1) If $\mathrm{i} < p$ then $A_{r,s} \rightleftharpoons A_{i,j}$
(2) If $p = i$ then $A_{r,s} \rightleftharpoons A_{k,j}$
(3) If $p < i$ then $A_{r,s} \rightleftharpoons A_{k,j}^{-1} A_{i-1,j} A_{k,j}$.

These relations are consequences of (A).

Case 4, $k = i$

(1) $A_{r,s} \rightleftharpoons A_{i,j}^{-1} A_{i-1,j} A_{i,j}$

Case 5, $k < i$

(1) $A_{r,s} \rightleftharpoons A_{i,j}$

In cases 4 and 5, the relations follow from (A).

In other case $r < s < i < j$, we can find this relation:

$$A_{r,s} \rightleftharpoons A_{i,j}\,.$$

Now let us set $R_p = N_p\left(A_{r,s}A_{r,j}A_{r,s}^{-1}A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}\right)N_p^{-1}$. Then $\tau\left(R_p\right)=1$ yields the following relations:

Case 1, $r<k<s<j$

(1) If $r<p\le k$ then $A_{r,s}A_{r,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}A_{s,j}$

(2) If $p=r$ then $A_{k,s}A_{k,j}A_{k,s}^{-1} = A_{s,j}^{-1}A_{k,j}A_{s,j}$

(3) If $p<r$ then

$A_{k,s}^{-1}A_{r-1,s}A_{k,s}A_{k,j}^{-1}A_{r-1,j}A_{k,j}A_{k,s}^{-1}A_{r-1,s}^{-1}A_{k,s} = A_{s,j}^{-1}A_{k,j}^{-1}A_{r-1,j}A_{k,j}A_{s,j}$.

Case 2, $r=k$

(1) $A_{k,s}^{-1}A_{k-1,s}A_{k,s}A_{k,j}^{-1}A_{k-1,j}A_{k,j}A_{k,s}^{-1}A_{k-1,s}^{-1}A_{k,s} = A_{s,j}^{-1}A_{k,j}^{-1}A_{k-1,j}A_{k,j}A_{s,j}$

Case 3, $r<k$

(1) $A_{r,s}A_{r,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}A_{s,j}$

Therefore, we only need to show that Case 1(3) and Case 2 are consequences of relations$(A)-(D)$.

Case 1(3) For $(l=)\ r-1<k<s<j$

$$\boxed{A_{k,s}^{-1}A_{l,s}A_{k,s}}\boxed{A_{k,j}^{-1}A_{l,j}A_{k,j}}\boxed{A_{k,s}^{-1}A_{l,s}^{-1}A_{k,s}} = A_{s,j}^{-1}\boxed{A_{k,j}^{-1}A_{l,j}A_{k,j}}A_{s,j}$$

$$\overbrace{\Leftrightarrow}^{(B)} A_{l,k}A_{l,s}A_{l,k}^{-1}A_{l,k}A_{l,j}A_{l,k}^{-1}A_{l,k}A_{l,s}^{-1}A_{l,k}^{-1} = \boxed{A_{s,j}^{-1}A_{l,k}}A_{l,j}\boxed{A_{l,k}^{-1}A_{s,j}}$$

$$\overbrace{\Leftrightarrow}^{(A)} A_{l,k}A_{l,s}A_{l,j}A_{l,s}^{-1}A_{l,k}^{-1} = A_{l,k}A_{s,j}^{-1}A_{l,j}A_{s,j}A_{l,k}^{-1}$$

$$\Leftrightarrow A_{l,s}A_{l,j}A_{l,s}^{-1} = A_{s,j}^{-1}A_{l,j}A_{s,j}$$

$$\Leftrightarrow (B)$$

The proof of Case 2 follows along similar lines to the above proof if instead we set k-1=l.

Now, let us set $R_p = N_p\left(A_{r,s}A_{s,j}A_{r,s}^{-1}A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}^{-1}A_{r,j}A_{s,j}\right)N_p^{-1}$ for $1 \le r < s < j \le n$. Then $\tau\left(R_p\right)=1$ yields the following relations:

Case 1, $r<k<s<j$

(1) If $r<p\le k$ then the relation is unchanged.

(2) If $p=r$ then change r into k in the orginal relation.

(3) If $p<r$ then

$$A_{k,s}^{-1}A_{r-1,s}A_{k,s}A_{s,j}A_{k,s}^{-1}A_{r-1,s}^{-1}A_{k,s} = A_{s,j}^{-1}A_{k,j}^{-1}A_{r-1,j}^{-1}A_{k,j}A_{s,j}A_{k,j}^{-1}A_{r-1,j}A_{k,j}A_{s,j}.$$

Case 2, $r=k$

$$(1)\, A_{k,s}^{-1}A_{k-1,s}A_{k,s}A_{s,j}A_{k,s}^{-1}A_{k-1,s}^{-1}A_{k,s} = A_{s,j}^{-1}A_{k,s}^{-1}A_{k-1,s}^{-1}A_{k,s}A_{s,j}A_{k,s}^{-1}A_{k-1,s}A_{k,s}A_{s,j}$$

Case 3, $k<r$

This is the same as the orginal relation.

Therefore, we need only show that Case 1(3) (and hence Case 2) is a consequence of $(A)-(D)$.

For $l=r-1<k<s<j$,

$$\begin{aligned}
&\boxed{A_{k,s}^{-1}A_{l,s}A_{k,s}}A_{s,j}\boxed{A_{k,s}^{-1}A_{l,s}A_{k,s}} = A_{s,j}^{-1}\boxed{A_{k,j}^{-1}A_{l,j}^{-1}A_{k,j}}A_{s,j}\boxed{A_{k,j}^{-1}A_{l,j}A_{k,j}}A_{s,j}\\
&\overbrace{\Leftrightarrow}^{(B)} A_{l,k}A_{l,s}\boxed{A_{l,k}^{-1}A_{s,j}}A_{l,k}A_{l,s}^{-1}A_{l,k}^{-1} = \boxed{A_{s,j}^{-1}A_{l,k}}A_{l,j}^{-1}A_{l,k}^{-1}\boxed{A_{s,j}A_{l,k}}A_{l,j}\boxed{A_{l,k}^{-1}A_{s,j}}\\
&\overbrace{\Leftrightarrow}^{(A)} A_{l,k}A_{l,s}A_{s,j}A_{l,s}^{-1}A_{l,k}^{-1} = A_{l,k}A_{s,j}^{-1}A_{l,j}^{-1}A_{s,j}A_{l,j}A_{s,j}A_{l,k}^{-1}\\
&\Leftrightarrow A_{l,s}A_{s,j}A_{l,s}^{-1} = A_{s,j}^{-1}A_{l,j}^{-1}A_{s,j}A_{l,j}A_{s,j}\\
&\Leftrightarrow (C)
\end{aligned}$$

Finally, let us set $R_p = N_p\left(A_{i,j}^{-1}A_{s,j}A_{i,j}A_{r,i}A_{i,j}^{-1}A_{s,j}^{-1}A_{i,j}A_{r,i}^{-1}\right)N_p^{-1}$. For $1 \le r < s < i < j \le n$, $\tau\left(R_p\right)=1$ yields the following relations:

Case 1, $s < k < i$

(1) If $s < p < k$ then the same relation is obtained.

(2) If $s = p$ then $A_{i,j}^{-1} A_{k,j} A_{i,j} \rightleftharpoons A_{r,i}$.

(3) If $r < p < s$ then $A_{i,j}^{-1} A_{k,j}^{-1} A_{s-1,j} A_{k,j} A_{i,j} \rightleftharpoons A_{r,i}$.

(4) If $p = r$ then $A_{i,j}^{-1} A_{k,j}^{-1} A_{s-1,j} A_{k,j} A_{i,j} \rightleftharpoons A_{k,i}$.

(5) If $p < r$ then $A_{i,j}^{-1} A_{k,j}^{-1} A_{s-1,j} A_{k,j} A_{i,j} \rightleftharpoons A_{k,i}^{-1} A_{r-1,i} A_{k,i}$.

Case 2, $k = s$

(1) If $r < p < s$ then $A_{i,j}^{-1} A_{k,j}^{-1} A_{k-1,j} A_{k,j} A_{i,j} \rightleftharpoons A_{r,i}$.

(2) If $p = r$ then $A_{i,j}^{-1} A_{k,j}^{-1} A_{k-1,j} A_{k,j} A_{i,j} \rightleftharpoons A_{k,i}$.

(3) If $p < r$ then $A_{i,j}^{-1} A_{k,j}^{-1} A_{k-1,j} A_{k,j} A_{i,j} \rightleftharpoons A_{k,i}^{-1} A_{r-1,i} A_{k,i}$.

Case 3, $r < k < s$

(1) If $r < p$ then $A_{i,j}^{-1} A_{s,j} A_{i,j} \rightleftharpoons A_{r,i}$.

(2) If $p = r$ then $A_{i,j}^{-1} A_{s,j} A_{i,j} \rightleftharpoons A_{k,i}$.

(3) If $p < r$ then $A_{i,j}^{-1} A_{s,j} A_{i,j} \rightleftharpoons A_{k,i}^{-1} A_{r-1,i} A_{k,i}$.

Case 4, $k = r$

(1) $A_{i,j}^{-1} A_{s,j} A_{i,j} \rightleftharpoons A_{k,i}^{-1} A_{k-1,i} A_{k,i}$.

Case 5, $k < r$

(1) $A_{i,j}^{-1} A_{s,j} A_{i,j} \rightleftharpoons A_{r,i}$.

Now, to complete the proof, we need to show that the following relaitons are consequences of $(A)-(D)$.

$$
\begin{aligned}
&(1)\quad A^{-1}_{i,j}A^{-1}_{k,j}A_{m,j}A_{k,j}A_{i,j} \rightleftharpoons A_{l,i} \quad \text{if } l<m<k<i<j \\
&(2)\quad A^{-1}_{i,j}A^{-1}_{k,j}A_{m,j}A_{k,j}A_{i,j} \rightleftharpoons A^{-1}_{k,i}A_{l,i}A_{k,i} \quad \text{if } l<m<k<i<j \\
&(3)\quad A^{-1}_{i,j}A^{-1}_{k,j}A_{m,j}A_{k,j}A_{i,j} \rightleftharpoons A_{k,i} \quad \text{if } m<k<i<j \\
&(4)\quad A^{-1}_{i,j}A_{s,j}A_{i,j} \rightleftharpoons A^{-1}_{k,i}A_{l,i}A_{k,i} \quad \text{if } l<k<s<i<j
\end{aligned}
\tag{3.88}
$$

We shall prove separately each of the four parts in the above relations.

Proof of (1)

$$A^{-1}_{i,j}\boxed{A^{-1}_{k,j}A_{m,j}A_{k,j}}A_{i,j}A_{l,i} = A_{l,i}A^{-1}_{i,j}\boxed{A^{-1}_{k,j}A_{m,j}A_{k,j}}A_{i,j}$$

$$\overbrace{\Leftrightarrow}^{(B)} \boxed{A^{-1}_{i,j}A_{m,k}}A_{m,j}\boxed{A^{-1}_{m,k}A_{i,j}A_{l,i}} = \boxed{A_{l,i}A^{-1}_{i,j}A_{m,k}}A_{m,j}\boxed{A^{-1}_{m,k}A_{i,j}}$$

$$\overbrace{\Leftrightarrow}^{(A)} A_{m,k}A^{-1}_{i,j}A_{m,j}A_{i,j}A_{l,i}A^{-1}_{m,k} = A_{m,k}A_{l,i}A^{-1}_{i,j}A_{m,j}A_{i,j}A^{-1}_{m,k}$$

$$\Leftrightarrow A^{-1}_{i,j}A_{m,j}A_{i,j} \rightleftharpoons A_{l,i}$$

$$\Leftrightarrow (D)$$

Proof of (2)

$$A^{-1}_{i,j}A^{-1}_{k,j}A_{m,j}A_{k,j}A_{i,j}\boxed{A^{-1}_{k,i}A_{l,i}A_{k,i}} = \boxed{A^{-1}_{k,i}A_{l,i}A_{k,i}}A^{-1}_{i,j}A^{-1}_{k,j}A_{m,j}A_{k,j}A_{i,j}$$

$$\overbrace{\Leftrightarrow}^{(B)} A^{-1}_{i,j}A^{-1}_{k,j}A_{m,j}A_{k,j}\boxed{A_{i,j}A_{l,k}}A_{l,i}A^{-1}_{l,k} = A_{l,k}A_{l,i}\boxed{A^{-1}_{l,k}A^{-1}_{i,j}}A^{-1}_{k,j}A_{m,j}A_{k,j}A_{i,j}$$

$$\overbrace{\Leftrightarrow}^{(A)} A^{-1}_{i,j}\boxed{A^{-1}_{k,j}A_{m,j}A_{k,j}}A_{l,k}A_{i,j}A_{l,i}A^{-1}_{l,k} = A_{l,k}A_{l,i}A^{-1}_{i,j}A^{-1}_{l,k}\boxed{A^{-1}_{k,j}A_{m,j}A_{k,j}}A_{i,j}$$

$$
\begin{aligned}
\overbrace{\Leftrightarrow}^{(D)\ (A)}\quad & \boxed{A_{l,k}}A^{-1}_{i,j}\boxed{A^{-1}_{k,j}A_{m,j}A_{k,j}}A_{i,j}A_{l,i}\boxed{A^{-1}_{l,k}} \\
&= \boxed{A_{l,k}}A_{l,i}A^{-1}_{i,j}\boxed{A^{-1}_{k,j}A_{m,j}A_{k,j}}A_{i,j}\boxed{A^{-1}_{l,k}}
\end{aligned}
$$

$$\overbrace{\Leftrightarrow}^{(B)} \boxed{A^{-1}_{i,j}A_{m,k}}A_{m,j}\boxed{A^{-1}_{m,k}A_{i,j}A_{l,i}} = \boxed{A_{l,i}A^{-1}_{i,j}A_{m,k}}A_{m,j}\boxed{A^{-1}_{m,k}A_{i,j}}$$

$$\overbrace{\Leftrightarrow}^{(A)} A_{m,k}A^{-1}_{i,j}A_{m,j}A_{i,j}A_{l,i}A^{-1}_{m,k} = A_{m,k}A_{l,i}A^{-1}_{i,j}A_{m,j}A_{i,j}A^{-1}_{m,k}$$

$$\Leftrightarrow (D)$$

Proof of (3)

$$A_{i,j}^{-1}A_{k,j}^{-1}A_{m,j}A_{k,j}\boxed{A_{i,j}A_{k,i}^{-1}} = A_{k,i}^{-1}A_{i,j}^{-1}A_{k,j}^{-1}A_{m,j}A_{k,j}A_{i,j}$$

$$\overbrace{\Leftrightarrow}^{(C)} A_{i,j}^{-1}A_{k,j}^{-1}A_{m,j}A_{k,j}A_{k,i}^{-1}\boxed{A_{i,j}^{-1}A_{k,j}^{-1}A_{i,j}}\boxed{A_{k,j}A_{i,j}}$$

$$= \boxed{A_{k,i}^{-1}A_{i,j}^{-1}A_{k,j}^{-1}}A_{m,j}\boxed{A_{k,j}A_{i,j}}$$

$$\Leftrightarrow A_{i,j}^{-1}A_{k,j}^{-1}A_{m,j}\boxed{A_{k,j}A_{k,i}^{-1}A_{k,i}A_{k,j}^{-1}}A_{k,i}^{-1} = A_{i,j}^{-1}A_{k,j}^{-1}A_{k,i}^{-1}A_{m,j}$$

(By (B) and Lemma 3.28(3))

$$\Leftrightarrow A_{m,j}A_{k,i}^{-1} = A_{k,i}^{-1}A_{m,j}$$

$$\Leftrightarrow (A)$$

Proof of (4)

$$\boxed{A_{i,j}^{-1}A_{s,j}A_{i,j}}A_{k,i}^{-1}A_{l,i}A_{k,i} = A_{k,i}^{-1}A_{l,i}A_{k,i}\boxed{A_{i,j}^{-1}A_{s,j}A_{i,j}}$$

$$\overbrace{\Leftrightarrow}^{(D)} A_{k,i}^{-1}\ A_{i,j}^{-1}A_{s,j}A_{i,j}A_{l,i}A_{k,i} = A_{k,i}^{-1}A_{l,i}A_{i,j}^{-1}A_{s,j}A_{i,j}A_{k,i}$$

$$\Leftrightarrow A_{i,j}^{-1}A_{s,j}A_{i,j} \rightleftharpoons A_{l,i}$$

$$\Leftrightarrow (D)$$

This now completes the proof of Theorem 3.25.

$\square$

CHAPTER 4

THE FUNDAMENTAL GROUP OF CONFIGURATION SPACE

In this chapter, we introduce configuration space to obtain the different view for braid group.

4.1 Configuration Space

4.1.1 Definitions

Definition 4.1 Let M be a manifold of dimension ≥ 2, let $\prod_{i=1}^{n} M$ denote the n-fold product space, and let $F_{0,n}M$ denote the subspace of $\prod_{i=1}^{n} M$

$$F_{0,n}M = \left\{ (z_1, z_2, ..., z_n) \in \prod_{i=1}^{n} M \middle| z_i \neq z_j \text{ if } i \neq j \right\} \tag{4.1}$$

(We will give the meaning of subscript "0" later.) The fundamental group $\pi_1 F_{0,n}M$ of the space $F_{0,n}M$ is the pure braid with n strings of manifold M [9].

Definition 4.2 Two points z and $z^{'}$ of $F_{0,n}M$ are said to be equivalent if the coordinates $(z_1, z_2, ..., z_n)$ of z differ from the coordinates $(z_1^{'}, z_2^{'}, ..., z_n^{'})$ of $z^{'}$ by permutation.

Let $B_{0,n}M$ denote the identification space of $F_{0,n}M$ under this equivalence relation.

Definition 4.3 The fundamental group $\pi_1 B_{0,n}M$ of the space $B_{0,n}M$ is called the full braid group of M , or simply, the braid group of M .

Claim 4.4 The natural projection $p: F_{0,n}M \to B_{0,n}M$ is a regular (normal) covering map.

Proof Firstly, we show that p is a covering map.

Let $z^0 = \left(z_1^0, z_2^0, \ldots, z_n^0\right)$ be an element of $F_{0,n}M$ and $\overline{z^0}$ (the equivalence class of z^0) be element of $B_{0,n}M$.

$$\overline{z^0} = \left\{ z^i \,\middle|\, z^0 \sim z^i, z^i \in F_{0,n}M \right\} \quad (4.2)$$

Now, we can define a metric on $F_{0,n}M$ such that

$$d: F_{0,n}M \times F_{0,n}M \to R$$

$$d(z^i, z^j) = \left\| z^i - z^j \right\| \quad (4.3)$$

If we take the elements of $\overline{z^0}$, then there is a positive real number $\varepsilon > 0$;

$$\varepsilon \le 2 \min_{i,j \in I} d\left(z^i, z^j\right) \quad (4.4)$$

where I is index set and $z^i, z^j \in \overline{z^0}$.

Therefore, $U_\varepsilon = \left\{ z^i \,\middle|\, d\left(z^0, z^i\right) < \varepsilon \right\}$ is a neighborhood of $\overline{z^0}$.

Hence, we can easily see that,

$$V_\delta = \left\{ z^i \,\middle|\, d\left(z^0, z^i\right) < \delta(\varepsilon) \right\} \quad (4.5)$$

where $\delta > 0$ is a real number; V_δ are open sets and

$$p^{-1}\left(U_\varepsilon\right) = \bigcup_\delta V_\delta \quad (4.6)$$

So, *p* is a covering mapping. The equivalence relation shows a deck transfomation. With this deck transformation, *p* is a regular(normal) covering map [6].

□

The classical braid group of Artin is the braid group is the braid group $\pi_1 B_{0,n} R^2$. Artin's geometric definiton of $\pi_1 B_{0,n} R^2$ can be recovered from the definition above as follows:

Choose a base point $z^0 = \left(z_1^0,...,z_n^0\right) \in F_{0,n} R^2$ for $\pi_1 F_{0,n} R^2$ and a point $\overline{z^0} \in B_{0,n} R^2$ such that $p\left(z^0\right) = \overline{z^0}$. Any element in $\pi_1 B_{0,n} R^2 = \pi_1\left(B_{0,n} R^2, \overline{z^0}\right)$ is represented by a loop

$$l : I,\{0,1\} \to B_{0,n} R^2, \overline{z^0} \tag{4.7}$$

which lifts uniquely to a path

$$l : I,\{0\} \to F_{0,n} R^2, z^0 \tag{4.8}$$

If $l(t) = \left(l_1(t),...,l_n(t)\right)$, $t \in I$, then each of the coordinate functions l_i defines (via its graph) an arc $\mathcal{P}_i = \left(l_i(t), t\right)$ in $R^2 \times I$. Since $l(t) \in F_{0,n} R^2$, the arcs $\mathcal{P}_1,...,\mathcal{P}_n$ are disjoint. Their union $\mathcal{P} = \mathcal{P}_1 \cup ... \cup \mathcal{P}_n$ is called a geometric braid. The arc is called the ith string.

A geometric braid is a representative of a path class in the fundamental group $\pi_1 B_{0,n} R^2$. Thus if $\mathcal{P}$ and $\mathcal{P}'$ are geometric braids, then $\mathcal{P} \sim \mathcal{P}'$ (that is, they represent the same element of $\pi_1 B_{0,n} R^2$) if the paths l and l' which define these braids are homotopic relative to base point $z^0 = \left(z_1^0,...,z_n^0\right)$ in the space $F_{0,n} R^2$. Thus we require the existence of a continuous mapping $\mathcal{F} : I \times I \to F_{0,n} R^2$ with

$$\begin{aligned}
\mathcal{F}(t,0) &= \left(\mathcal{F}_1(t,0),...,\mathcal{F}_n(t,0)\right) = \left(l_1(t),...,l_n(t)\right) \\
\mathcal{F}(t,1) &= \left(\mathcal{F}_1(t,1),...,\mathcal{F}_n(t,1)\right) = \left(l_1'(t),...,l_n'(t)\right) \\
\mathcal{F}(0,s) &= \left(\mathcal{F}_1(0,s),...,\mathcal{F}_n(0,s)\right) = \left(z_1^0,...,z_n^0\right) \\
\mathcal{F}(1,s) &= \left(\mathcal{F}_1(1,s),...,\mathcal{F}_n(1,s)\right) = \left(z_{\mu_1}^0,...,z_{\mu_n}^0\right)
\end{aligned} \tag{4.9}$$

where $(\mu_1,...,\mu_n)$ is a permutation of array $(1,...,n)$. The homotopy $\mathscr{T}$ defines a continous a sequence of geometric braids $\mathscr{P}(s)=\mathscr{P}_1(s)\cup...\cup\mathscr{P}_n(s)$, $s\in I$, where $\mathscr{P}_i(s)=(\mathscr{T}_i(t,s),t)$, such that $\mathscr{P}(0)=\mathscr{P}$ and $\mathscr{P}(1)=\mathscr{P}'$.

Definition 4.5 Let $Q_m=\{q_1,...,q_m\}$ be a set of fixed distinguished points of M. The configuration space $F_{m,n}M$ is the space of $F_{0,n}(M-Q_m)$.

Note that the topological type of $F_{m,n}M$ does not depend on the choice of the particular points Q_m, since one may always find an isotopy of M which deforms any one such point set Q_m into any other $Q_m^{'}$. Note that $F_{m,1}M=M-Q_m$.

We are interested in the relationship between the configuration spaces $F_{m,n}M$ and $F_{0,n}M$.

The key observation is the following theorem:

Theorem 4.6 (Fadell and Nuewirth [7]) Let $\pi: F_{m,n}M\to F_{m,r}M$ be defined by

$$\pi(z_1,...,z_n)=(z_1,...,z_r),\ 1\le r<n \tag{4.10}$$

Then π exhibits $F_{m,n}M$ as a locally trivial fibre space over the base space $F_{m,r}M$, with fibre $F_{m+r,n-r}M$.

Proof First consider, for some base point $(z_1^0,...,z_r^0)$ in $F_{m,r}M$ the fibre $\pi^{-1}(z_1^0,...,z_r^0)$

$$\pi^{-1}(z_1^0,...,z_r^0)=\{(z_1^0,...,z_r^0,y_{r+1},...,y_n),\ \text{where}\ z_1^0,...,z_r^0,y_{r+1},...,y_n\ \text{are distinct and in}\ M-Q_m\}$$

If we select Q_{m+r} equal to $Q_m\cup\{z_1^0,...,z_r^0\}$, then

$F_{m+r,n-r}M=\{(y_{r+1},...,y_n),\ \text{where}\ y_{r+1},...,y_n\ \text{are distinct and in}\ M-Q_{m+r}\}$,

and there is an obvious homeomorphism

$$h: F_{m+r,n-r}M\to\pi^{-1}(z_1^0,...,z_r^0)$$

defined by

$$h\left(y_{r+1},\ldots,y_n\right)=\left(z_1^0,\ldots,z_r^0,y_{r+1},\ldots,y_n\right)$$

The proof of the local triviality of π will be carried out, for notational and descriptive convenience, only in the case of $r=1$. For the other cases [3]. Fix for consideration, therefore, a point $x_0\in M-Q_m=F_{m,1}M=F_{m,r}M$. Add another point q_{m+1} to the set Q_m to form Q_{m+1} and pick a homeomorphism $\alpha:M\to M$, fixed on Q_m, such that $\alpha\left(q_{m+1}\right)=x_0$. Let U denote a neighborhood of x_0 in $M-Q_m$ which is homeomorphic to an open ball, let $\overline{U}$ denote the closure of U. Define a map $\theta:U\times\overline{U}\to\overline{U}$ with the following properties. Setting $\theta_z\left(y\right)=\theta\left(z,y\right)$ we require:

(i) $\theta_z:\overline{U}\to\overline{U}$ is a homeomorphism which fixes $\partial\overline{U}$.

(ii) $\theta_z\left(z\right)=x_0$.

By (i), θ can be extended to $\theta:U\times M\to M$ be defining $\theta\left(z,y\right)=y$ for $y\notin U$. The required local product representation

$$U\times F_{m+1,n-1}M\underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}}\pi^{-1}\left(U\right)$$

is given by

$$\phi\left(z,z_2,\ldots,z_n\right)=\left(z,\theta_z^{-1}\alpha\left(z_2\right),\ldots,\theta_z^{-1}\alpha\left(z_n\right)\right).$$

$$\phi^{-1}\left(z,z_2,\ldots,z_n\right)=\left(z,\alpha^{-1}\theta_z\left(z_2\right),\ldots,\alpha^{-1}\theta_z\left(z_n\right)\right).$$

$\square$

Proposition 4.7 If $\pi_2\left(M-Q_m\right)=\pi_3\left(M-Q_m\right)=0$ for each $m\geq 0$, then $\pi_2F_{0,n}M=0$.

Proof The exact homotopy sequence of the fibration $\pi:F_{m,n}M\to F_{m,1}M=M-Q_m$ of theorem 4.6 gives an exact sequence

$$\ldots \to \underbrace{\pi_3\left(M-Q_m\right)}_{0} \to \pi_2 F_{m+1,n-1}M \to \pi_2 F_{m,n}M \to \underbrace{\pi_2\left(M-Q_m\right)}_{0} \to \ldots$$

Since $\pi_2\left(M-Q_m\right)=\pi_3\left(M-Q_m\right)=0$, it follows that $\pi_2 F_{m+1,n-1}M$ and $\pi_2 F_{m,n}M$ are isomorphic.

By induction,

$$\begin{aligned}
&\pi_2 F_{m+2,n-2}M \to \pi_2 F_{m,n}M \to \pi_2 F_{m,2}M \\
&\pi_2 F_{m+3,n-3}M \to \pi_2 F_{m,n}M \to \pi_2 F_{m,3}M \\
&\vdots \\
&\pi_2 F_{m+n-1,1}M \to \pi_2 F_{m,n}M \to \pi_2 F_{m,n-1}M
\end{aligned} \tag{4.11}$$

and if we take as m=0, we obtain that,

$$\pi_2 F_{n-1,1}M \to \pi_2 F_{0,n}M \to \pi_2 F_{0,n-1}M \tag{4.12}$$

$$\pi_2 F_{0,n}M \approx \pi_2 F_{n-1,1}M = \pi_2 F_{0,n-1}M = 0 \tag{4.13}$$

□

Theorem 4.8 If $\pi_2\left(M-Q_m\right)=\pi_3\left(M-Q_m\right)=\pi_0\left(M-Q_m\right)=1$ for every $m \geq 0$, then the following sequence of the groups and homomorphism is exact:

$$1 \to \pi_1\left(F_{n-1,1}M, z_1^0\right) \xrightarrow{j_*} \pi_1\left(F_{0,n}M, \left(z_1^0, \ldots, z_n^0\right)\right) \xrightarrow{\pi_*} \pi_1\left(F_{0,n-1}M, \left(z_1^0, \ldots, z_{n-1}^0\right)\right) \to 1 \tag{4.14}$$

where π_* and j_* are the homomorhism induced by the mapping π and j.

Proof

$$\ldots \to \underbrace{\pi_3\left(M - Q_m, z_1^0\right)}_{1} \to \pi_2\left(F_{m+1,n-1}M, \left(z_1^0, \ldots, z_{n-1}^0\right)\right) \to \pi_2\left(F_{m,n}M, z^0\right) \to \underbrace{\pi_2\left(M - Q_m, z_1^0\right)}_{1}$$
$$\to \pi_1\left(F_{m+1,n-1}M, \left(z_1^0, \ldots, z_{n-1}^0\right)\right) \to \pi_1\left(F_{m,n}M, z^0\right) \to \pi_1\left(M - Q_m, z_1^0\right)$$
$$\to \underbrace{\pi_0\left(F_{m+1,n-1}M, \left(z_1^0, \ldots, z_{n-1}^0\right)\right)}_{1} \to \pi_0\left(F_{m,n}M, z^0\right) \to \underbrace{\pi_0\left(M - Q_m, z_1^0\right)}_{1}$$

By induction, we obtain that

$$1 \to \pi_1\left(F_{n-1,1}M, z_1^0\right) \xrightarrow{j_*} \pi_1\left(F_{0,n}M, \left(z_1^0, \ldots, z_n^0\right)\right) \xrightarrow{\pi_*} \pi_1\left(F_{0,n-1}M, \left(z_1^0, \ldots, z_{n-1}^0\right)\right) \to 1 \qquad (4.15)$$

□

Theorem 4.9 (Artin [1]) The group $\pi_1 B_{0,n} R^2$ admits a presentation with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and defining relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for all } i, j = 1, \ldots, n-1 \text{ with } |i-j| \geq 2 \qquad (4.16)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } i = 1, \ldots, n-2 \qquad (4.17)$$

Proof (Fadell and Van Buskirk [8]) We introduced B_n with generators and relations in chapter 2. Until we established the isomorphism between B_n and $\pi_1 B_{0,n} R^2$, we will use the symbols $\overline{\sigma}_1, \overline{\sigma}_2, \ldots, \overline{\sigma}_{n-1}$ for elements of $\pi_1 B_{0,n} R^2$ with $\iota : \mathrm{B}_n \to \pi_1 B_{0,n} R^2$ defined by $\iota(\sigma_i) = \overline{\sigma}_i$, $1 \leq i \leq n-1$. We now give a definition for $\overline{\sigma}_i$. Recall the covering projection $p : F_{0,n} R^2 \to B_{0,n} R^2$. Choose the point $p((1,0),(2,0),\ldots,(n,0)) = \overline{z^0}$ as base point for the group $\pi_1 B_{0,n} R^2$. Lift loops based at $p((1,0),(2,0),\ldots,(n,0))$ in $B_{0,n} R^2$ to paths in $F_{0,n} R^2$ with initial point $p((1,0),(2,0),\ldots,(n,0)) = \overline{z^0}$. Then the generator $\overline{\sigma}_i \in \pi_1 B_{0,n} R^2$ is represented by by the path $l(t)$ in $F_{0,n} R^2$ given by

$$l(t) = \left((1,0),(2,0),\ldots,(i-1,0), l_i(t), l_{i+1}(t), (i+2,0),\ldots,(n,0)\right) \qquad (4.18)$$

where $l_i(t)=\left(i+t,-\sqrt{t-t^2}\right)$ and $l_{i+1}(t)=\left(i+1-t,\sqrt{t-t^2}\right)$. That is, $l_i(t)$ is constant on all but the ith and $i+1$ st strings and interchanges those two in a nice way.

The proof of theorem 3.9 will be by induction on n, and will exploit the relationship between $\pi_1 B_{0,n}R^2$ and $\pi_1 F_{0,n}R^2$. Let

$$\bar{v}:\pi_1\left(B_{0,n}R^2,\overline{z^0}\right)\to\sum_n$$

be defined as follows: Let $\bar{\alpha}\in\pi_1 B_{0,n}R^2$ be reprensented by a loop

$$\bar{\alpha}:\left(I,\{0,1\}\right)\to\left(B_{0,n}R^2,\overline{z^0}\right)$$

and let $\alpha=\left(\alpha_1,\ldots,\alpha_n\right):\left(I,\{0\}\right)\to\left(F_{0,n}R^2,z^0\right)$ be the unique lift of $\bar{\alpha}$. Define

$$(\alpha)=\begin{pmatrix}\alpha_1(0),\ldots,\alpha_n(0)\\ \alpha_1(1),\ldots,\alpha_n(1)\end{pmatrix}\in\sum_n$$

The kernel of the homomorphism $\bar{v}$ is the pure braid group, $\pi_1 F_{0,n}R^2$. Corresponding to the homomorphism $\bar{v}$ is the homomorphism (we mentioned in chapter 2.)

$$v:\mathrm{B}_n\to\mathfrak{S}_n$$

from the n-braid group B_n to the symmetric group $\mathfrak{S}_n$ on n letters defined by:

$$v(\sigma_i)=(i,i+1)\qquad 1\le \mathrm{i}\le \mathrm{n}\text{-}1. \tag{4.19}$$

And we know that $P_n=\ker v$.

Lemma 4.10 The homomorphism $\imath:\mathrm{B}_n\to\pi_1 B_{0,n}R^2$ is an isomorphism onto if $\imath|_{P_n}$ is an isomorphism onto $\pi_1 F_{0,n}R^2$.

Proof of Lemma 4.10 The homomorphism v is clearly surjective, since the transpositions $\{v(\sigma_i)\ 1 \le \mathrm{i} \le \mathrm{n}\text{-}1\}$ generate $\mathfrak{S}_n$. Hence we have a commutative diagram;

$$\begin{array}{ccccccccc}
1 & \to & P_n & \to & \mathrm{B}_n & \xrightarrow{v} & \mathfrak{S}_n & \to & 1 \\
 & & \downarrow{\scriptstyle \iota_n = \iota|_{P_n}} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle 1} & & \\
1 & \to & \pi_1 F_{0,n} R^2 & \to & \pi_1 B_{0,n} R^2 & \xrightarrow{\bar{v}} & \sum_n & \to & 1
\end{array}$$

with exact rows. ι is an isomorphism with five lemma [5].

Now, we must show that $\iota|_{P_n}$ is an isomorphism onto $\pi_1 F_{0,n} R^2$. For this purpose, we introduced a representation for a subgroup $\mathbb{H}_n$ in B_n. With the help of the Reidemeister-Schreier method, $\mathbb{H}_n$ is exhibited as the semi-direct product of $\mathbb{A}_n$ (which is an invariant free group generated by free generators $A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}$ [10]) and B_{n-1}. And we presented a representation for P_n. Now, we refine P_n and find a relation between P_n and $\pi_1 F_{0,n} R^2$.

□

The group P_{n-1} can be regarded as the subgroup of P_n which is generated by $\{A_{i,j},\ 1 \le i < j \le n-1\}$. Note that a natural homomorphism $\eta : P_n \to P_{n-1}$ may be defined by the rule $\eta(A_{i,j}) = A_{i,j}$ if $1 \le i < j \le n-1$, while $\eta(A_{i,n}) = 1$, $1 \le i < n$. Thus $\ker \eta = \mathbb{A}_n$.

Corresponding to the homomorphism $\eta : P_n \to P_{n-1}$, we have the homomorphism $\pi_* : \pi_1 F_{0,n} R^2 \to \pi_1 F_{0,n-1} R^2$ of theorem 4.8. By theorem 4.8 we also know that $\ker \pi_* = \pi_1 F_{n-1,1} R^2 = \pi_1 \left(R^2 - Q_{n-1} \right)$, which is a free group of rank n-1.

It is easy to see that the following diagram is commutative:

$$
\begin{array}{ccccccc}
1 \to \mathbb{A}_n & \to & P_n & \xrightarrow{\eta} & P_{n-1} & \to 1 \\
\downarrow \iota_n|\mathbb{A}_n & & \downarrow \iota_n & & \downarrow \iota_{n-1} & \\
1 \to \pi_1 F_{n-1,1} R^2 & \to & \pi_1 F_{0,n} R^2 & \xrightarrow{\pi_*} & \pi_1 F_{0,n-1} R^2 & \to 1
\end{array}
$$

with exact rows. In the bottom row, the base point for $\pi_1 F_{0,n} R^2$ is $\left(z_1^0,\ldots,z_n^0\right)$, so that z_n^0 is the base point for $\pi_1 F_{n-1,1} R^2 = \pi_1\left(R^2 - z_1^0 \cup \ldots \cup z_{n-1}^0\right)$. Now, we may identify the image $\iota_n\left(A_{j,n}\right)$ of the generator $A_{j,n}$ of $\mathbb{A}_n$ as being represented by a loop based at z_n^0 which encircles the point z_j^0 once and separates it from $z_1^0,\ldots,z_{j-1}^0, z_{j+1}^0,\ldots,z_{n-1}^0$. Clearly the image set $\left\{\iota_n\left(A_{j,n}\right),\ 1 \le j < n\right\}$ is a free basis for the free group $\pi_1 F_{n-1,1} R^2$. And we know that $\mathbb{A}_n$ is a free group, hence $\iota_n|\mathbb{A}_n$ is an isomorphism onto. Now observe that $P_1 = 1$ and $\pi_1 F_{0,1} R^2 = 1$. Therefore ι_1 is an isomorphism. Assume inductively therefore that ι_{n-1} is an isomorphism. Then, since $\iota_n|\mathbb{A}_n$ is an isomorphism for each n , ι_n is an isomorphism by five lemma [5]. This completes the proof of Therorem 4.9.

□

CHAPTER 5

CONCLUSIONS

In this master thesis, we focused on two groups: Braid groups and the fundamental group of configuration space. Firstly, we introduced braids with their algebraic and geometric defitinitions. And we refined generators and relations of B_n and we defined an operation for braids. Therefore, we obtained a group presentation for B_n. After that, we must show that when are two n-braids equivalent? For this purpose, we used word problem with solution. We applied Reidemeister-Schreier method to obtain generators and relations for subgroups of braid groups. With the help of these relations and generators and some properties of free groups, we presented an invariant free subgroup $\mathbb{A}_n$ of B_n. In addition, we introduced pure braid P_n.

Secondly, we gave information about configuration spaces. And we defined the fundamental group of configuration spaces. Especially, we indicated relations between Artin braid groups and the fundamental group of configuration space on R^2. Consequently, we obtained an isomorphism between them.

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