REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

# HIGH ORDER STABLE DIFFERENCE SCHEMES FOR MULTIPOINT NONLOCAL BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS

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MSc. THESIS
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İSTANBUL, 2014

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A thesis submitted by Meltem UZUN in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE** is approved by the committee on 25.06.2014 in Department of Mathematics, Mathematics Program.

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## ACKNOWLEDGEMENTS

I would like to express my great appreciation to my supervisor Assist. Prof. Dr. Özgür YILDIRIM for his valuable informations and comments and also for his help, motivation, encouragement and patience during this research.

I would also like to thank to my family for their support and understanding during my education and all my life.

Haziran, 2014

Meltem UZUN

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## LIST OF SYMBOLS

| | |
|---|---|
| H | Hilbert Space |
| sup f(x) | supremum of f(x) |
| D(A) | Domain of A |
| $\|\ \|_{H \mapsto H}$ | norm of an operator defined on a Hilbert space |
| $\|\ \|_{H}$ | norm of a function defined on a Hilbert space |
| $R^m$ | m-dimensional Euclidean space |

## LIST OF ABBREVIATIONS

| | |
|---|---|
| BVP | Boundary Value Problem |
| NBVP | Nonlocal Boundary Value Problem |

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# ABSTRACT

## HIGH ORDER STABLE DIFFERENCE SCHEMES FOR MULTIPOINT NONLOCAL BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS

Meltem UZUN

Department of Mathematics
MSc. Thesis

Adviser: Assist. Prof. Dr. Özgür YILDIRIM

In this work the multipoint nonlocal boundary value problem for hyperbolic equation in a Hilbert space H with the self-adjoint positive definite operator A is considered. The third and fourth order of accuracy difference schemes generated by the integer power of A approximately solving this abstract nonlocal boundary value problem are presented. Stability estimates for the solution of these difference schemes are obtained. The theoretical statements for the solution of this difference schemes are supported by the numerical results using MATLAB.

**Keywords:** Multipoint nonlocal boundary value problem, Difference schemes, Convergence, Stability, Numerical analysis.

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# ÖZET

## ÇOK NOKTALI YEREL  OLMAYAN SINIR KOŞULLU HİPERBOLİK DENKLEMLER İÇİN YÜKSEK MERTEBELİ KARARLI FARK ŞEMALARI

Meltem UZUN

Matematik Anabilim Dalı

Yüksek Lisans Tezi

Danışman: Yrd. Doç. Dr. Özgür YILDIRIM

Bu çalışmada, H Hilbert uzayında pozitif tanımlı öz-eşlenik A operatörü ile tanımlanan çok noktalı yerel olmayan sınır değer problemleri ele alınmıştır. Bu tür problemlerin çözümü için A operatörünün tamsayı kuvvetleri ile tanımlanan üçüncü ve dördüncü mertebeden kararlılık fark şamaları oluşturulmuştur. Fark şemalarının kararlılık kestirimleri elde edilmiştir. Teorik sonuçlar MATLAB programı kullanılarak sayısal örneklerle desteklenmiştir .

**Anahtar Kelimeler**: Çok noktalı yerel olmayan sınır değer problemleri,  Fark şemaları,Yakınsaklık, Kararlılık ,Sayısal analiz.

# CHAPTER 1

# INTRODUCTION

## 1.1 Literature Review

Hyperbolic partial differential equations play an important role in modelling most problems arising in many branches of science and engineering and can be used to describe a wide variety of phenomena such as acoustics, electromagnetics, thermodynamics, elasticity, wave propagation, fluid mechanics and other areas of physics (see [1]- [6] and the references given therein).

While applying mathematical modelling, there often arise problems with nonclassical boundary conditions, which the values of unknown function on the boundary are connected with inside of the given domain. Such type of boundary conditions are called nonlocal boundary conditions. Over the last decades, boundary value problems with nonlocal boundary conditions have become a rapidly growing area of research and they were addressed by many authors. (see, *e.g.*, [17]- [37] )

In this section, some of the classical methods such as Fourier series method, Laplace transform method and Fourier transform method are presented for the solution of nonlocal hyperbolic differential equations.

Consider the nonlocal boundary value problem for hyperbolic equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = (t^2 - 2t + 3)\sin x, \ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \frac{1}{8}u(1,x) + \frac{1}{8}u(\frac{1}{2},x) + \frac{31}{32}\sin x, \ 0 \le x \le \pi, \\ u_t(0,x) = \frac{1}{8}u_t(1,x) + \frac{1}{8}u_t(\frac{1}{2},x) - \frac{15}{8}\sin x, \ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \ 0 \le t \le 1. \end{cases} \tag{1.1}$$

Let us solve problem (1.1), using the Fourier series method. Assume that the solution is

$$u(t,x) = v(t,x) + w(t,x). \tag{1.2}$$

Here $v(t,x)$ is the solution of problem

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = 0, \ 0 < t < 1, \ 0 < x < \pi, \\ v(0,x) - \frac{1}{8}v(1,x) - \frac{1}{8}v(\frac{1}{2},x) = \frac{31}{32}\sin x, \ 0 \le x \le \pi, \\ v_t(0,x) - \frac{1}{8}v_t(1,x) - \frac{1}{8}v_t(\frac{1}{2},x) = -\frac{15}{8}\sin x, \ 0 \le x \le \pi, \\ v(t,0) = v(t,\pi) = 0, \ 0 \le t \le 1, \end{cases}$$

(1.3)

and $w(t,x)$ is the solution of problem

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = (t^2 - 2t + 3)\sin x, 0 < t < 1, \ 0 < x < \pi, \\ w(0,x) = \frac{1}{8}w(1,x) + \frac{1}{8}w(\frac{1}{2},x), 0 \le x \le \pi, \\ w_t(0,x) = \frac{1}{8}w_t(1,x) + \frac{1}{8}w_t(\frac{1}{2},x), 0 \le x \le \pi, \\ w(t,0) = w(t,\pi) = 0, 0 \le t \le 1. \end{cases} \tag{1.4}$$

First, we solve problem (1.3). By the method of separation of variables we have

$$v(t,x) = T(t)X(x) \ne 0.$$

Substituting $v(t,x)$ into (1.3) we obtain

$$T''(t)X(x) - T(t)X''(x) = 0$$

or

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -k^2 = \lambda. \qquad (1.5)$$

The boundary conditions in (1.3), require $X(0) = X(\pi) = 0.$ Then, from (1.5) we have the second order ordinary differential equation

$$X''(x) = \lambda X(x), \; X(0) = X(\pi) = 0. \qquad (1.6)$$

For $\lambda = 0$ and $\lambda > 0$ the boundary value problem (1.6) has only trivial solution $X(x) = 0.$ For $\lambda < 0,$ we obtain the solutions of the boundary value problem (1.6) as

$$X_k(x) = \sin kx, \text{ where } k = 1, 2, \ldots, \; \lambda = -k^2.$$

The ordinary differential equation with respect to $t$ obtained from (1.5) is

$$T''(t) = \lambda T(t), \text{where } \lambda = -k^2, \; k = 1, 2, \ldots$$

and solution of this ordinary differential equation is

$$T_k(t) = c_{1k}\cos kt + c_{2k}\sin kt, \quad \text{where } k = 1, 2, \ldots$$

Thus, we obtained the solution as

$$v(t,x) = \sum_{k=1}^{\infty} v_k(t,x) = \sum_{k=1}^{\infty} (c_{1k}\cos kt + c_{2k}\sin kt)\sin kx. \qquad (1.7)$$

Using the nonlocal boundary condition

$$v(0,x) = \frac{1}{8}v(1,x) + \frac{1}{8}v(\frac{1}{2},x) + \frac{31}{32}\sin x, \; 0 \le x \le \pi,$$

we obtain the following equation

$$\sum_{k=1}^{\infty} c_{1k} \sin kx = \frac{1}{8} \sum_{k=1}^{\infty} (c_{1k} \cos k + c_{2k} \sin k) \sin kx$$

$$+\frac{1}{8} \sum \left( c_{1k} \cos \frac{k}{2} + c_{2k} \sin \frac{k}{2} \right) \sin kx + \frac{31}{32} \sin x$$

or

$$\sum_{k=1}^{\infty} \left[ c_{1k} - \frac{1}{8}(c_{1k} \cos k + c_{2k} \sin k) - \frac{1}{8}\left( c_{1k} \cos \frac{k}{2} + c_{2k} \sin \frac{k}{2} \right) \right] \times \sin kx = \frac{31}{32} \sin x.$$

Equating the coefficients of $\sin \quad kx,$ we get

$$c_{1k} - \frac{1}{4}(c_{1k} \cos k + c_{2k} \sin k) + \frac{1}{4}\left( c_{1k} \cos \frac{k}{2} + c_{2k} \sin \frac{k}{2} \right) = 0, \text{ for } k = 2, 3, \ldots$$

or

$$c_{11}\left(1 - \frac{1}{8}\left(\cos \frac{1}{2} + \cos 1\right)\right) - \frac{1}{8} c_{21} \left(\sin 1 + \sin \frac{1}{2}\right) = \frac{31}{32}. \tag{1.8}$$

Using the nonlocal boundary condition

$$v_t(0, x) = \frac{1}{8} v_t(1, x) + \frac{1}{8} v_t(\frac{1}{2}, x) - \frac{15}{8} \sin x, \ 0 \le x \le \pi,$$

we get the following equation

$$\sum_{k=1}^{\infty} \left[ kc_{2k} + \frac{1}{8}(kc_{1k} \sin k - kc_{2k} \cos k) + \frac{1}{8}\left( kc_{1k} \sin \frac{k}{2} - c_{2k} k \sin \frac{k}{2} \right) \right] \sin kx = -\frac{15}{8} \sin x.$$

Equating the coefficients of $\sin \quad kx$ ,we get

$$kc_{2k} + \frac{1}{8}(kc_{1k} \sin k - kc_{2k} \cos k) + \frac{1}{8}\left( kc_{1k} \sin \frac{k}{2} + kc_{2k} \sin \frac{k}{2} \right) = 0, \text{ for } k = 2, 3, \ldots,$$

or

$$\frac{1}{8}c_{11}\left(\sin\frac{1}{2}+\sin 1\right)+c_{21}\left(1-\frac{1}{8}\left(\cos\frac{1}{2}+\cos 1\right)\right)=-\frac{15}{8}. \tag{1.9}$$

So, we have

$$\begin{cases} c_{11}\left(1+\frac{1}{4}\left(\cos\frac{1}{2}-\cos 1\right)\right)+c_{21}\left(\frac{\sin\frac{1}{2}}{4}-\frac{\sin 1}{4}\right)=-\frac{1}{8}, \\ \\ c_{11}\left(\frac{\sin 1}{4}-\frac{\sin\frac{1}{2}}{4}\right)+c_{21}\left(1-\frac{\cos 1}{4}+\frac{\cos\frac{1}{2}}{4}\right)=1 \end{cases}$$

for $k=1$ and

$$\begin{cases} c_{1k}-\frac{1}{8}(c_{1k}\cos k+c_{2k}\sin k)-\frac{1}{8}\left(c_{1k}\cos\frac{k}{2}+c_{2k}\sin\frac{k}{2}\right)=0, \\ \\ kc_{2k}+\frac{1}{8}(kc_{1k}\sin k-kc_{2k}\cos k)+\frac{1}{8}(kc_{1k}\sin\frac{k}{2}-kc_{2k}\cos\frac{k}{2})=0 \end{cases}$$

for $k\geq 2$ . Using Crammer's rule we deduce that

$$\Delta=\begin{vmatrix} 1-\frac{1}{8}(\cos 1+\cos\frac{1}{2}) & -\frac{1}{8}(\sin 1+\sin\frac{1}{2} \\ \frac{1}{8}(\sin 1+\sin\frac{1}{2}) & 1-\frac{1}{8}(\cos 1+\cos\frac{1}{2}) \end{vmatrix}$$

$$=\left[1-\frac{1}{8}\left(\cos 1+\cos\frac{1}{2}\right)\right]^2+\left[\frac{1}{8}\left(\sin 1+\sin\frac{1}{2}\right)\right]^2\neq 0.$$

Thus, the coefficients are

$$c_{11}=\frac{\begin{vmatrix} \frac{31}{32} & -\frac{1}{8}(\sin 1+\sin\frac{1}{2}) \\ -\frac{15}{8} & 1-\frac{1}{8}(\cos 1+\cos\frac{1}{2}) \end{vmatrix}}{\Delta},$$

$$c_{21}=\frac{\begin{vmatrix} 1-\frac{1}{8}(\cos 1+\cos\frac{1}{2}) & \frac{31}{32} \\ \frac{1}{8}(\sin 1+\sin\frac{1}{2}) & -\frac{15}{8} \end{vmatrix}}{\Delta}.$$

Substituting $c_{11}$ and $c_{21}$ in (1.7), the following equation can be written as

$$v_1(t,x)=\left[\left(\frac{\frac{31}{32}-\frac{31}{256}(\cos 1+\cos\frac{1}{2})-\frac{15}{64}(\sin 1+\sin\frac{1}{2})}{\Delta}\right)\cos t\right.$$

$$\left.+\left(\frac{-\frac{15}{8}+\frac{15}{64}(\cos 1+\cos\frac{1}{2})-\frac{31}{256}(\sin 1+\sin\frac{1}{2})}{\Delta}\right)\sin t\right]\sin x.$$

For the $w(t,x)$ the solution is

$$w(t,x)=\sum_{k=1}^{\infty}A_k(t)\sin kx.$$

Putting these solution into (1.4), we get

$$\sum_{k=1}^{\infty}A_k''(t)\sin kx+\sum_{k=1}^{\infty}A_k(t)k^2\sin kx=(t^2-2t+3)\sin x+\sum_{k=2}^{\infty}(A_k''(t)+A_k(t)k^2)\sin kx.$$

The complementary solution of the equation

$$A_k''(t)+A_k(t)k^2=t^2-2t+3$$

is obtained by solving

$$A_k''(t)+A_k(t)k^2=0, A_k^c(t):\ m^2+k^2=0\Rightarrow m=\pm ik,$$

$$A_k^c(t):\ \sum_{k=1}^{\infty}(d_{1k}\cos kt+d_{2k}\sin kt)\quad A_p(t)=at^2+bt+c$$

$$2a+k^2(at^2+bt+c)=t^2-2t+3\Rightarrow ak^2=1\Rightarrow a=\frac{1}{k^2}\qquad,\qquad bk^2=-2\Rightarrow b=-\frac{2}{k^2}$$

$$2a+ck^2=3\Rightarrow c=0.$$

Thus,

$$w_k(t,x)=(d_{11}\cos t+d_{21}\sin t+t^2-2t+1)\sin x$$

$$+\sum_{k=2}^{\infty}\left(d_{1k}\cos kt+d_{2k}\sin kt+\frac{1}{k^2}t^2-\frac{2}{k^2}t+\frac{3k^2-2}{k^4}\right)\sin kx. \tag{1.10}$$

Using nonlocal boundary conditions for $k=1$ we have

$$(d_{11}+1)\sin x=\frac{1}{8}(d_{11}\cos 1+d_{21}\sin 1)\sin x+\frac{1}{8}(d_{11}\cos\frac{1}{2}+d_{21}\sin\frac{1}{2}+\frac{1}{4})\sin x,$$

$$(d_{21}\ -2\ )\sin x=\ \frac{1}{8}(-d_{11}\sin 1+d_{21}\cos 1+1)\ \sin x+\frac{1}{8}(-d_{11}\sin\frac{1}{2}+d_{21}\cos\frac{1}{2}-1)$$

and

$$d_{11}(1-\frac{1}{8}(\cos\frac{1}{2}+\cos 1))-\frac{1}{8}d_{21}(\sin\frac{1}{2}+\sin 1)=-\frac{31}{32}, \tag{1.11}$$

$$\frac{1}{8}d_{11}(\sin\frac{1}{2}+\sin 1)+d_{21}(1-\frac{1}{8}(\cos 1+\cos\frac{1}{2}))=\frac{15}{8} \tag{1.12}$$

for $k\geq 2$

$$d_{1k}(1-\frac{1}{8}(\cos\frac{k}{2}+\cos k))-\frac{1}{8}d_{2k}(\sin\frac{k}{2}+\sin k)=0$$

$$\frac{1}{8}d_{1k}(\sin\frac{k}{2}+\sin k)+d_{2k}(1-\frac{1}{8}(\cos k+\cos\frac{k}{2}))=0.$$

Using Crammer's rule we obtain

$$\Delta=\begin{vmatrix}1-\frac{1}{8}(\cos\frac{1}{2}+\cos 1) & -\frac{1}{8}(\cos 1+\cos\frac{1}{2}) \\ \frac{1}{8}(\sin\frac{1}{2}+\sin 1) & 1-\frac{1}{8}(\cos\frac{1}{2}+\cos 1)\end{vmatrix}$$

$$=\left[1-\frac{1}{8}\left(\cos\frac{1}{2}+\cos 1\right)\right]^2+\left[\frac{1}{8}\left(\sin\frac{1}{2}+\sin 1\right)\right]^2\neq 0.$$

From that we have

$$d_{11}=\frac{\begin{vmatrix}-\frac{31}{32} & -\frac{1}{8}(\sin\frac{1}{2}+\sin 1)\,\frac{15}{8} & 1-\frac{1}{8}(\cos\frac{1}{2}+\cos 1)\end{vmatrix}}{\Delta}$$
$$=\frac{-\frac{31}{32}+\frac{31}{256}(\cos\frac{1}{2}+\cos 1)+\frac{15}{64}(\sin\frac{1}{2}+\sin 1)}{\Delta}$$

and

$$d_{21}=\frac{\begin{vmatrix}1-\frac{1}{8}(\cos\frac{1}{2}+\cos 1) & -\frac{31}{32}\frac{15}{8}(\sin 1+\sin\frac{1}{2}) & \frac{15}{8}\end{vmatrix}}{\Delta}$$
$$=\frac{\frac{15}{8}(1-\frac{1}{8}(\cos\frac{1}{2}+\cos 1))+\frac{31}{256}(\sin 1+\sin\frac{1}{2})}{\Delta}.$$

Putting $d_{11}$ and $d_{21}$ in equation (1.10), $w(t,x)$ can be written as

$$w(t,x)=\left[\left(\frac{-\frac{31}{32}+\frac{31}{256}(\cos\frac{1}{2}+\cos 1)+\frac{15}{64}(\sin\frac{1}{2}+\sin 1)}{\Delta}\right)\cos t\right.$$

$$\left.+\left(\frac{\frac{15}{8}\left(1-\frac{1}{8}\left(\cos\frac{1}{2}+\cos 1\right)\right)+\frac{31}{256}\left(\sin 1+\sin\frac{1}{2}\right)}{\Delta}\right)\sin t+t^2-2t+1\right]\sin x.$$

Then, substituting $v(t,x)$ and $w(t,x)$ in (1.2), we obtain

$$u(t,x)=\left(\frac{\frac{31}{32}-\frac{31}{256}(\cos\frac{1}{2}+\cos 1)-\frac{15}{64}(\sin\frac{1}{2}+\sin 1)}{\Delta}\right.$$

$$\left.+\frac{-\frac{31}{32}+\frac{31}{256}(\cos\frac{1}{2}++\cos 1)+\frac{15}{64}(\sin\frac{1}{2}+\sin 1)}{\Delta}\right)\cos t$$

$$+\left(\frac{-\frac{15}{8}+\frac{15}{64}(\cos\frac{1}{2}+\cos 1)-\frac{31}{256}(\sin\frac{1}{2}+\sin 1)}{\Delta}\right.$$

$$\left.+\frac{\frac{15}{8}-\frac{15}{64}(\cos\frac{1}{2}+\cos 1)+\frac{31}{256}(\sin\frac{1}{2}+\sin 1)}{\Delta}\right)\sin t+(1-t)^2\sin x$$

$$=0+(1-t)^2\sin x. \tag{1.13}$$

Thus, $u(t,x)=(1-t)^2\sin x$ is the solution of the equation obtained by the Fourier series method.

Next, we will solve another hyperbolic equation by Laplace transformation method.

Consider the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = -3te^{-2x}, 0 < t < 1, 0 \le x < \infty, \\ u(0,x) = \frac{1}{8}u(1,x) - \frac{1}{8}u(\frac{1}{2},x) - \frac{1}{16}e^{-2x}, 0 < x < \infty, \\ u_t(0,x) = \frac{1}{8}u_t(1,x) - \frac{1}{8}u_t(\frac{1}{2},x) + e^{-2x}, 0 < x < \infty, \\ u_t(t,0) = t,\ u_x(t,0) = -2t,\ 0 \le t \le 1. \end{cases} \tag{1.14}$$

Applying the Laplace transform

$$\mathrm{L}\{u(t,x)\} = u(t,s)$$

to equation (1.14), we have

$$L\{u_{tt}\} - L\{u_{xx}\} + L\{u\} = -3tL\{e^{-x}\}.$$

From that it follows

$$u_{tt}(t,s) - s^2 L\{u\} + su(t,0) + u_x(t,0) + u(t,s) = \frac{-3t}{s+2},$$

$$u_{tt}(t,s) + (1-s^2)u(t,s) + st - 2t + u(t,s) = \frac{-3t}{s+2}$$

or

$$u_{tt}(t,s) + (1-s^2)u(t,s) = \frac{-3t}{s+2} + t(2-s) = \frac{1-s^2}{s+2}t. \tag{1.15}$$

Using the method of undetermined coefficients, we can write the solution as

$$u(t,s) = u_c(t,s) + u_p(t,s).$$

Here $u_c(t,s)$ is the complementary solution and $u_p(t,s)$ is the particular solution. First, let us obtain the complementary solution. The auxiliary equation is

$$u_c(t,s):\ m^2 = s^2 - 1$$

$$m^2 + \left(1 - s^2\right) = 0,$$

$$m_{1,2} = \pm\sqrt{s^2 - 1}.$$

Then the complementary solution is

$$u_c(t,s) = c_1 e^{\sqrt{s^2-1}t} + c_2 e^{-\sqrt{s^2-1}t}.$$

For the particular solution we assume that

$$u_p(t,s) = a(s)t + b(s)$$

and substituting $u_p(t,s)$ into equation (1.15), we get

$$0 + (1 - s^2)\left[a(s)t + b(s)\right] = \frac{1 - s^2}{s+2} t$$

$$a(s) = \frac{1}{s+2},\ b(s) = 0,$$

$$u_p(t,s) = \frac{t}{s+2}.$$

Thus we obtain $u(t,s)$ as

$$u(t,s) = c_1 e^{\sqrt{s^2-1}t} + c_2 e^{-\sqrt{s^2-1}t} + \frac{t}{s+2}.$$

Using the first nonlocal boundary condition

$$u(0,x) = \frac{1}{8}u(1,x) - \frac{1}{8}u(\frac{1}{2},x) - \frac{1}{16}e^{-2x}$$

we get

$$c_1\left[1 - \frac{1}{8}\left(e^{\sqrt{s^2-1}} - e^{\frac{\sqrt{s^2-1}}{2}}\right)\right] + c_2\left[1 - \frac{1}{8}\left(e^{-\sqrt{s^2-1}} - e^{\frac{-\sqrt{s^2-1}}{2}}\right)\right] = 0 \qquad (1.16)$$

and using the second nonlocal boundary condition

$$u_t(0,x)=\frac{1}{8}u_t(1,x)-\frac{1}{8}u_t(\frac{1}{2},x)+e^{-2x},$$

we get

$$c_1\left[1-\frac{1}{8}\left(e^{\sqrt{s^2-1}}-e^{\frac{\sqrt{s^2-1}}{2}}\right)\right]-c_2\left[1-\frac{1}{8}\left(e^{-\sqrt{s^2-1}}-e^{\frac{-\sqrt{s^2-1}}{2}}\right)\right]=0. \tag{1.17}$$

From (1.16) and (1.17) it follows that

$c_1=0$, $c_2=0$,

Thus $u(t,s)$ is

$$u(t,s)=\frac{t}{s+2}.$$

By applying the inverse Laplace operator

$$L^{-1}\{u(t,s)\}=tL^{-1}\left\{\frac{1}{s+2}\right\},$$

we get the solution $u(t,x)=te^{-2x}$ .

In the final step, let us solve a nonlocal boundary value problem for hyperbolic equation by Fourier Transform method.

Consider the following problem

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+u=\left[2+(3-4x^2)t^2\right]e^{-x^2} \quad ,0<t<1 \quad ,0\le x<\infty,\\ \\ u(0,x)=\frac{1}{8}u(1,x)-\frac{1}{8}u(\frac{1}{2},x)-\frac{3}{32}e^{-x^2} \quad ,0<x<\infty\\ \\ u_t(0,x)=\frac{1}{8}u_t(1,x)-\frac{1}{8}u_t(\frac{1}{2},x)-\frac{1}{8}e^{-x^2} \quad ,0<x<\infty.\end{cases}$$

First, applying the Fourier transform method, we get

$$F\{u_{tt}(t,x)\}-F\{u_{xx}(t,x)\}+F\{u(t,x)\}=F\left\{\left(2+(3-4x^2)t^2\right)e^{-x^2}\right\},$$

$$F\left\{u_{tt}(t,x)\right\}-F\left\{u_{xx}(t,x)\right\}+F\left\{u(t,x)\right\}=2F\left\{e^{-x^2}\right\}+F\left\{(3-4x^2)t^2e^{-x^2}\right\}.$$

From that it follows

$$u_{tt}(t,s)-(is)^2u(t,s)+u(t,s)=2F\left\{e^{-x^2}\right\}+F\left\{e^{-x^2}-(e^{-x^2})''\right\}t^2,$$

or

$$u_{tt}(t,s)+(s^2+1)u(t,s)=\left[(s^2+1)t^2+2\right]F\left\{e^{-x^2}\right\}. \tag{1.18}$$

Assume that the solution is

$$u(t,s)=u_c(t,s)+u_p(t,s). \tag{1.19}$$

Here $u_c(t,s)$ is the complementary solution and $u_p(t,s)$ is the particular solution. For the complementary solution, the auxiliary equation is

$$u_c(t,s):\ m^2+(s^2+1)=0,$$

$$m=\pm i\sqrt{(s^2+1)},$$

and the complementary solution is

$$u_c(t,s)=c_1\cos\sqrt{(s^2+1)}t+c_2\sin\sqrt{(s^2+1)}t\ .$$

Using the method of undetermined coefficients, we obtain the particular solution

$$u_p(t,s)=\left[a(s)t^2+b(s)t+c(s)\right]F\left\{e^{-x^2}\right\}.$$

Substituting this solution into (1.18), we have

$$2a(s)F\left\{e^{-x^2}\right\}+(s^2+1)\left[a(s)t^2+b(s)t+c(s)\right]F\left\{e^{-x^2}\right\}=2F\left\{e^{-x^2}\right\}+(s^2+1)F\left\{e^{-x^2}\right\}t^2$$

$$2a(s)+(s^2+1)a(s)t^2+(s^2+1)b(s)t+(s^2+1)c(s)=2+(s^2+1)t^2$$

$$a(s)=1\ ,\ b(s)=0\ ,\ c(s)=0,$$

$$u_p(t,s)=t^2F\left\{e^{-x^2}\right\}.$$

Thus, the general solution is

$$u_c(t,s)=c_1\cos\sqrt{(s^2+1)}t+c_2\sin\sqrt{(s^2+1)}t+t^2F\left\{e^{-x^2}\right\}.$$

Using the nonlocal boundary condition

$$u(0,x)=\frac{1}{8}u(1,x)-\frac{1}{8}u(\frac{1}{2},x)-\frac{3}{32}e^{-x^2}\ ,0<x<\infty\ ,$$

we get

$$c_1=\frac{1}{8}\left[c_1\cos\sqrt{(s^2+1)}+c_2\sin\sqrt{(s^2+1)}+F\left\{e^{-x^2}\right\}\right]$$

$$-\frac{1}{8}\left[c_1\cos\frac{\sqrt{(s^2+1)}}{2}+c_2\sin\frac{\sqrt{(s^2+1)}}{2}+\frac{1}{4}F\left\{e^{-x^2}\right\}\right]-\frac{3}{32}F\left\{e^{-x^2}\right\},$$

$$c_1=\frac{1}{8}\left[c_1\cos\sqrt{(s^2+1)}+c_2\sin\sqrt{(s^2+1)}\right]-\frac{1}{8}\left[c_1\cos\frac{\sqrt{(s^2+1)}}{2}+c_2\sin\frac{\sqrt{(s^2+1)}}{2}\right]. \qquad (1.20)$$

Using the other nonlocal boundary condition

$$u_t(0,x)=\frac{1}{8}u_t(1,x)-\frac{1}{8}u_t(\frac{1}{2},x)-\frac{1}{8}e^{-x^2}\ ,0<x<\infty,$$

we get

$$\sqrt{s^2+1}c_2=\frac{1}{8}\left[-c_1\sqrt{s^2+1}\sin\sqrt{s^2+1}+c_2\sqrt{s^2+1}\cos\sqrt{s^2+1}+2F\left\{e^{-x^2}\right\}\right]$$

$$-\frac{1}{8}\left[-c_1\sqrt{s^2+1}\sin\frac{\sqrt{s^2+1}}{2}+c_2\sqrt{s^2+1}\cos\frac{\sqrt{s^2+1}}{2}+F\left\{e^{-x^2}\right\}\right]-\frac{1}{8}F\left\{e^{-x^2}\right\}$$

$$c_2=\frac{1}{8}\left[-c_1\sin\sqrt{s^2+1}+c_2\cos\sqrt{s^2+1}\right]-\frac{1}{8}\left[-c_1\sin\frac{\sqrt{s^2+1}}{2}+c_2\cos\frac{\sqrt{s^2+1}}{2}\right]. \qquad (1.21)$$

Solving (1.20) and (1.21) for $c_1$ and $c_2$, we obtain $c_1=c_2=0.$ Thus the complementary solution is $u_c(t,s)=0$ and the particular solution is

$$u_p(t,s)=t^2\mathbf{F}\left\{e^{-x^2}\right\}.$$

Substituting $u_c(t,s)$ and $u_p(t,s)$ in (1.19) we get

$$u(t,s)=0+t^2\mathbf{F}\left\{e^{-x^2}\right\}.$$

Finally, applying the inverse Fourier transform

$$u(t,x)=\mathbf{F}^{-1}\left\{t^2\mathbf{F}\left\{e^{-x^2}\right\}\right\},$$

we obtain the solution as

$$u(t,x)=t^2e^{-x^2}.$$

## 1.2 Objective of the Thesis

Unfortunately, one can use these methods only when the hyperbolic partial differential equation has constant coefficients and it is well-known that the most useful method for solving partial differential equations with dependent coefficients in time variable $t$ and in the space variables is difference method.

In this work we obtain the stability estimates for solution of nonlocal boundary problem. In applications this result allows us to obtain the stability estimates for the solution of nonlocal boundary value problem for the hyperbolic equations. The third and fourth order of accuracy difference schemes generated by the integer power of $A$ approximately solving this abstract nonlocal boundary value problem are presented. Stability estimates for the solution of these difference schemes are obtained. The theoretical statements for the solution of this difference schemes are supported by the numerical results.

## 1.3 Hypothesis

Let us briefly describe the contents of the chapters of the present work. It consists of four chapters.

First chapter is the introduction.

Second chapter consists of three sections. In the first section we introduce the problem. Then we present stability estimates for the third order of accuracy difference schemes generated by the integer power of $A$ approximately solving multipoint nonlocal boundary value problem in the second section. In the third section, stability estimates for the fourth order of accuracy difference schemes generated by the integer power of $A$ approximately solving multipoint nonlocal boundary value problem are obtained.

Third chapter is devoted to the study of numerical results. The applications of third and fourth order of accuracy difference schemes for approximately solving multipoint nonlocal boundary value problem are studied and solved via MATLAB.

In the fourth chapter we conclude with our theoretical and numerical results.

**CHAPTER 2**

# DIFFERENCE SCHEMES OF MULTIPOINT NONLOCAL BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS

## 2.1 The Multipoint Nonlocal Boundary Value Problem of the Hyperbolic Type

In papers [24] and [25] the first and second order of accuracy difference schemes and in [21] the third order of accuracy difference scheme generated by the integer powers of $A$ for approximately solving the nonlocal boundary-value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) \ \ 0 \le t \le 1, \\ u(0) = \sum\limits_{r=1}^{n} \alpha_r u(\lambda_r) + \phi, \\ u_t(0) = \sum\limits_{r=1}^{n} \beta_r u_t(\lambda_r) + \psi, \\ 0 < \lambda_1 < \lambda_2 < ... < \lambda_k \le 1, \end{cases} \tag{2.1}$$

in a Hilbert space $H$ with self -adjoint positive definite operator $A$ were presented. The stability estimates for the solution of problem (2.1) and of first and second order difference schemes were established. In applications, the stability estimates for the solutions of the difference schemes of the mixed type nonlocal boundary-value problems for hyperbolic equations were obtained.

In the present work the third and fourth order of approximation two-step difference scheme for approximately solving nonlocal boundary value problem (2.1) are presented. In this paper for simplicity $\lambda_1 > 2\tau$ and $\lambda_k < 1$ will be considered.

The well-posedness of the nonlocal boundary value problems for parabolic, elliptic equations and equations of mixed types have been studied in [40]-[51].

## 2.2 The Third Order of Accuracy Difference Scheme Subject to Multipoint Nonlocal Conditions

In this section we consider the third order of accuracy difference scheme for the numerical solution of nonlocal boundary-value problem (2.1). We associate boundary value problem (2.1) with the corresponding third order of accuracy difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{2}{3}Au_k+\frac{1}{6}A\left(u_{k+1}+u_{k-1}\right) \\ +\frac{1}{12}\tau^2A^2u_{k+1}=f_k,\ f_k=\frac{2}{3}f(t_k)+\frac{1}{6}\left(f\left(t_{k+1}\right)+f\left(t_{k-1}\right)\right) \\ -\frac{1}{12}\tau^2\left(-Af\left(t_{k+1}\right)+f''(t_{k+1})\right), t_k=k\tau, 1\le k\le N-1, N\tau=1, \\ \left(I-i\tau A^2\right)u_0=\sum\limits_{k=1}^{n}\alpha_k\left\{u_{\left[\frac{\lambda_k}{\tau}\right]}+\tau^{-1}\left(u_{\left[\frac{\lambda_k}{\tau}\right]}-u_{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)\right. \\ +\frac{3}{2}\left(f_{\left[\frac{\lambda_k}{\tau}\right]}-Au_{\left[\frac{\lambda_k}{\tau}\right]}\right)\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^2 \\ \left.+\frac{7}{6}\left(f'_{\left[\frac{\lambda_k}{\tau}\right]}-\tau^{-1}A\left(u_{\left[\frac{\lambda_k}{\tau}\right]}-u_{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\right)\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^3\right\}+\phi, \\ \left(I+i\tau A^2\right)\tau^{-1}\left(u_1-u_0\right) \\ =\sum\limits_{k=1}^{n}\beta_k\left\{\tau^{-1}\left(u_{\left[\frac{\lambda_k}{\tau}\right]}-u_{\left[\frac{\lambda_k}{\tau}\right]-1}\right)+\left(f_{\left[\frac{\lambda_k}{\tau}\right]}-Au_{\left[\frac{\lambda k}{\tau}\right]}\right)\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)\right. \\ +\frac{1}{2!}\left(f'_{\left[\frac{\lambda_k}{\tau}\right]}-\tau^{-1}A\left(u_{\left[\frac{\lambda_k}{\tau}\right]}-u_{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\right)\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^2 \\ \left.+\frac{1}{3!}\left(f''_{\left[\frac{\lambda_k}{\tau}\right]}-Af_{\left[\frac{\lambda_k}{\tau}\right]}+A^2u_{\left[\frac{\lambda_k}{\tau}\right]}\right)\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^3\right\}+\psi+f_{1,1}\tau \end{cases} \tag{2.2}$$

where

$$f_{1,1}=\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\tau^{-1}f_{1,1}^3=\left\{f(0)+\left(-f(0)+\tau f'(0)\right)\frac{1}{2}-2f'(0)\frac{\tau}{6}\right\}.$$

We are interested in to study the stability of solutions of difference scheme (2.2) under the assumption

$$\sum_{k=1}^{n}|\alpha_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{7}{12}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4+\frac{7}{36}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6\right\}<1 \qquad (2.3)$$

Throughout this section for sake of simplicity we denote $B_n^\tau$ for

$$B_n^\tau=\left(I+\tau^2A^4\right)^{-1}\left\{1-\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left(\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\right.\right.$$

$$\times\left(-\frac{1}{\tau}\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)+\frac{7}{6\tau}A\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^3\right)+\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}-R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)$$

$$\times\left.\left(1+\frac{1}{\tau}\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)-\frac{3}{2}A\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^2-\frac{7}{6\tau}A\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^3\right)\right)$$

$$-\frac{1}{2}R_2\sum_{k=1}^{n}\beta_k\left(\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\right.$$

$$\times\left(\frac{1}{\tau}-A\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)-\frac{1}{2\tau}A\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^2+\frac{1}{6}A^2\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^3\right)$$

$$+\left.\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left(\frac{1}{\tau}-A\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)-\frac{1}{2\tau}A\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^2+\frac{1}{6}A^2\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^3\right)\right)$$

$$+\frac{1}{4}R_2\sum_{k=1}^{n}\sum_{k=1}^{n}\alpha_k\beta_k\left(-\frac{1}{\tau}+A\tau\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^2\right.$$

$$\left.+\frac{7}{12}A^2\tau^3\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^4-\frac{7}{36}A^3\tau^5\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^6\right)$$

$$\times\left(\tilde{R}_1-R_1\right)\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}(R-I)-R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}(\tilde{R}-I)\right].$$

We consider the following operators

$$R=\left(I-\frac{1}{3}\tau^2 A+i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4 A^2}\right)\left(I+\frac{1}{6}\tau^2 A+\frac{1}{12}\tau^4 A^2\right)^{-1}$$

and its conjugate $\tilde{R}$

$$\begin{aligned}R_1 &=\left(-\tfrac{5\tau^4}{144}A^2+\tfrac{7\tau^6}{216}A^3-i\tau A^{1/2}\left(I+\tfrac{\tau^2}{12}A+\tfrac{\tau^4}{144}A^2\right)\sqrt{I+\tfrac{1}{72}\tau^4 A^2}\right)\\ &\times\left(-i\tau A^{1/2}\left(\sqrt{I+\tfrac{1}{72}\tau^4 A^2}\right)\left(I+\tfrac{\tau^2}{12}A+\tfrac{\tau^4}{144}A^2\right)\right)^{-1},\end{aligned}$$

and its conjugate $\tilde{R}_1$,

$$\begin{aligned}R_2 &=\left(I-\tfrac{\tau^2}{12}A\right)\left(I+\tfrac{\tau^2}{6}A+\tfrac{\tau^4}{12}A^2\right)\\ &\times\left(-iA^{1/2}\left(I+\tfrac{\tau^2}{12}A+\tfrac{\tau^4}{144}A^2\right)\sqrt{I+\tfrac{1}{72}\tau^4 A^2}\right)^{-1},\end{aligned}$$

$$R_3=\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\left(\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\left(-i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4 A^2}\right)\right)^{-1},$$

$$\begin{aligned}R_4 &=\left(I+\tfrac{\tau^2}{3}A+\tfrac{\tau^4}{9}A^2+\tfrac{\tau^6}{72}A^3\right)\\ &\times\left(-iA^{1/2}\left(\sqrt{I+\tfrac{1}{72}\tau^4 A^2}\right)\left(I+\tfrac{\tau^2}{6}A+\tfrac{\tau^4}{12}A^2\right)\left(I+\tfrac{\tau^2}{6}A\right)\right)^{-1},\end{aligned}$$

$$R_5=\left(-\frac{\tau^2}{2}A-\frac{\tau^4}{12}A^2+i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4 A^2}\right)\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)^{-1},$$

and its conjugate $\tilde{R}_5$,

$$\begin{aligned}R_6 &=\left(I-\tfrac{1}{3}\tau^2 A+i\tau A^{1/2}\sqrt{I+\tfrac{1}{72}\tau^4 A^2}\right)\\ &\times\left(\tfrac{\tau^2}{2}A+\tfrac{\tau^4}{12}A^2-i\tau A^{1/2}\sqrt{I+\tfrac{1}{72}\tau^4 A^2}\right)^{-1},\end{aligned}$$

and its conjugate $\tilde{R}_6$,

$$R_7 = \left(-2i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)^{-1}.$$

We will use the estimates of the following lemma presented in [21].

**Lemma 2.1** The following estimates hold:

$$\begin{cases} \|R\|_{H\mapsto H} \le 1, \|\tilde{R}\|_{H\mapsto H} \le 1, \\ \|R_1\|_{H\mapsto H} \le 1, \|\tilde{R}_1\|_{H\mapsto H} \le 1, \\ \|A^{1/2}R_2\|_{H\mapsto H} \le 1, \|\tau A^{1/2}R_3\|_{H\mapsto H} \le 1, \\ \|A^{1/2}R_4\|_{H\mapsto H} \le 1, \|A^{-1/2}R_5\|_{H\mapsto H} \le \tau, \\ \|A^{-1/2}R_5\|_{H\mapsto H} \le \tau, \|\tau A^{1/2}R_6\|_{H\mapsto H} \le 1, \\ \|\tau A^{1/2}R_6\|_{H\mapsto H} \le 1, \|A^{-1/2}R_7\|_{H\mapsto H} \le \tau \end{cases} \tag{2.4}$$

**Proof.** Using spectral property of the self-adjoint positive definite operators, we can write

$$\left\|\left(I-\frac{1}{3}\tau^2A\pm i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\left(I+\frac{1}{6}\tau^2A+\frac{1}{12}\tau^4A^2\right)^{-1}\right\|_{H\mapsto H}$$

$$\le \sup_{\delta\le\lambda<\infty}\left|\frac{1-\frac{1}{3}\tau^2\lambda+i\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}}{1+\frac{1}{6}\tau^2\lambda+\frac{1}{12}\tau^4\lambda^2}\right|.$$

Since

$$\frac{\left(1-\frac{1}{3}\tau^2\lambda\right)^2+\left(\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}{\left(1+\frac{1}{6}\tau^2\lambda+\frac{1}{12}\tau^4\lambda^2\right)^2} = \frac{1+\frac{1}{3}\tau^2\lambda+\frac{1}{9}\tau^4\lambda^2+\frac{1}{72}\tau^6\lambda^3}{1+\frac{1}{3}\tau^2\lambda+\frac{7}{36}\tau^4\lambda^2+\frac{1}{36}\tau^6\lambda^3+\frac{1}{144}\tau^8\lambda^4} \le 1,$$

we have that

$$\|R\|_{H\to H} \le 1, \|\tilde{R}\|_{H\to H} \le 1.$$

Using the spectral property of the self-adjoint positive definite operators, we obtain

$$\left\| \left( -\frac{5\tau^4}{144}A^2 + \frac{7\tau^6}{216}A^3 \pm i\tau A^{1/2}\left( I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2 \right)\sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \right.$$

$$\left. \times \left( -i\tau A^{1/2}\left( \sqrt{I + \frac{1}{72}\tau^4 A^2} \right)\left( I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2 \right) \right)^{-1} \right\|_{H\mapsto H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left| \frac{\left( -\frac{5\tau^4}{144}\lambda^2 + \frac{7\tau^6}{216}\lambda^3 \pm i\tau\lambda^{1/2}\left( 1 + \frac{\tau^2}{12}\lambda + \frac{\tau^4}{144}\lambda^2 \right)\sqrt{1 + \frac{1}{72}\tau^4\lambda^2} \right)}{i\tau\lambda^{1/2}\left( \sqrt{1 + \frac{1}{72}\tau^4\lambda^2} \right)\left( 1 + \frac{\tau^2}{12}\lambda + \frac{\tau^4}{144}\lambda^2 \right)} \right|.$$

Since

$$\frac{\left( -\frac{5\tau^4}{144}\lambda^2 + \frac{7\tau^6}{216}\lambda^3 \right)^2 + \left( \tau\lambda^{1/2}\left( 1 + \frac{\tau^2}{12}\lambda + \frac{\tau^4}{144}\lambda^2 \right)\sqrt{1 + \frac{1}{72}\tau^4\lambda^2} \right)^2}{\left( \tau\lambda^{1/2}\left( \sqrt{1 + \frac{1}{72}\tau^4\lambda^2} \right)\left( 1 + \frac{\tau^2}{12}\lambda + \frac{\tau^4}{144}\lambda^2 \right) \right)^2} \leq 1,$$

we have that

$\|R_1\|_{H\mapsto H} \leq 1$ and $\|\tilde{R}_1\|_{H\mapsto H} \leq 1$.

Similarly,

$$\left\| A^{1/2}\left( I - \frac{\tau^2}{12}A \right)\left( I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2 \right) \times \left( -iA^{1/2}\left( I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2 \right)\sqrt{I + \frac{1}{72}\tau^4 A^2} \right)^{-1} \right\|_{H\to H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left| \frac{\lambda^{1/2}\left( 1 + \frac{\tau^2}{12}\lambda - \frac{5\tau^4}{72}\lambda^2 - \frac{\tau^6}{144}\lambda^3 \right)}{i\lambda^{1/2}\left( 1 + \frac{\tau^2}{12}\lambda + \frac{\tau^4}{144}\lambda^2 \right)\sqrt{1 + \frac{1}{72}\tau^4\lambda^2}} \right|.$$

Since

$$\frac{\lambda\left( 1 + \frac{\tau^2}{12}\lambda - \frac{5\tau^4}{72}\lambda^2 - \frac{\tau^6}{144}\lambda^3 \right)^2}{\left( \lambda^{1/2}\left( 1 + \frac{\tau^2}{12}\lambda + \frac{\tau^4}{144}\lambda^2 \right)\sqrt{1 + \frac{1}{72}\tau^4\lambda^2} \right)^2} \leq 1,$$

we have

$$\left\|A^{1/2}R_2\right\|_{H\mapsto H}\le 1.$$

In the same manner, we can write

$$\left\|\tau A^{1/2}\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\left(i\tau A^{1/2}\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\sqrt{I+\frac{1}{72}\tau^4A^2}\right)^{-1}\right\|_{H\to H}$$

$$\le \sup_{\delta\le\lambda<\infty}\left|\frac{\tau\lambda^{1/2}\left(1+\frac{\tau^2}{6}\lambda+\frac{\tau^4}{12}\lambda^2\right)}{i\tau\lambda^{1/2}\left(1+\frac{\tau^2}{12}\lambda+\frac{\tau^4}{144}\lambda^2\right)\sqrt{1+\frac{1}{72}\tau^4\lambda^2}}\right|.$$

Since

$$\frac{\tau^2\lambda\left(1+\frac{\tau^2}{6}\lambda+\frac{\tau^4}{12}\lambda^2\right)^2}{\left(\tau\lambda^{1/2}\left(1+\frac{\tau^2}{12}\lambda+\frac{\tau^4}{144}\lambda^2\right)\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}\le 1,$$

we have that

$$\left\|\tau A^{1/2}R_3\right\|_{H\mapsto H}\le 1.$$

Using the similar way, we obtain

$$\left\|\left(I+\frac{\tau^2}{3}A+\frac{\tau^4}{9}A^2+\frac{\tau^6}{72}A^3\right)\times\left(-iA^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\left(I+\frac{\tau^2}{6}A\right)\right)^{-1}\right\|_{H\mapsto H}$$

$$\le \sup_{\delta\le\lambda<\infty}\left|\frac{1+\frac{\tau^2}{3}\lambda+\frac{\tau^4}{9}\lambda^2+\frac{\tau^6}{72}\lambda^3}{i\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\left(1+\frac{\tau^2}{6}\lambda+\frac{\tau^4}{12}\lambda^2\right)\left(1+\frac{\tau^2}{6}\lambda\right)}\right|.$$

Since

$$\frac{\left(1+\frac{\tau^2}{3}\lambda+\frac{\tau^4}{9}\lambda^2+\frac{\tau^6}{72}\lambda^3\right)^2}{\left(\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\left(1+\frac{\tau^2}{6}\lambda+\frac{\tau^4}{12}\lambda^2\right)\left(1+\frac{\tau^2}{6}\lambda\right)\right)^2}\le 1,$$

we have

$$\left\|A^{1/2}R_4\right\|_{H\mapsto H}\leq 1.$$

In same manner, one can obtain

$$\left\|\left(-\frac{\tau^2}{2}A-\frac{\tau^4}{12}A^2\pm i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\left(A^{1/2}\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\right)^{-1}\right\|_{H\mapsto H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{-\frac{\tau^2}{2}\lambda-\frac{\tau^4}{12}\lambda^2\pm i\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}}{\lambda^{1/2}\left(1+\frac{1}{6}\tau^2\lambda+\frac{1}{12}\tau^4\lambda^2\right)}\right|.$$

Since

$$\frac{\left(-\frac{\tau^2}{2}\lambda-\frac{\tau^4}{12}\lambda^2\right)^2+\left(\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}{\lambda\left(1+\frac{1}{6}\tau^2\lambda+\frac{1}{12}\tau^4\lambda^2\right)^2}=\frac{\tau^2\lambda+\frac{\tau^4}{4}\lambda^2+\frac{7\tau^6}{72}\lambda^3+\frac{\tau^8}{144}\lambda^4}{\lambda+\frac{1}{3}\tau^2\lambda^2+\frac{7}{36}\tau^4\lambda^3+\frac{\tau^6}{36}\lambda^4+\frac{\tau^8}{144}\lambda^5}\leq\tau^2,$$

we have that

$$\left\|A^{-1/2}R_5\right\|_{H\mapsto H}\leq\tau \text{ and } \left\|A^{-1/2}\tilde{R}_5\right\|_{H\mapsto H}\leq\tau.$$

In a similar manner, we establish

$$\left\|\tau A^{1/2}\left(I-\frac{1}{3}\tau^2A+i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\times\left(\frac{\tau^2}{2}A+\frac{\tau^4}{12}A^2-i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)^{-1}\right\|_{H\mapsto H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{\tau\lambda^{1/2}\left(1-\frac{1}{3}\tau^2\lambda+i\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)}{\frac{\tau^2}{2}\lambda+\frac{\tau^4}{12}\lambda^2-i\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}}\right|.$$

Since

$$\frac{\tau^2\lambda\left(1-\frac{1}{3}\tau^2\lambda\right)^2+\tau^2\lambda\left(\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}{\left(\frac{\tau^2}{2}\lambda+\frac{\tau^4}{12}\lambda^2\right)^2+\left(\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}=\frac{\lambda\left(1+\frac{1}{3}\tau^2\lambda+\frac{1}{9}\tau^4\lambda^2+\frac{1}{72}\tau^6\lambda^3\right)}{\lambda+\frac{1}{4}\tau^2\lambda^2+\frac{7}{72}\tau^4\lambda^3+\frac{1}{144}\tau^6\lambda^4}$$

$$= \frac{1 + \frac{1}{3}\tau^2\lambda + \frac{1}{9}\tau^4\lambda^2 + \frac{1}{72}\tau^6\lambda^3}{1 + \frac{1}{4}\tau^2\lambda + \frac{7}{72}\tau^4\lambda^2 + \frac{1}{144}\tau^6\lambda^3} \le 1,$$

we have that

$$\left\| \tau A^{1/2} R_6 \right\|_{H \to H} \le 1 \text{ and } \left\| \tau A^{1/2} \tilde{R}_6 \right\|_{H \to H} \le 1.$$

Similarly, we have

$$\left\| \left( -2i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \left( A^{1/2} \left( I + \frac{\tau^2}{6} A + \frac{\tau^4}{12} A^2 \right) \right)^{-1} \right\|_{H \mapsto H}$$
$$\le \sup_{\delta \le \lambda < \infty} \left| \frac{-2i\tau\lambda^{1/2} \sqrt{1 + \frac{1}{72}\tau^4\lambda^2}}{\lambda^{1/2} \left( 1 + \frac{1}{6}\tau^2\lambda + \frac{1}{12}\tau^4\lambda^2 \right)} \right|.$$

Since

$$\frac{\left( 2\tau\lambda^{1/2} \sqrt{1 + \frac{1}{72}\tau^4\lambda^2} \right)^2}{\lambda \left( 1 + \frac{1}{6}\tau^2\lambda + \frac{1}{12}\tau^4\lambda^2 \right)^2} = \frac{4\tau\lambda + \frac{\tau^5}{18}\lambda^3}{\lambda + \frac{1}{3}\tau^2\lambda^2 + \frac{7}{36}\tau^4\lambda^3 + \frac{\tau^6}{36}\lambda^4 + \frac{\tau^8}{144}\lambda^5} \le \tau,$$

we have that

$$\left\| A^{-1/2} R_7 \right\|_{H \mapsto H} \le \tau \ .$$

Lemma 2.1 is proved.

**Lemma 2.2** Suppose that assumption (2.3) holds. Then the operator $I - B_n^\tau$ has an inverse $T_\tau = \left( I - B_n^\tau \right)^{-1}$. From symmetry and positivity properties of the operator $A$ the following estimate holds:

$$\left\| T_\tau \right\|_{H \mapsto H} \le M \tag{2.5}$$

**Proof.** Using the definitions of $B_n^\tau, R, \tilde{R},$ estimates in Lemma 2.1 and the triangle inequality, we get

$$B_n^\tau \le \frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\{\left(\left\|\tilde{R}_1 R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right.\right.$$

$$+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left(\left\|\tilde{R}_1 R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\tau^2A\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right)+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\tau^2A\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\Bigg)$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left\|\tau^2A\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\left(\left\|\tilde{R}_1 R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right)$$

$$\left.+\frac{3}{2}\left\|\tau^2A\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\left(\left\|\tilde{R}_1 R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right\}$$

$$-\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\|A^{\frac{1}{2}}R_2\right\|_{H\to H}\left\{\left\|\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\tau^{-1}\tilde{R}_5\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\tau^{-1}R_5\right\|_{H\to H}\right)$$

$$+\frac{1}{2}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\tau A^{\frac{1}{2}}\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right)\Bigg)$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\tau A^{\frac{1}{2}}\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|A^{\frac{1}{2}}\tau\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$\left.+\frac{1}{6}\left\|\tau^3A^{\frac{3}{2}}\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$-\frac{1}{4}\sum_{k=1}^{n}\sum_{k=1}^{n}|\alpha_k\beta_k|\left\|A^{\frac{1}{2}}R_2\right\|_{H\to H}\left\{\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(\tilde{R}_1-R_1\right)R_5\tau^{-1}A^{-\frac{1}{2}}\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right.\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(\tilde{R}_1-R_1\right)\tilde{R}_5\tau^{-1}A^{-\frac{1}{2}}\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\Bigg)$$

$$+\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(\tilde{R}_1-R_1\right)R_5\tau^{-1}A^{\frac{1}{2}}\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(\tilde{R}_1-R_1\right)\tilde{R}_5\tau A^{\frac{1}{2}}\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\frac{7}{12}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(\tilde{R}_1-R_1\right)R_5A^{\frac{3}{2}}\tau^3\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(\tilde{R}_1-R_1\right)\tilde{R}_5\tau^3A^{\frac{3}{2}}\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4$$

$$+\frac{7}{36}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(\tilde{R}_1-R_1\right)R_5\tau^5A^{\frac{5}{2}}\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(\tilde{R}_1-R_1\right)\tilde{R}_5\tau^5A^{\frac{5}{2}}\left(I+\tau^2A^4\right)^{-1}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6\right\}\le q,$$

where

$$q=\left\{\sum_{k=1}^{n}\left|\alpha_k\right|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}\right.$$

$$+\sum_{k=1}^{n}\left|\beta_k\right|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$\left.+\frac{1}{2}\sum_{k=1}^{n}\left|\alpha_k\right|\sum_{k=1}^{n}\left|\beta_k\right|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{7}{12}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4+\frac{7}{36}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6\right\}\right\}$$

Since $q<1,$ the operator $I-B_n^{\tau}$ has a bounded inverse, i.e.,

$$\left\|\left(I-B_n^{\tau}\right)^{-1}\right\|_{H\to H} \le \frac{1}{1-q}.$$

Lemma 2.2 is proved.

Now, using the following formula for the solution of problem (2.2) which is obtained in [22],

$$u_0=\mu, u_1=\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}\times\left(\left(I-\frac{5}{12}\tau^2A+\frac{\tau^4}{144}A^2\right)\mu+\tau\left(I-\frac{\tau^2}{12}A\right)\omega+\tau^2 f_{1,1}\right),$$

$$u_k=\frac{1}{2}\left[R_1R^k-R_1\tilde{R}^k\right]\mu+\frac{1}{2}\left[\tilde{R}^k-R^k\right]R_2\omega$$
$$+\frac{1}{2}\left[\tilde{R}^k-R^k\right]R_3\tau^2 f_{1,1}+\frac{1}{2}R_4\sum_{s=1}^{k-1}\left[\tilde{R}^{k-s}-R^{k-s}\right]f_s\tau^2. \tag{2.6}$$

and applying formula (2.6) and the nonlocal boundary conditions in (2.2), we obtain

$$\mu=T_\tau\left(I+i\tau^2A^4\right)^{-1}\times\left\{\left(I-\frac{1}{2}R_2\sum_{k=1}^{n}\beta_k\left\{\tau^{-1}\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\right.\right.\right.$$

$$+\frac{1}{2}\tau A\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2-A\tau\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$\left.\left.+\frac{1}{6}\tau^3A^2\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}\right)$$

$$+\left(\sum_{k=1}^{n}\frac{1}{2}\alpha_kR_3\tau^2 f_{1,1}\left\{\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$-\frac{7}{6}A\tau^2\left(R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]$$

$$\left.-\frac{3}{2}\tau^2A\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right\}$$

$$+\frac{1}{2}R_4\sum_{k=1}^{n}\alpha_k\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}f_s\tau^2\left\{\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$\left.-\frac{7}{6}A\tau^2\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\frac{1}{2}R_4\sum_{k=1}^{n}\alpha_k\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}f_s\tau^2\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}+R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\left(1-\frac{3}{2}A\tau^2\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)$$

$$+\frac{1}{2}R_4\sum_{k=1}^{n}\alpha_kR_7\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|-\frac{7}{6}A\tau^2\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau^2+\sum_{k=1}^{n}\alpha_k\left\{\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2f_{\left[\frac{\lambda_k}{\tau}\right]}\tau^2\right.$$

$$\left.\left.+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3f'_{\left[\frac{\lambda_k}{\tau}\right]}\tau^3\right\}+\phi\right)-\left(-\frac{1}{2}R_2\sum_{k=1}^{n}\alpha_k\left\{\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$-\frac{7}{6}\tau^2A\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)$$

$$\left.\left.-\frac{3}{2}A\tau^2\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right|\right\}\right)$$

$$\times\left(R_3\sum_{k=1}^{n}\frac{1}{2}\beta_kf_{1,1}\tau^2\left\{\tau^{-1}\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)-\frac{1}{2}\tau A\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.\right.$$

$$\left.-A\tau\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{6}A^2\tau^3\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}R_4\beta_k\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5\right)\left(\frac{1}{\tau}-\frac{1}{2}A\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)f_s\tau^2$$

$$+\frac{1}{2}\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}R_4\beta_k\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}+R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\times\left(-A\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{6}A\tau^2\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)f_s\tau^2$$

$$+\frac{1}{2}\sum_{k=1}^{n}R_4\beta_k R_7\left(\frac{1}{\tau}-\frac{1}{2}A\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau^2+\sum_{k=1}^{n}\beta_k\left\{-\frac{1}{6}A\tau^3\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3 f_{\left[\frac{\lambda_k}{\tau}\right]}\right.$$

$$\left.+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right| f_{\left[\frac{\lambda_k}{\tau}\right]}\tau+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2 f'_{\left[\frac{\lambda_k}{\tau}\right]}\tau^2+\frac{1}{6}\tau^3\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3 f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\}+\psi\Bigg)\Bigg\} \tag{2.7}$$

and

$$\omega=T_\tau\left(I+i\tau^2A^4\right)^{-1}\times\left\{\left(\left(1-\frac{1}{2}\sum_{k=1}^{n}a_k\left\{\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.\right.\right.$$

$$-\frac{7}{6}A\tau^2\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$\left.\left.\left.+\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left(1-\frac{3}{2}A\tau^2\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)\right\}\right)\right)$$

$$\times\left(R_3\sum_{k=1}^{n}\frac{1}{2}\beta_k f_{1,1}\tau^2\left\{\tau^{-1}\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)-\frac{1}{2}\tau A\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.\right.$$

$$\left.-A\tau\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{6}A^2\tau^3\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}R_4\beta_k f_s\tau^2\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5\right)\times\left(\frac{1}{\tau}-\frac{1}{2}A\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)$$

$$+\frac{1}{2}\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}R_4\beta_k\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}+R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\times\left(-A\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{6}A\tau^2\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)f_s\tau^2$$

$$+\frac{1}{2}\sum_{k=1}^{n}R_4\beta_k R_7\left(\frac{1}{\tau}-\frac{1}{2}A\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau^2+\sum_{k=1}^{n}\beta_k\left\{-\frac{1}{6}A\tau^3\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3 f_{\left[\frac{\lambda_k}{\tau}\right]}\right)$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right| f_{\left[\frac{\lambda_k}{\tau}\right]}\tau+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2 f'_{\left[\frac{\lambda_k}{\tau}\right]}\tau^2+\frac{1}{6}\tau^3\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3 f''_{\left[\frac{\lambda_k}{\tau}\right]}\Bigg\}+\psi\Bigg)\Bigg\}\Bigg)$$

$$-\Bigg(-\frac{1}{2}\sum_{k=1}^{n}\beta_k\Bigg\{\frac{1}{\tau}\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)+\frac{1}{2}A\tau\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$-A\tau\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{6}A^2\tau^3\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg\}\Bigg)$$

$$\times\Bigg(\sum_{k=1}^{n}\frac{1}{2}\alpha_kR_3\tau^2f_{1,1}\Bigg\{\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}A\tau^2\left(R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]$$

$$-\frac{3}{2}\tau^2A\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg\}$$

$$+\frac{1}{2}R_4\sum_{k=1}^{n}\alpha_k\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}f_s\tau^2\Bigg\{\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}A\tau^2\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg\}$$

$$+\frac{1}{2}R_4\sum_{k=1}^{n}\alpha_k\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}f_s\tau^2\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}+R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\left(1-\frac{3}{2}A\tau^2\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)$$

$$\frac{1}{2}R_4\sum_{k=1}^{n}\alpha_kR_7\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|-\frac{7}{6}A\tau^2\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau^2+\sum_{k=1}^{n}\alpha_k\Bigg\{\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2 f_{\left[\frac{\lambda_k}{\tau}\right]}\tau^2$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3 f'_{\left[\frac{\lambda_k}{\tau}\right]}\tau^3\Bigg\}+\phi\Bigg)\Bigg\}. \tag{2.8}$$

Thus, formulas (2.6), (2.7) and (2.8) give a solution of problem (2.2).

**Theorem 2.1** Suppose that assumptions (2.3), (2.4) hold and $\phi \in D(A^{3/2}),\ \psi \in D(A^{\frac{1}{2}})$ Then, for the solution of difference scheme (2.2) the following stability estimates hold:

$$\max_{1\le k\le N}\|u_k\|_H \le M\left\{\sum_{s=1}^{k-1}\left\|A^{-1/2}f_s\right\|_H\tau+\|\phi\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\}, \tag{2.9}$$

$$\max_{1\le k\le N}\left\|A^{1/2}u_k\right\|_H \le M\left\{\sum_{s=1}^{k-1}\|f_s\|_H\tau+\left\|A^{1/2}\phi\right\|_H+\|\psi\|_H+\tau\left\|f_{1,1}\right\|_H\right\}, \tag{2.10}$$

$$\max_{1\le k\le N}\|Au_k\|_H \le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\phi\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}, \tag{2.11}$$

where $M$ does not depend on $\tau$, $\phi$, $\psi$, $f_{1,1}$ and $f_s$, $1\le s\le N-1$.

**Proof.** Using formulas (2.7), (2.8) and estimates (2.4), (2.5), we get

$$\|\mu\|_H \le \|T_\tau\|_{H\to H}\times\Bigg\{\Bigg(\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}+\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\|A^{\frac{1}{2}}R_2\right\|_{H\to H}\Bigg(\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}$$

$$\times\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\tau^{-1}\tilde{R}_5\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\tau^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{2}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|A^{\frac{1}{2}}\tau\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|\tau^3A^{\frac{3}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg]\Bigg)\Bigg)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}|\alpha_k|\left\|\tau A^{\frac{1}{2}}R_3\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{1,1}\right\|_H\tau\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right)\right.\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$+\frac{7}{6}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}+\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$\left.+\frac{3}{2}\left\|\tau^2A\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right\}$$

$$+\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\times\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}\right.\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\left]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$+\frac{7}{6}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$\times\left\{\left\|\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\right.$$

$$\left.+\frac{3}{2}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\times\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\|A^{-\frac{1}{2}}R_7\right\|_{H\to H}\times\left(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$\left.+\frac{7}{6}\left\|\tau^3A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau$$

$$+\sum_{k=1}^{n}|\alpha_k|\left\{\frac{3}{2}\left\|A^{\frac{1}{2}}\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$\left.+\frac{7}{6}\left\|\tau^2A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|A^{-\frac{1}{2}}f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\|\phi\|_H\Bigg)$$

$$+\left(\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\|A^{\frac{1}{2}}R_2\right\|_{H\to H}\left\{\left(\left(\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|\tilde{R}_5\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left(\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I+i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I+i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$+\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$\left.+\frac{3}{2}\left\|\tau^2A\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right|\Bigg)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}|\beta_k|\left\|A^{\frac{1}{2}}R_3\tau\right\|_{H\to H}\tau\left\|A^{-\frac{1}{2}}f_{1,1}\right\|_H\times\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]+\frac{1}{2}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|\tau^3A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}|\beta_k|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\Bigg\{\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]+\frac{1}{2}\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg\}\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$+\sum_{k=1}^{n}\frac{1}{2}|\beta_k|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\Bigg(\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}R_7\right\|_{H\to H}$$

$$+\frac{1}{2}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}R_7\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau+\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}|\beta_k|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}$$

$$\times\Bigg\{\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|A^{\frac{1}{2}}\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg\}\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$+\sum_{k=1}^{n}|\beta_k|\Bigg[\frac{1}{6}\left\|A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$+\left\|\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H}\tau+\frac{1}{2}\left\|\tau\left(I-i\tau A^{2}\right)^{-1}\right\|\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\left\|A^{-\frac{1}{2}}f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H}\tau$$

$$+\frac{1}{6}\left\|\tau^{2}\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{3}\left\|A^{-\frac{1}{2}}f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H}\tau\Bigg]+\left\|\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_{H}\Bigg)\Bigg\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\left\|A^{-1/2}f_s\right\|_{H}\tau+\|\phi\|_{H}+\left\|A^{-1/2}\psi\right\|_{H}+\tau\left\|A^{-1/2}f_{1,1}\right\|_{H}\right\}, \tag{2.12}$$

$$\left\|A^{-\frac{1}{2}}\omega\right\|_{H}\leq\|T_\tau\|_{H\to H}\times\Bigg\{\Bigg(\left\|\left(I+i\tau A^{2}\right)^{-1}\right\|_{H\to H}+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|$$

$$\Bigg(\Bigg[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I+i\tau A^{2}\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\left(I+i\tau A^{2}\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\Bigg[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\tau^{2}A\left(I+i\tau A^{2}\right)^{-1}\right\|_{H\to H}\Bigg)+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\tau^{2}A\left(I+i\tau A^{2}\right)^{-1}\right\|_{H\to H}\Bigg]$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{3}+\left\|\left(I+i\tau A^{2}\right)^{-1}\right\|_{H\to H}\Bigg[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\Bigg]$$

$$+\frac{3}{2}\left\|\tau^{2}A\left(I+i\tau A^{2}\right)^{-1}\right\|_{H\to H}\Bigg[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\Bigg)\Bigg)$$

$$\times\Bigg(\sum_{k=1}^{n}\frac{1}{2}|\beta_k|\left\|A^{\frac{1}{2}}R_3\tau\right\|_{H\to H}\tau\left\|A^{-\frac{1}{2}}f_{1,1}\right\|_{H}\times\Bigg\{\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\Bigg]+\frac{1}{2}\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|A^{\frac{1}{2}}\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|A^{\frac{3}{2}}\tau^3\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg\}$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\sum_{k=1}^{n}\frac{1}{2}|\beta_k|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\Bigg\{\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\Bigg]\Bigg\}\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$+\sum_{k=1}^{n}\frac{1}{2}|\beta_k|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\Bigg\{\left\|\tau^{-1}A^{-\frac{1}{2}}R_7\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}$$

$$+\frac{1}{2}\left\|A\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}R_7\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg\}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau$$

$$+\sum_{k=1}^{n}|\beta_k|\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\Bigg\{\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]$$

$$+\frac{1}{6}\left\|A^{\frac{1}{2}}\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\Bigg\}\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$+\sum_{k=1}^{n}|\beta_k|\left\{\frac{1}{6}\left\|A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$\left.+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right)$$

$$+\frac{1}{2}\left\|\tau\left(I-i\tau A^2\right)^{-1}\right\|\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|A^{-\frac{1}{2}}f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$\left.+\frac{1}{6}\left\|\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|A^{-\frac{1}{2}}f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_H\Bigg)$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\tilde{R}_1\tilde{R}_5\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}R_1R_5\right\|_{H\to H}$$

$$\left.+\frac{1}{2}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\tilde{R}_5\tilde{R}_1\right\|_{H\to H}\right)\right)$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}R_5R_1\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$\left.+\frac{1}{6}\left\|\tau^3A^{\frac{3}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}|\alpha_k|\left\|\tau A^{\frac{1}{2}}R_3\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{1,1}\right\|_H\tau\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right)$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5 A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}+\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5 A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left\|\left(I-i\tau A^2\right)^{-1}\right\|\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{3}{2}\left\|\tau^2 A\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg\}$$

$$+\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$\times\left\{\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right)\right.$$

$$\left.+\frac{3}{2}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\times\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\|A^{-\frac{1}{2}}R_7\right\|_{H\to H}\times\left(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$\left.+\frac{7}{6}\left\|A^{\frac{3}{2}}\tau^3\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau$$

$$+\sum_{k=1}^{n}\left|\alpha_k\right|\left\{\frac{3}{2}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$\left.+\frac{7}{6}\left\|\tau^2 A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|A^{-\frac{1}{2}}f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|\phi\right\|_H\Bigg)\Bigg\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|\phi\right\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\}. \tag{2.13}$$

Applying $A^{\frac{1}{2}}$ to formulas (2.7) and (2.8), in a similar way, we get

$$\left\|A^{1/2}\mu\right\|_H\leq\left\|T_\tau\right\|_{H\to H}\times\left\{\left(\left(\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}+\frac{1}{2}\sum_{k=1}^{n}\left|\beta_k\right|\left\|A^{\frac{1}{2}}R_2\right\|_{H\to H}\left(\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right.\right.$$

$$\times\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\tau^{-1}\tilde{R}_5\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\tau^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{2}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|A^{\frac{1}{2}}\tau\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$\left.\left.\left.+\frac{1}{6}\left\|\tau^3 A^{\frac{3}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right]\right)\right)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}\left|\alpha_k\right|\left\|\tau A^{\frac{1}{2}}R_3\right\|_{H\to H}\left\|f_{1,1}\right\|_H\tau\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right)$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5 A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}+\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5 A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{3}{2}\left\|\tau^2 A\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg\}$$

$$+\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\|f_s\|_H\,\tau\,\Bigg\{\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg\}$$

$$+\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\|f_s\|_H\,\tau$$

$$\times\Bigg\{\left\|\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]+\frac{3}{2}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}$$

$$\times\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg\}+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\|A^{-\frac{1}{2}}R_7\right\|_{H\to H}$$

$$\times\left(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{7}{6}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau$$

$$+\sum_{k=1}^{n}\left|\alpha_k\right|\left\{\frac{3}{2}\left\|A^{\frac{1}{2}}\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$\left.+\frac{7}{6}\left\|\tau^2A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\phi\right\|_H\Bigg)$$

$$+\left(\frac{1}{2}\sum_{k=1}^{n}\left|\alpha_k\right|\left\|A^{\frac{1}{2}}R_2\right\|_{H\to H}\left\{\left(\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|\tilde{R}_5\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left(\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I+i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I+i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$+\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$\left.+\frac{3}{2}\left\|\tau^2A\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right|\Bigg)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}\left|\beta_k\right|\left\|A^{\frac{1}{2}}R_3\tau\right\|_{H\to H}\tau\left\|f_{1,1}\right\|_H\times\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]+\frac{1}{2}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|\tau^{3}A^{\frac{3}{2}}\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{3}\Bigg\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left|\beta_k\right|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\Bigg\{\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\Bigg]+\frac{1}{2}\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}R_5\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\Bigg\}\|f_s\|_H\,\tau$$

$$+\sum_{k=1}^{n}\frac{1}{2}\left|\beta_k\right|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\Bigg(\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I-i\tau A^{2}\right)^{-1}R_7\right\|_{H\to H}$$

$$+\frac{1}{2}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^{2}\right)^{-1}R_7\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\Bigg)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau+\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left|\beta_k\right|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}$$

$$\times\Bigg\{\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|A^{\frac{1}{2}}\tau^{2}\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{3}\Bigg\}\|f_s\|_H\,\tau$$

$$+\sum_{k=1}^{n}\left|\beta_k\right|\Bigg[\frac{1}{6}\left\|A\tau^{2}\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{3}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$+\left\|\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau+\frac{1}{2}\left\|\tau\left(I-i\tau A^{2}\right)^{-1}\right\|\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$+\frac{1}{6}\left\|\tau^{2}\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{3}\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\Bigg]+\left\|\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\|\psi\|_H\Bigg)\Bigg\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\|f_s\|_H\,\tau+\left\|A^{1/2}\phi\right\|_H+\|\psi\|_H+\tau\,\|f_{1,1}\|_H\right\}, \tag{2.14}$$

$$\|\omega\|_H\leq\|T_\tau\|_{H\to H}\times\left\{\left(\left(\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left(\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right.\right.$$

$$\left.+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\tau^2A\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right)+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\tau^2A\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right]$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$\left.\left.+\frac{3}{2}\left\|\tau^2A\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)\right)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}|\beta_k|\left\|A^{\frac{1}{2}}R_3\tau\right\|_{H\to H}\tau\,\|f_{1,1}\|_H\;\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]+\frac{1}{2}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|A^{\frac{1}{2}}\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$\left.+\frac{1}{6}\left\|A^{\frac{3}{2}}\tau^3\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\sum_{k=1}^{n}\frac{1}{2}|\beta_k|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\right]\right\}\|f_s\|_H\,\tau$$

$$+\sum_{k=1}^{n}\frac{1}{2}|\beta_k|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\{\left\|\tau^{-1}A^{-\frac{1}{2}}R_7\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.+\frac{1}{2}\left\|A\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}R_7\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right\}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\,\tau$$

$$+\sum_{k=1}^{n}|\beta_k|\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\{\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\right.$$

$$\left.+\frac{1}{6}\left\|A^{\frac{1}{2}}\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\right\}\|f_s\|_H\,\tau$$

$$+\sum_{k=1}^{n}|\beta_k|\left\{\frac{1}{6}\left\|A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\,\tau\right.$$

$$+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\,\tau\Big)+\frac{1}{2}\left\|\tau\left(I-i\tau A^2\right)^{-1}\right\|\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\,\tau$$

$$\left.+\frac{1}{6}\left\|\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\,\tau\right\}+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\|\psi\|_H\Big)$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\tilde{R}_1\tilde{R}_5\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}R_1R_5\right\|_{H\to H}$$

$$+\frac{1}{2}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\tilde{R}_5\tilde{R}_1\right\|_{H\to H}\right)\Bigg)$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}R_5R_1\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|\tau^3A^{\frac{3}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}|\alpha_k|\left\|\tau A^{\frac{1}{2}}R_3\right\|_{H\to H}\left\|f_{1,1}\right\|_H\,\tau\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right)\right.\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}+\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left\|\left(I-i\tau A^2\right)^{-1}\right\|\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{3}{2}\left\|\tau^2A\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg\}$$

$$+\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\|f_s\|_H\,\tau\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\sum_{k=1}^{n}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\|f_s\|_H\,\tau$$

$$\times\left\{\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right)\right.$$

$$\left.+\frac{3}{2}\left\|A\tau^3\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\times\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\|A^{-\frac{1}{2}}R_7\right\|_{H\to H}\times\left(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$\left.+\frac{7}{6}\left\|A^{\frac{3}{2}}\tau^3\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau$$

$$+\sum_{k=1}^{n}|\alpha_k|\left\{\frac{3}{2}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$\left.+\frac{7}{6}\left\|\tau^2A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\phi\right\|_H\Bigg)\Bigg\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\|f_s\|_H\tau+\left\|A^{1/2}\phi\right\|_H+\|\psi\|_H+\tau\|f_{1,1}\|_H\right\}. \tag{2.15}$$

Then, applying Abel's formula to (2.7) and (2.8), we can obtain the following formulas

$$\mu=T_\tau\left(I+i\tau^2A^4\right)^{-1}\times\Bigg\{\Bigg(\left(I-\frac{1}{2}R_2\sum_{k=1}^{n}\beta_k\left\{\tau^{-1}\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\right.\right.$$

$$+\frac{1}{2}\tau A\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2-A\tau\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$\left.\left.+\frac{1}{6}\tau^3A^2\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}\right)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}\alpha_kR_3\tau^2f_{1,1}\left\{\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$-\frac{7}{6}A\tau^2\left(R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]$$

$$\left.-\frac{3}{2}\tau^2A\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right\}$$

$$+\frac{1}{2}R_4\tau^2\sum_{k=1}^{n}\alpha_k\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left(\left(R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$\left.-\frac{7}{6}A\tau^2\left(R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left(f_s-f_{s-1}\right)$$

$$+\left(\left(\tilde{R}_5R_6-R_5R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|-\frac{7}{6}A\tau^2\left(\tilde{R}_5R_6-R_5R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)f_{\left[\frac{\lambda_k}{\tau}\right]-2}$$

$$-\left(\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$+\frac{7}{6}A\tau^2\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg)f_1\Bigg]$$

$$+\frac{1}{2}R_4\tau^2\sum_{k=1}^{n}\alpha_k\Bigg[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg(\Bigg(R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\Bigg)$$

$$-\frac{3}{2}A\tau^2\left(R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)(f_s-f_{s-1})+\Big(\big(R_6-R_6\big)$$

$$-\frac{3}{2}A\tau^2\left(R_6-R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)f_{\left[\frac{\lambda_k}{\tau}\right]-1}-\Bigg(\Bigg(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg)$$

$$+\frac{3}{2}A\tau^2\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)f_1\Bigg]$$

$$+\frac{1}{2}R_4\sum_{k=1}^{n}\alpha_kR_7\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|-\frac{7}{6}A\tau^2\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau^2$$

$$+\sum_{k=1}^{n}\alpha_k\left\{\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2f_{\left[\frac{\lambda_k}{\tau}\right]}\tau^2+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3f'_{\left[\frac{\lambda_k}{\tau}\right]}\tau^3\right\}+\phi\Bigg)$$

$$-\Bigg(-\frac{1}{2}R_2\sum_{k=1}^{n}\alpha_k\Bigg\{\Bigg(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$-\frac{7}{6}\tau^2A\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)$$

$$-\frac{3}{2}A\tau^2\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg|\Bigg\}\Bigg)$$

$$\times\Bigg(R_3\sum_{k=1}^{n}\frac{1}{2}\beta_kf_{1,1}\tau^2\Bigg\{\tau^{-1}\Bigg(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\Bigg)-\frac{1}{2}\tau A\Bigg(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$-A\tau\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{6}A^2\tau^3\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg\}$$

$$+\frac{1}{2}R_4\tau^2\sum_{k=1}^{n}\beta_k\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left(\frac{1}{\tau}\left(R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)\right.\right.$$

$$\left.-\frac{1}{2}A\tau\left(R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)\left(f_s-f_{s-1}\right)+\left(\frac{1}{\tau}\left(\tilde{R}_5R_6-R_5R_6\right)\right.$$

$$\left.-\frac{1}{2}A\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left(\tilde{R}_5R_6-R_5R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)f_{\left[\frac{\lambda_k}{\tau}\right]-2}-\left(\frac{1}{\tau}\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\right.$$

$$\left.\left.-\frac{1}{2}A\tau\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)f_1\right]$$

$$+\frac{1}{2}R_4\tau^2\sum_{k=1}^{n}\beta_k\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(-A\tau\left(R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$\left.+\frac{1}{6}A\tau^2\left(R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left(f_s-f_{s-1}\right)+\left(-A\tau\left(R_6-R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$\left.+\frac{1}{6}A\tau^2\left(R_6-R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}-\left(-A\tau\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$\left.\left.+\frac{1}{6}A\tau^2\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)f_1\right]$$

$$+\frac{1}{2}\sum_{k=1}^{n}R_4\beta_kR_7\left(\frac{1}{\tau}-\frac{1}{2}A\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau^2+\sum_{k=1}^{n}\beta_k\left\{-\frac{1}{6}A\tau^3\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3f_{\left[\frac{\lambda_k}{\tau}\right]}\right.$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right| f_{\left[\frac{\lambda_k}{\tau}\right]}\tau+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2 f'_{\left[\frac{\lambda_k}{\tau}\right]}\tau^2+\frac{1}{6}\tau^3\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3 f''_{\left[\frac{\lambda_k}{\tau}\right]}\Bigg\}+\psi\Bigg)\Bigg\} \tag{2.16}$$

and

$$\omega=T_\tau\left(I+i\tau^2A^4\right)^{-1}\Bigg\{\Bigg(\Bigg(1-\frac{1}{2}\sum_{k=1}^{n}a_k\Bigg\{\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$-\frac{7}{6}A\tau^2\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$+\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left(1-\frac{3}{2}A\tau^2\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)\Bigg\}\Bigg)$$

$$\times\Bigg(R_3\sum_{k=1}^{n}\frac{1}{2}\beta_k f_{1,1}\tau^2\Bigg\{\tau^{-1}\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)-\frac{1}{2}\tau A\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$-A\tau\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{6}A^2\tau^3\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg\}$$

$$+\frac{1}{2}R_4\tau^2\sum_{k=1}^{n}\beta_k\Bigg[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg(\frac{1}{\tau}\left(R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)$$

$$-\frac{1}{2}A\tau\left(R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)\left(f_s-f_{s-1}\right)+\Bigg(\frac{1}{\tau}\left(\tilde{R}_5R_6-R_5R_6\right)$$

$$-\frac{1}{2}A\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left(\tilde{R}_5R_6-R_5R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)f_{\left[\frac{\lambda_k}{\tau}\right]-2}-\Bigg(\frac{1}{\tau}\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)$$

$$-\frac{1}{2}A\tau\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)f_1\Bigg]$$

$$+\frac{1}{2}R_4\tau^2\sum_{k=1}^{n}\beta_k\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(-A\tau\left(R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$\left.+\frac{1}{6}A\tau^2\left(R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left(f_s-f_{s-1}\right)$$

$$+\left(-A\tau\left(R_6-R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{6}A\tau^2\left(R_6-R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}$$

$$-\left(-A\tau\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$
$$\left.\left.+\frac{1}{6}A\tau^2\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)f_1\right]$$

$$+\frac{1}{2}\sum_{k=1}^{n}R_4\beta_kR_7\left(\frac{1}{\tau}-\frac{1}{2}A\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau^2+\sum_{k=1}^{n}\beta_k\left\{-\frac{1}{6}A\tau^3\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3f_{\left[\frac{\lambda_k}{\tau}\right]}\right)$$

$$\left.\left.\left.+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|f_{\left[\frac{\lambda_k}{\tau}\right]}\tau+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2f'_{\left[\frac{\lambda_k}{\tau}\right]}\tau^2+\frac{1}{6}\tau^3\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\}+\psi\right)\right\}\right)$$

$$-\left(-\frac{1}{2}\sum_{k=1}^{n}\beta_k\left\{\frac{1}{\tau}\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)+\frac{1}{2}A\tau\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.\right.$$

$$\left.\left.-A\tau\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{6}A^2\tau^3\left(\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}+R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}\right)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}\alpha_kR_3\tau^2f_{1,1}\left\{\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$+\frac{7}{6}A\tau^2\left(R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]$$

$$-\frac{3}{2}\tau^2 A\left[\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg\}$$

$$+\frac{1}{2}R_4\tau^2\sum_{k=1}^{n}\alpha_k\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left(\left(R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$\left.-\frac{7}{6}A\tau^2\left(R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left(f_s-f_{s-1}\right)$$

$$+\left(\left(\tilde{R}_5R_6-R_5R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|-\frac{7}{6}A\tau^2\left(\tilde{R}_5R_6-R_5R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)f_{\left[\frac{\lambda_k}{\tau}\right]-2}$$

$$-\left(\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$\left.\left.-\frac{7}{6}A\tau^2\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)f_1\right]$$

$$+\frac{1}{2}R_4\tau^2\sum_{k=1}^{n}\alpha_k\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(\left(R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\right.\right.$$

$$\left.-\frac{3}{2}A\tau^2\left(R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)\left(f_s-f_{s-1}\right)+\left(\left(R_6-R_6\right)\right.$$

$$\left.-\frac{3}{2}A\tau^2\left(R_6-R_6\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}-\left(\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\right.$$

$$\left.\left.-\frac{3}{2}A\tau^2\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)f_1\right]$$

$$+\frac{1}{2}R_4\sum_{k=1}^{n}\alpha_kR_7\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|-\frac{7}{6}A\tau^2\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau^2$$

$$+\sum_{k=1}^{n}\alpha_k\left\{\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2 f_{\left[\frac{\lambda_k}{\tau}\right]}\tau^2+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3 f'_{\left[\frac{\lambda_k}{\tau}\right]}\tau^3\right\}+\phi\Bigg)\Bigg\}. \tag{2.17}$$

Now we obtain the estimates for $\|A\mu\|_H$ and $\|A^{1/2}\omega\|_H$ . First, applying $A$ to formula (2.16), using estimates (2.4), (2.5) and the triangle inequality, we get

$$\|A\mu\|_H\le\|T_\tau\|_{H\to H}\Bigg\{\Bigg(\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}+\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\|A^{\frac{1}{2}}R_2\right\|_{H\to H}\Bigg(\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}$$

$$\times\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\tau^{-1}\tilde{R}_5\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\tau^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{2}\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}$$
$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|A^{\frac{1}{2}}\tau\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|\tau^3A^{\frac{3}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg]\Bigg)\Bigg)$$

$$\times\Bigg(\sum_{k=1}^{n}\frac{1}{2}|\alpha_k|\left\|\tau A^{\frac{1}{2}}R_3\right\|_{H\to H}\left\|A^{\frac{1}{2}}f_{1,1}\right\|_H\tau\Bigg\{\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg)$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}+\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{3}{2}\left\|\tau^2 A\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\left|\alpha_k\right|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\times\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left(\left(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}R_5R_6\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)$$

$$\times\left\|f_s-f_{s-1}\right\|_H+\left(\left(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}+\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}R_5R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$+\frac{7}{6}\left(\left(\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\right.\right.$$

$$\left.\left.\left.+\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}R_5R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\right\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left(\left(\left(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}R_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left(\left(\left(\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right.\right.\right.$$

$$+\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_5 R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg)\|f_1\|_H\Bigg]$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\|\tau A^{\frac{1}{2}} R_4\right\|_{H\to H}\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(\left(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right)+\frac{3}{2}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)\|f_s-f_{s-1}\|_H$$

$$+\left(\left(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}+\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right)\right.$$

$$+\frac{3}{2}\left(\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}+\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right)$$

$$\left.\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H+\left(\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right)+\frac{3}{2}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right.$$

$$\left.\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right)\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)\|f_1\|_H\Bigg]$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\|A^{\frac{1}{2}} R_4\right\|_{H\to H}\left\|A^{-\frac{1}{2}} R_7\right\|_{H\to H}\times\left(\left\|\tau A\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$\left.+\frac{7}{6}\left\|\tau^3 A^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau$$

$$+\sum_{k=1}^{n}\left|\alpha_k\right|\left\{\frac{3}{2}\left\|A\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$\left.\left.+\frac{7}{6}\left\|\tau^2 A\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|A\phi\right\|_H\right)$$

$$+\left(\frac{1}{2}\sum_{k=1}^{n}\left|\alpha_k\right|\left\|A^{\frac{1}{2}}R_2\right\|_{H\to H}\left\{\left(\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|\tilde{R}_5\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left(\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I+i\tau A^2\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I+i\tau A^2\right)^{-1}\tilde{R}_5\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$+\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$\left.\left.+\frac{3}{2}\left\|\tau^2A\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right|\right)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}\left|\beta_k\right|\left\|A^{\frac{1}{2}}R_3\tau\right\|_{H\to H}\tau\left\|A^{\frac{1}{2}}f_{1,1}\right\|_H\times\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]+\frac{1}{2}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|\tau^{3}A^{\frac{3}{2}}\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{3}\Bigg\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\left|\beta_k\right|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\times\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left(\left(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\right)+\frac{1}{2}\left(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\right)$$

$$\times\left\|f_s-f_{s-1}\right\|_H+\left(\left(\left\|\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}+\left\|\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}\right)\right.$$

$$\left.+\frac{1}{2}\left(\left(\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}+\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\right)\right)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left(\left(\left\|\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}+\left\|\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right)\right.$$

$$+\frac{1}{2}\left(\left(\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right.\right.$$

$$\left.\left.\left.+\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\right)\left\|f_1\right\|_H\right]$$

$$+\frac{1}{2}\sum_{k=1}^{n}\left|\beta_k\right|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\times\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(\left(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^{2}\right)^{-1}R_6\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^{2}\right)^{-1}R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3 A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3 A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg)\left\|f_s-f_{s-1}\right\|_H$$

$$+\Bigg(\left(\left\|\tau^2 A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}+\left\|\tau^2 A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left(\left\|\tau^3 A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}+\left\|\tau^3 A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right)$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H+\Bigg(\Bigg(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right)\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg)\left\|f_1\right\|_H\Bigg]$$

$$+\sum_{k=1}^{n}\frac{1}{2}\left|\beta_k\right|\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\left(\left\|\tau A R_7\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.+\frac{1}{2}\left\|A^2\tau^3 R_7\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H$$

$$+\sum_{k=1}^{n}\left|\beta_k\right|\left[\frac{1}{6}\left\|A^{\frac{3}{2}}\tau^3\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\right.$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H+\frac{1}{2}\left\|\tau^2 A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H$$

$$+\frac{1}{6}\left\|\tau^3 A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\Bigg]+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H\Bigg)\Bigg\}$$

$$\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\phi\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}. \tag{2.18}$$

Next, applying $A^{1/2}$ to formula (2.17) and using estimates (2.4), (2.5) and the triangle inequality, we get

$$\left\|A^{1/2}\omega\right\|_H\leq\|T_\tau\|_{H\to H}\Bigg\{\Bigg(\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|$$

$$\Bigg(\Bigg[\left\|\tilde{R}_1 R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\Bigg[\left\|\tilde{R}_1 R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\tau^2 A\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg)+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\tau^2 A\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left\|\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1 R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{3}{2}\left\|\tau^2 A\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1 R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)\Bigg)$$

$$\times\Bigg(\sum_{k=1}^{n}\frac{1}{2}|\beta_k|\left\|A^{\frac{1}{2}}R_3\tau\right\|_{H\to H}\tau\left\|A^{\frac{1}{2}}f_{1,1}\right\|_H\times\Bigg\{\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]+\frac{1}{2}\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\Bigg]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|A^{\frac{1}{2}}\tau\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|A^{\frac{3}{2}}\tau^{3}\left(I-i\tau A^{2}\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{3}\Bigg\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\left|\beta_k\right|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\times\Bigg[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg(\Bigg(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\Bigg)+\frac{1}{2}\Bigg(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\Bigg)$$

$$\times\left\|f_s-f_{s-1}\right\|_H+\Bigg(\Bigg(\left\|\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}+\left\|\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}\Bigg)$$

$$+\frac{1}{2}\Bigg(\Bigg(\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}+\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\Bigg)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\Bigg(\Bigg(\left\|\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}+\left\|\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\Bigg)$$

$$+\frac{1}{2}\Bigg(\Bigg(\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}$$

$$+\left\|\tau^{2}A\left(I-i\tau A^{2}\right)^{-1}R_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\Bigg)\left\|f_1\right\|_H\Bigg]$$

$$+\frac{1}{2}\sum_{k=1}^{n}\left|\beta_k\right|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\times\Bigg[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg(\Bigg(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^{2}\right)^{-1}R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg)\left\|f_s-f_{s-1}\right\|_H$$

$$+\left(\left(\left\|\tau^2A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}+\left\|\tau^2A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$\left.+\frac{1}{6}\left(\left\|\tau^3A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}+\left\|\tau^3A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right)\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H$$

$$+\left(\left(\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A\left(I-i\tau A^2\right)^{-1}R_6\right\|_{H\to H}\right)\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left\|f_1\right\|_H\Bigg]$$

$$+\sum_{k=1}^{n}\frac{1}{2}\left|\beta_k\right|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\left\{\left\|\tau^{-1}A^{-\frac{1}{2}}R_7\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right.$$

$$\left.+\frac{1}{2}\left\|A\tau\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}R_7\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right\}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau$$

$$+\sum_{k=1}^{n}\left|\beta_k\right|\left\{\frac{1}{6}\left\|A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right)+\frac{1}{2}\left\|\tau\left(I-i\tau A^2\right)^{-1}\right\|\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$+\frac{1}{6}\left\|\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H\right)$$

$$+\frac{1}{2}\sum_{k=1}^{n}\left|\beta_k\right|\left(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\tilde{R}_1\tilde{R}_5\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}R_1R_5\right\|_{H\to H}$$

$$+\frac{1}{2}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\tilde{R}_5\tilde{R}_1\right\|_{H\to H}\right)\right)$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}R_5R_1\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{1}{6}\left\|\tau^3A^{\frac{3}{2}}\left(I+i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}_1R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R_1\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)$$

$$\times\left(\sum_{k=1}^{n}\frac{1}{2}\left|\alpha_k\right|\left\|\tau A^{\frac{1}{2}}R_3\right\|_{H\to H}\left\|A^{\frac{1}{2}}f_{1,1}\right\|_H\tau\left\{\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right)\right.\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|R_5A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}+\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}_5A\tau^2\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\right]$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\left\|\left(I-i\tau A^2\right)^{-1}\right\|\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{3}{2}\left\|\tau^2 A\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\left|\alpha_k\right|\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\times\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left(\left(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}R_5R_6\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^3A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^3A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5\tilde{R}_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)$$

$$\times\left\|f_s-f_{s-1}\right\|_H+\left(\left(\left(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}R_5R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{7}{6}\left(\left(\left\|\tau^3A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\right.\right.$$

$$\left.\left.+\left\|\tau^3A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}R_5R_6\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left(\left(\left(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}R_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left(\left(\left(\left\|\tau^3A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1}\tilde{R}_5R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right.\right.\right.$$

$$+\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_5 R_6\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg)\|f_1\|_H\Bigg]$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\|\tau A^{\frac{1}{2}} R_4\right\|_{H\to H}\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg(\Bigg(\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\Bigg)+\frac{3}{2}\Bigg(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)\|f_s-f_{s-1}\|_H$$

$$+\Bigg(\Bigg(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}+\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\Bigg)$$

$$+\frac{3}{2}\Bigg(\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}+\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\Bigg)$$

$$\times\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H+\Bigg(\Bigg(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\Bigg)+\frac{3}{2}\Bigg(\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-i\tau A^2\right)^{-1} R_6\right\|_{H\to H}\Bigg)\Bigg)\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg)\|f_1\|_H\right]$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\|A^{\frac{1}{2}} R_4\right\|_{H\to H}\left\|A^{-\frac{1}{2}} R_7\right\|_{H\to H}\times\Bigg(\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|$$

$$+\frac{7}{6}\left\|A^{\frac{3}{2}}\tau^3\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\Bigg)\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau$$

$$+\sum_{k=1}^{n}\left|\alpha_k\right|\left\{\frac{3}{2}\left\|\tau A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$\left.+\frac{7}{6}\left\|\tau^2 A^{\frac{1}{2}}\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}+\left\|\left(I-i\tau A^2\right)^{-1}\right\|_{H\to H}\left\|A\phi\right\|_H\Big)\Big\}$$

$$\leq M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\phi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}. \tag{2.19}$$

We will prove the estimates (2.9), (2.10) and (2.11). Using formula (2.6) and estimates (2.4), (2.12), (2.13), (2.14), (2.15) and the triangle inequality, we obtain

$$\left\|u_k\right\|_H\leq\frac{1}{2}\left(\left\|\tilde{R}_1R^k\right\|_{H\to H}+\left\|R_1\tilde{R}^k\right\|_{H\to H}\right)\left\|\mu\right\|_H$$

$$+\frac{1}{2}\left(\left\|A^{1/2}R_2\right\|_{H\to H}\left\|\tilde{R}^k\right\|_{H\to H}+\left\|A^{1/2}R_2\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)\left\|A^{-1/2}\omega\right\|_H$$

$$+\frac{1}{2}\left(\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|\tilde{R}^k\right\|_{H\to H}+\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)\tau\left\|A^{-1/2}f_{1,1}\right\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\sum_{s=1}^{k-1}\left[\left\|\tilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right]\left\|A^{-1/2}f_s\right\|_H\tau$$

$$\leq M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|\phi\right\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\}$$

for any $k\geq 2$. Applying $A^{1/2}$ to (2.6), we get

$$\left\|A^{1/2}u_k\right\|_H\leq\frac{1}{2}\left(\left\|R_1R^k\right\|_{H\to H}+\left\|R_1\tilde{R}^k\right\|_{H\to H}\right)\left\|A^{1/2}\mu\right\|_H$$

$$+\frac{1}{2}\left(\left\|A^{1/2}R_2\right\|_{H\to H}\left\|\tilde{R}^k\right\|_{H\to H}+\left\|A^{1/2}R_2\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)\left\|\omega\right\|_H$$

$$+\frac{1}{2}\left(\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|\tilde{R}^k\right\|_{H\to H}+\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)\tau\left\|f_{1,1}\right\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\sum_{s=1}^{k-1}\left(\left\|\tilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right)\|f_s\|_H\,\tau$$

$$\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\,\tau+\left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\phi\right\|_H+\|\psi\|_H+\tau\left\|f_{1,1}\right\|_H\right\}$$

for $k\ge 2$. Now, applying Abel's formula to (2.6), we have

$$u_k=\frac{1}{2}\left[\tilde{R}_1R^k-R_1\tilde{R}^k\right]\mu+\frac{1}{2}\left[\tilde{R}^k-R^k\right]R_2\omega+\frac{1}{2}\left[\tilde{R}^k-R^k\right]R_3\tau^2 f_{1,1}$$

$$+\tau^2R_4\frac{1}{2}\left(\sum_{s=2}^{k-1}\left[R_6R^{k-s}-\tilde{R}_6\tilde{R}^{k-s}\right](f_s-f_{s-1})\right.$$
$$\left.+\left(\tilde{R}_6-R_6\right)f_{k-1}-\left[\tilde{R}_6\tilde{R}^{k-1}-R_6R^{k-1}\right]f_1\right),2\le k\le N. \qquad (2.20)$$

Applying $A$ to formula (2.20) and using estimates (2.4) and the triangle inequality, we obtain

$$\|Au_k\|_H\le\frac{1}{2}\left(\left\|R_1R^k\right\|_{H\to H}+\left\|R_1\tilde{R}^k\right\|_{H\to H}\right)\|A\mu\|_H$$

$$+\frac{1}{2}\left(\left\|A^{1/2}R_2\right\|_{H\to H}\left\|\tilde{R}^k\right\|_{H\to H}+\left\|A^{1/2}R_2\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)\left\|A^{1/2}\omega\right\|_H$$

$$+\frac{1}{2}\left(\left\|\tau A^{-1/2}R_3\right\|_{H\to H}\left\|\tilde{R}^k\right\|_{H\to H}+\left\|\tau A^{-1/2}R_3\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)\tau\left\|A^{1/2}f_{1,1}\right\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\left(\sum_{s=2}^{k-1}\left[\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|R^{k-s}\right\|_{H\to H}+\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|\tilde{R}^{k-s}\right\|_{H\to H}\right]\right.$$

$$\times\|f_s-f_{s-1}\|_H+\left(\left\|\tau A^{1/2}R_6\right\|_{H\to H}+\left\|\tau A^{1/2}R_6\right\|_{H\to H}\right)\|f_{k-1}\|_H$$

$$\left.+\left[\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right]\|f_1\|_H\right)$$

$$\le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\left\|A\left(I+i\tau A^{1/2}\right)\phi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}$$

for $k\ge 2$. Theorem 2.1 is proved.

Thus, stability estimates for the solutions of these difference schemes are obtained. We denote the mesh function of approximation by $a^{\tau}=\{a_k\}_{k=0}^{N}$ . if we assume that $\tau\|Aa_1\|_H$ tends to $0$ as $\tau\to 0$ , then $\|(I+\tau^2A^4)^{-1}a_1\|_H \sim \|a_1\|_H \;=o(\tau)$ not slower than $\|a_1\|_H$ . It takes place in many applications by supplementary restriction of the smooth property of the data of space variables. Let us give corollaries of Theorem 2.1.

First, the mixed problem for hyperbolic equation

$$\begin{cases} u_{tt}-(a(x)u_x)_x+\delta u=f(t,x),\ 0<t<1,\ 0<x<1, \\ u(0,x)=\sum\limits_{j=1}^{n}\alpha_j u(\lambda_j,x)+\phi(x), 0\le x\le 1, \\ u_t(0,x)=\sum\limits_{j=1}^{n}\beta_j u_t(\lambda_j,x)+\psi(x), 0\le x\le 1, \\ u(t,0)=u(t,1),\ u_x(t,0)=u_x(t,1),\ 0\le t\le 1 \end{cases} \tag{2.21}$$

is considered. Problem (2.21) has a unique smooth solution $u(t,x)$ when $\delta>0$ and the smooth functions $a(x)\ge a>0$ $(x\in(0,1))$, $\phi(x),\psi(x)$ $(x\in[0,1])$ and $f(t,x)$ $(t,x\in[0,1])$. This permits us to reduce mixed problem (2.21) to nonlocal boundary value problem (2.1) in a Hilbert space $H=L_2[0,1]$ with a self-adjoint positive definite operator $A^x$ defined by (2.21).

The discretization of (2.21) is carried out in two steps. In the first step, let us define the grid space

$$[0,1]_h=\{x\ :\ x_r=rh, 0\le r\le K, Kh=1\}.$$

We introduce the Hilbert space $L_{2h}=L_2\left([0,1]_h\right)$, $W_{2h}^1=W_{2h}^1\left([0,1]_h\right)$ and $W_{2h}^2=W_{2h}^2\left([0,1]_h\right)$ of the grid functions $\phi^h(x)=\{\phi_r\}_1^{K-1}$ defined on $[0,1]_h$, equipped with the norms defined by

$$\|\phi^h\|_{L_{2h}}=\left(\sum_{r=1}^{K-1}\left|\phi^h(x)\right|^2 h\right)^{1/2},$$

$$\left\|\phi^h\right\|_{W_{2h}^1} = \left\|\phi^h\right\|_{L_{2h}} + \left(\sum_{r=1}^{K-1}\left|\left(\phi^h\right)_{\bar{x},j}\right|^2 h\right)^{1/2}$$

and

$$\left\|\phi^h\right\|_{W_{2h}^2} = \left\|\phi^h\right\|_{L_{2h}} + \left(\sum_{r=1}^{K-1}\left|\left(\phi^h\right)_{\bar{x},j}\right|^2 h\right)^{1/2} + \left(\sum_{r=1}^{K-1}\left|\left(\phi^h\right)_{x\bar{x},j}\right|^2 h\right)^{1/2},$$

respectively. To the differential operator $A$ generated by the problem (2.21), we assign the difference operator $A_h^x$ by the formula

$$A_h^x \phi^h(x) = \left\{-(a(x)\phi_{\bar{x}})_{x,r} + \delta\phi_r\right\}_1^{K-1}, \tag{2.22}$$

acting in the space of grid functions $\phi^h(x) = \{\phi_r\}_0^K$ satisfying the conditions $\phi_0 = \phi_K$, $\phi_1 - \phi_0 = \phi_K - \phi_{K-1}$. With the aid of $A_h^x$ we obtain the nonlocal boundary value problem

$$\begin{cases} \frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x),\ 0 < t < 1,\ x \in [0,1]_h, \\ v^h(0,x) = \sum\limits_{j=1}^{n} \alpha_j v^h\left(\lambda_j, x\right) + \phi^h(x), x \in [0,1]_h, \\ v_t^h(0,x) = \sum\limits_{j=1}^{n} \beta_j v_t^h(\lambda_j, x) + \psi^h(x), x \in [0,1]_h \end{cases} \tag{2.23}$$

for an infinite system of ordinary differential equations.In the second step, we replace problem (2.23) by difference scheme (2.24)

$$
\begin{cases}
\tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\frac{2}{3}A_h^x u_k^h(x)+\frac{1}{6}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)\\
+\frac{1}{12}\tau^2(A_h^x)^2u_{k+1}^h(x)=f_k^h(x),\\
f_k^h(x)=\frac{2}{3}f^h(t_k,x)+\frac{1}{6}(f^h(t_{k+1},x)+f^h(t_{k-1},x))\\
-\frac{1}{12}\tau^2\left(-Af^h(t_{k+1},x)+f_{tt}^h(t_{k+1},x)\right),x\in[0,1]_h,\\
t_k=k\tau,N\tau=1,1\le k\le N-1,\\
u_0^h(x)=\sum_{j=1}^{n}\alpha_j\left\{u_{[\lambda_j/\tau]}^h(x)+\tau^{-1}\left(u_{[\lambda_j/\tau]}^h(x)-u_{[\lambda_j/\tau]-1}^h(x)\right)(\lambda_j-[\lambda_j/\tau]\tau)\right.\\
+\frac{3}{2}\left(f_{[\lambda_j/\tau]}-A_h^x u_{[\lambda_j/\tau]}^h(x)\right)(\lambda_j-[\lambda_j/\tau]\tau)^2\\
\left.+\frac{7}{6}\left(f'_{[\lambda_j/\tau]}-\tau^{-1}A_h^x\left(u_{[\lambda_j/\tau]}^h(x)-u_{[\lambda_j/\tau]-1}^h(x)\right)\right)(\lambda_j-[\lambda_j/\tau]\tau)^3\right\}+\varphi^h(x),x\in[0,1]_h,\\
\left(I+\tau^2(A_h^x)^4\right)\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right)\\
=\sum_{j=1}^{n}\beta_j\left\{\tau^{-1}\left(u_{[\lambda_j/\tau]}^h(x)-u_{[\lambda_j/\tau]-1}^h(x)\right)+\left(f_{[\lambda_j/\tau]}-A_h^x u_{[\lambda_j/\tau]}^h(x)\right)(\lambda_j-[\lambda_j/\tau]\tau)\right.\\
+\frac{1}{2!}\left(f'_{[\lambda_j/\tau]}-\tau^{-1}A_h^x\left(u_{[\lambda_j/\tau]}^h(x)-u_{[\lambda_j/\tau]-1}^h(x)\right)\right)(\lambda_j-[\lambda_j/\tau]\tau)^2\\
\left.+\frac{1}{3!}\left(f''_{[\lambda_j/\tau]}-A_h^x f_{[\lambda_j/\tau]}+(A_h^x)^2u_{[\lambda_j/\tau]}^h(x)\right)(\lambda_j-[\lambda_j/\tau]\tau)^3\right\}+\psi^h(x),x\in[0,1]_h,\\
f_{1,1}^h(x)=\frac{1}{2}f^h(0,x)+\frac{\tau}{6}f_t^h(0,x).
\end{cases}
\tag{2.24}
$$

**Theorem 2.2** Let $\tau$ and $h$ be sufficiently small numbers. Then, the solution of difference scheme (2.24) satisfies the following stability estimates:

$$
\max_{0\le k\le N}\left\|u_k^h\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|u_k^h\right\|_{W_{2h}^1}\le M_1\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\phi^h\right\|_{W_{2h}^1}+\tau\left\|\phi^h\right\|_{W_{2h}^2}+\tau\left\|f_{1,1}^h\right\|_{L_{2h}}\right],
$$

$$
\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|u_k^h\right\|_{W_{2h}^2}
$$
$$
\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\phi^h\right\|_{W_{2h}^2}+\tau\left\|\phi^h\right\|_{W_{2h}^3}+\tau\left\|f_{1,1}^h\right\|_{W_{2h}^1}\right].
$$

Here $M_1$ does not depend on $\tau$, $h$, $\phi^h(x)$, $\psi^h(x)$, $f_{1,1}^h$ and $f_k^h, 1\le k<N$.

The proof of Theorem 2.2 is based on the proof abstract Theorem 2.1 and the symmetry property of the operator $A_h^x$ defined by (2.22).

Second, let $\Omega$ be the unit open cube in the $m$ -dimensional Euclidean space $R^m\left\{x=(x_1,\cdots,x_m):\quad 0<x_j<1, 1\le j\le m\right\}$ with boundary $S$, $\overline{\Omega}=\Omega\cup S$ . In

$[0,1]\times\Omega,$ the mixed boundary value problem for the multi-dimensional hyperbolic equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} = f(t,x), \\ x = (x_1, \dots, x_m) \in \Omega_h,\ 0 < t < 1, \\ u^h(0,x) = \sum\limits_{j=1}^{n} \alpha_j u^h\left(\lambda_j, x\right) + \phi^h(x), x \in \overline{\Omega}_h, \\ u_t^h(0,x) = \sum\limits_{j=1}^{n} \beta_j u_t^h(\lambda_j, x) + \psi^h(x), x \in \overline{\Omega}_h, \\ u^h(t,x) = 0,\ x \in S \end{cases} \tag{2.25}$$

is considered. Here $a_r(x),$ $(x\in\Omega$ $\phi(x$ $\psi$ , $x$ $(\in\overline{\Omega}_h)$ and $f(t,x)\,(t\in(0,1),\ x\in\Omega_h)$ are given smooth functions and $a_r(x)\geq a>0$ .We introduce the Hilbert space $L_2(\overline{\Omega})$ of the all square integrable functions defined on $\overline{\Omega},$ equipped with the norm

$$\| f \|_{L_2(\overline{\Omega})} = \left\{ \int \cdots \int\limits_{x\in\overline{\Omega}} | f(x)|^2 \, dx_1 \cdots dx_m \right\}^{\frac{1}{2}}.$$

The problem (2.25) has a unique smooth solution $u(t,x)$ and the smooth functions $\phi(x),$ $\psi(x),$ $a_r(x)$ and $f(t,x)$ . This allows us to reduce the mixed problem (2.25) to the nonlocal boundary value problem (2.1) in a Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator $A^x$ defined by (2.25).

The discretization of problem (2.25) is carried out in two steps. In the first step, let us define the grid sets

$$\Omega_h = \{x = x_r = (h_1 r_1, \cdots, h_m r_m), r = (r_1, \cdots, r_m),$$

$$0 \leq r_j \leq N_j, h_j N_j = 1, j = 1, \cdots, m\}, \Omega_h = \Omega_h \cap \Omega, S_h = \Omega_h \cap S.$$

We introduce the Banach space $L_{2h} = L_2(\Omega_h),$ $W_{2h}^1 = W_{2h}^1\left(\Omega_h\right)$ and $W_{2h}^2 = W_{2h}^2\left(\Omega_h\right)$ of the grid functions

$\phi^h(x) = \{\phi(h_1 r_1, \cdots, h_m r_m)\}$ defined on $\Omega_h,$ equipped with norms

$$\left\|\phi^h\right\|_{L_2(\Omega_h)} = \left( \sum_{x\in\Omega_h} \left|\phi^h(x)\right|^2 h_1 \cdots h_m \right)^{1/2},$$

$$\left\|\phi^h\right\|_{W_{2h}^1} = \left\|\phi^h\right\|_{L_{2h}} + \left( \sum_{x\in\Omega_h} \sum_{r=1}^{m} \left|\left(\phi^h\right)_{\bar{x}_r, j_r}\right|^2 h_1 \cdots h_m \right)^{1/2},$$

and

$$\left\|\phi^h\right\|_{W_{2h}^2} = \left\|\phi^h\right\|_{L_{2h}} + \left( \sum_{x\in\Omega_h} \sum_{r=1}^{m} \left|\left(\phi^h\right)_{\bar{x}_r}\right|^2 h_1 \cdots h_m \right)^{1/2} + \left( \sum_{x\in\Omega_h} \sum_{r=1}^{m} \left|\left(\phi^h\right)_{x_r \bar{x}_r, j_r}\right|^2 h_1 \cdots h_m \right)^{1/2},$$

respectively. To the differential operator $A$ generated by problem (2.25), we assign the difference operator $A_h^x$ by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{m} \left( a_r(x) u_{\bar{x}_r}^h \right)_{x_r, j_r} \tag{2.26}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that $A_h^x$ is a self-adjoint positive definite operator belongs to $L_2(\Omega_h)$ . With the aid of $A_h^x$ we arrive at the nonlocal boundary value problem

$$\begin{cases} \frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x), \ 0 < t < 1, \ x \in \Omega_h, \\ v^h(0,x) = \sum_{j=1}^{n} \alpha_j v^h\left(\lambda_j, x\right) + \phi^h(x), x \in \Omega_h, \\ \frac{dv^h(0,x)}{dt} = \sum_{j=1}^{n} \beta_j v_t^h(\lambda_j, x) + \psi^h(x), x \in \Omega_h \end{cases} \tag{2.27}$$

for an infinite system of ordinary differential equations.

In the second step, we replace (2.27) by difference scheme (2.28)

$$\left\{\begin{array}{l}\tau^{-2}\left(u_{k+1}^{h}(x)-2u_{k}^{h}(x)+u_{k-1}^{h}(x)\right)+\frac{2}{3}A_{h}^{x}u_{k}^{h}(x)+\frac{1}{6}A_{h}^{x}\left(u_{k+1}^{h}(x)+u_{k-1}^{h}(x)\right)\\+\frac{1}{12}\tau^{2}(A_{h}^{x})^{2}u_{k+1}^{h}(x)=f_{k}^{h}(x),f_{k}^{h}(x)=\frac{2}{3}f^{h}(t_{k},x)+\frac{1}{6}(f^{h}(t_{k+1},x)+f^{h}(t_{k-1},x))\\-\frac{1}{12}\tau^{2}\left(-Af^{h}(t_{k+1},x)+f_{tt}^{h}(t_{k+1},x)\right),t_{k}=k\tau,1\le k\le N-1,N\tau=1,x\in\Omega_{h},\\u_{0}^{h}(x)=\sum_{j=1}^{n}\alpha_{j}\left\{u_{[\lambda_{j}/\tau]}^{h}(x)+\tau^{-1}\left(u_{[\lambda_{j}/\tau]}^{h}(x)-u_{[\lambda_{j}/\tau]-1}^{h}(x)\right)(\lambda_{j}-[\lambda_{j}/\tau]\tau)\right.\\+\frac{3}{2}\left(f_{[\lambda_{j}/\tau]}-A_{h}^{x}u_{[\lambda_{j}/\tau]}^{h}(x)\right)(\lambda_{j}-[\lambda_{j}/\tau]\tau)^{2}\\\left.+\frac{7}{6}\left(f_{[\lambda_{j}/\tau]}^{'}-\tau^{-1}A_{h}^{x}\left(u_{[\lambda_{j}/\tau]}^{h}(x)-u_{[\lambda_{j}/\tau]-1}^{h}(x)\right)\right)(\lambda_{j}-[\lambda_{j}/\tau]\tau)^{3}\right\}+\varphi^{h}(x),x\in\Omega_{h},\\\left(I+\tau^{2}(A_{h}^{x})^{4}\right)\tau^{-1}\left(u_{1}^{h}(x)-u_{0}^{h}(x)\right)\\=\sum_{j=1}^{n}\beta_{j}\left\{\tau^{-1}\left(u_{[\lambda_{j}/\tau]}^{h}(x)-u_{[\lambda_{j}/\tau]-1}^{h}(x)\right)+\left(f_{[\lambda_{j}/\tau]}-A_{h}^{x}u_{[\lambda_{j}/\tau]}^{h}(x)\right)(\lambda_{j}-[\lambda_{j}/\tau]\tau)\right.\\+\frac{1}{2!}\left(f_{[\lambda_{j}/\tau]}^{\prime}-\tau^{-1}A_{h}^{x}\left(u_{[\lambda_{j}/\tau]}^{h}(x)-u_{[\lambda_{j}/\tau]-1}^{h}(x)\right)\right)(\lambda_{j}-[\lambda_{j}/\tau]\tau)^{2}\\\left.+\frac{1}{3!}\left(f_{[\lambda_{j}/\tau]}^{''}-A_{h}^{x}f_{[\lambda_{j}/\tau]}+(A_{h}^{x})^{2}u_{[\lambda_{j}/\tau]}^{h}(x)\right)(\lambda_{j}-[\lambda_{j}/\tau]\tau)^{3}\right\}+\psi^{h}(x),x\in\Omega_{h},\\f_{1,1}^{h}(x)=\frac{1}{2}f^{h}(0,x)+\frac{\tau}{6}f_{t}^{h}(0,x).\end{array}\right. \tag{2.28}$$

**Theorem 2.3** Let $\tau$ and $|h|$ be sufficiently small numbers. Then, the solution of difference scheme (2.28) satisfies the following stability estimates:

$$\max_{0\le k\le N}\left\|u_{k}^{h}\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|u_{k}^{h}\right\|_{W_{2h}^{1}}\le M_{1}\left[\max_{1\le k\le N-1}\left\|f_{k}^{h}\right\|_{L_{2h}}+\left\|\psi^{h}\right\|_{L_{2h}}+\left\|\phi^{h}\right\|_{W_{2h}^{1}}+\tau\left\|\phi^{h}\right\|_{W_{2h}^{2}}+\tau\left\|f_{1,1}^{h}\right\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^{h}-2u_{k}^{h}+u_{k-1}^{h}\right)\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|u_{k}^{h}\right\|_{W_{2h}^{2}}$$

$$\le M_{1}\left[\left\|f_{1}^{h}\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_{k}^{h}-f_{k-1}^{h}\right)\right\|_{L_{2h}}+\left\|\psi^{h}\right\|_{W_{2h}^{1}}+\left\|\phi^{h}\right\|_{W_{2h}^{2}}+\tau\left\|\phi^{h}\right\|_{W_{2h}^{3}}+\tau\left\|f_{1,1}^{h}\right\|_{W_{2h}^{1}}\right].$$

Here $M_1$ does not depend on $\tau$, $h$, $\phi^h(x)$, $\psi^h(x)$, $f_{1,1}^h$ and $f_k^h, 1\le k<N$.

The proof of Theorem 2.3 is based on Theorem 2.1, the symmetry property of the operator $A_h^x$ defined by formula (2.26) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in $L_{2h}$

**Theorem 2.4** For the solutions of the elliptic difference problem

$$A_h^x u^h(x) = \omega^h(x), x \in \Omega_h,$$

$$u^h(x) = 0, x \in S_h$$

the following coercivity inequality holds [28]:

$$\sum_{r=1}^{m} \left\| u^h_{x_r \bar{x}_r, j_r} \right\|_{L_{2h}} \leq M \| \omega^h \|_{L_{2h}} .$$

Here $M$ does not depend on $h$, $\omega^h(x)$.

## 2.3 The Fourth Order of Accuracy Difference Scheme Subject to Multipoint Nonlocal Conditions

In present section we consider the fourth order of accuracy difference scheme for the numerical solution of multipoint nonlocal boundary value problem (2.1). Let us associate boundary value problem (2.1) with the corresponding fourth order of accuracy difference scheme

$$\begin{cases}
\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{5}{6}Au_k+\frac{1}{12}A(u_{k+1}+u_{k-1})-\frac{\tau^2}{72}A^2u_k \\
+\frac{\tau^2}{144}A^2(u_{k+1}+u_{k-1}) = \frac{5}{6}f(t_k)+\frac{1}{12}(f(t_{k+1})+f(t_{k-1})) \\
+\frac{\tau^2}{72}\left(-Af(t_k)+f''(t_k)\right)-\frac{1}{144}\tau^2\left(-A(f(t_{k+1})+f(t_{k-1}))+f''(t_{k+1})+f''(t_{k-1})\right), \\
t_k = k\tau, 1\le k\le N-1, N\tau=1, \\
u_0 = \left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\sum_{k=1}^{n}\alpha_k\left\{\left(\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)-\frac{7A\tau^2}{6}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right)\right. \\
\times\left(u_{\left[\frac{\lambda_k}{\tau}\right]}-u_{\left[\frac{\lambda_k}{\tau}\right]-1}\right)+\left(1-\frac{3\tau^2A}{2}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{A^2\tau^4}{24}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right)u_{\left[\frac{\lambda_k}{\tau}\right]} \\
+\frac{3}{2}\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2 f_{\left[\frac{\lambda_k}{\tau}\right]}+\frac{7}{6}\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3 f'_{\left[\frac{\lambda_k}{\tau}\right]} \\
\left.+\frac{1}{24}\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4 f''_{\left[\frac{\lambda_k}{\tau}\right]}-\frac{1}{24}A\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4 f_{\left[\frac{\lambda_k}{\tau}\right]}\right\}+\varphi, \\
\tau^{-1}(u_1-u_0) = \left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1} \\
\sum_{k=1}^{n}\beta_k\left\{\left[\frac{1}{\tau}-\frac{\tau A}{2}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{\tau^3A^2}{24}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]\left(u_{\left[\frac{\lambda_k}{\tau}\right]}-u_{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\right. \\
+\left(-A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{A^2\tau^3}{6}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right)u_{\left[\frac{\lambda_k}{\tau}\right]} \\
+\tau f_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{\tau^2}{2}f'_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2 \\
+\frac{\tau^3}{6}f''_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3+\frac{\tau^4}{24}f'''_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4 \\
\left.-\frac{A\tau^3}{6}f_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3-\frac{A\tau^4}{24}f'_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right\}+\psi+f_{2,2}\tau
\end{cases} \tag{2.29}$$

for approximately solving nonlocal boundary value problem (2.1) where

$$f_{2,2}=\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\tau^{-1}f_{1,1}^4=\left\{\left(I-\frac{\tau^2A}{12}\right)f(0)+\frac{1}{2}\left(-\left(I-\frac{5\tau^2A}{12}\right)f(0)+\tau f'(0)\right)\right.$$

$$+\left(-A\tau f(0)-2f'(0)+\tau f''(0)\right)\frac{\tau}{6}+\left(Af(0)-3f''(0)\right)\frac{\tau^2}{24}\bigg\}.$$

We are interested in to study the stability of solution of difference scheme (2.29) under the assumption

$$\Bigg\{\sum_{k=1}^{n}|\alpha_k|\Bigg\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\Bigg\}$$

$$+\sum_{k=1}^{n}|\beta_k|\Bigg\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\Bigg\}$$

$$+\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|\Bigg\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4$$

$$+\frac{1}{9}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6+\frac{1}{576}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^8\Bigg\}<1. \tag{2.30}$$

Throughout this section for simplicity, we put

$$B_n^{\tau}=1-\frac{1}{2}\sum_{k=1}^{n}\alpha_k\Bigg\{\left[\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)-\frac{7A\tau^2}{6}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]$$

$$+\left[1-\frac{3}{2}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]\left[R^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\Bigg\}$$

$$-\frac{1}{2}\sum_{k=1}^{n}\beta_k\Bigg\{iA^{-\frac{1}{2}}\left[\frac{1}{\tau}-\frac{1}{2}A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]$$

$$\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]+iA^{-\frac{1}{2}}\left[-A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{6}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\Bigg\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\sum_{k=1}^{n}\alpha_k\beta_k\left\{\left(-\frac{1}{\tau}+A\tau\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^2+\frac{1}{2}A^2\tau^3\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^4\right.\right.$$

$$\left.\left.-\frac{1}{9}A^3\tau^5\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^6+\frac{1}{576}A^4\tau^7\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^8\right)\times\left(\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}_5\right)\right\}.$$

First of all let us prove two lemmas needed in the sequel. We will consider operators

$$R=\left(I-\frac{\tau^2A}{12}+\frac{i\tau A^{1/2}}{2}\right)\left(I-\frac{\tau^2A}{12}-\frac{i\tau A^{1/2}}{2}\right)^{-1},$$

$$\tilde{R}=\left(I-\frac{\tau^2A}{12}-\frac{i\tau A^{1/2}}{2}\right)\left(I-\frac{\tau^2A}{12}+\frac{i\tau A^{1/2}}{2}\right)^{-1}$$

and we will introduce the following operators

$$R_0=2i\tau A^{1/2}\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}$$

and

$$R_5=i\tau A^{1/2}\left(I-\frac{\tau^2A}{12}-\frac{i\tau A^{1/2}}{2}\right)^{-1}$$

$$R_6=i\tau A^{1/2}\left(I-\frac{\tau^2A}{12}-\frac{i\tau A^{1/2}}{2}\right),$$

$$J_1=\left(I-\frac{\tau^2A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-1},$$

$$J_1=\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-1},$$

$$J_2=\left(-i\tau A^{1/2}\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)\right)^{-1},$$

$$J_2 = \left( -i\tau A^{1/2} \left( I + \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right) \right)^{-1},$$

$$J_3 = i\tau A^{1/2} \left( I - \frac{\tau^2 A}{12} \right) \left( I - \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^{-2},$$

$$J_3 = -i\tau A^{1/2} \left( I - \frac{\tau^2 A}{12} \right) \left( I + \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^{-2},$$

$$J_4 = \left( I - \frac{\tau^2 A}{12} \right) \left( i\tau A^{1/2} \left( I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right) \right)^{-1},$$

$$J_4 = \left( I - \frac{\tau^2 A}{12} \right) \left( -i\tau A^{1/2} \left( I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right) \right)^{-1},$$

Note that $R_5, \tilde{R}_5$, and $R_6, \tilde{R}_6$, and $J_1, J_1$ , and $J_2, J_2$ and $J_3, J_3$ and $J_4, J_4$ are complex conjugates. Now, let us give a lemma in which some of the estimates were presented in [22].

**Lemma 2.3** The following estimates hold:

$$\begin{cases} \|R\|_{H \to H} \le 1, \|\tilde{R}\|_{H \to H} \le 1, \\ \|A^{-1/2} R_5\|_{H \to H} \le \tau, \|A^{-1/2} R_5\|_{H \to H} \le \tau, \\ \|A^{-1} R_5 R_6\|_{H \to H} \le \tau, \|A^{-1} R_5 \tilde{R}_6\|_{H \to H} \le \tau, \\ \|A^{-1/2} R_0\|_{H \to H} \le \tau, \|J_1\|_{H \to H} \le 1, \|J_1\|_{H \to H} \le 1, \\ \|\tau A^{1/2} J_2\|_{H \to H} \le 1, \|\tau A^{1/2} J_2\|_{H \to H} \le 1, \\ \|A^{-1/2} J_3\|_{H \to H} \le \tau, \|A^{-1/2} J_3\|_{H \to H} \le \tau, \\ \|\tau A^{1/2} J_4\|_{H \to H} \le 1, \|\tau A^{1/2} J_4\|_{H \to H} \le 1. \end{cases} \tag{2.31}$$

**Proof.** Using the spectral property of the self-adjoint positive definite operators, we write

$$\left\| \left( I \pm \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right) \left( I \mp \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^{-1} \right\|_{H \to H} \le \sup_{\delta \le \lambda < \infty} \left| \frac{I \pm \frac{1}{2} i\tau\lambda^{1/2} - \frac{1}{12}\tau^2\lambda}{I \mp \frac{1}{2} i\tau\lambda^{1/2} - \frac{1}{12}\tau^2\lambda} \right|.$$

Since

$$\frac{\left(1 - \frac{1}{12}\tau^2\lambda\right)^2 + \frac{1}{4}\tau^2\lambda}{\left(1 - \frac{1}{12}\tau^2\lambda\right)^2 + \frac{1}{4}\tau^2\lambda} \le 1, \text{ we have that } \|R\|_{H \to H} \le 1, \|\tilde{R}\|_{H \to H} \le 1.$$

In similar way, we can write

$$\left\| i\tau A^{1/2} \left( I \mp \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^{-1} \right\|_{H \to H} \le \sup_{\delta \le \lambda < \infty} \left| \frac{\tau^2 \lambda}{1 \mp \frac{i\tau\lambda^{1/2}}{2} - \frac{\tau^2\lambda}{12}} \right|.$$

Since

$$\frac{\left(\tau^2\lambda\right)^2}{\left(1 - \frac{\tau^2\lambda}{12}\right)^2 + \frac{\tau^2\lambda}{4}} = \frac{\tau^4\lambda^2}{1 + \frac{\tau^2\lambda}{12} + \frac{\tau^4\lambda^2}{144}} \le \tau, \text{ we have that } \|R_5\|_{H \to H} \le 1, \|R_5\|_{H \to H} \le 1.$$

And similarly,

$$\left\| A^{-1/2} 2i\tau A^{1/2} \left( I - \frac{\tau^2 A}{12} \right) \left( I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \right\|_{H \to H} \le \sup_{\delta \le \lambda < \infty} \left| \frac{2\tau\left(I - \frac{\tau^2\lambda}{12}\right)}{1 + \frac{\tau^2\lambda}{12} + \frac{\tau^4\lambda^2}{144}} \right|.$$

Since $\dfrac{2\tau\left(I - \frac{\tau^2\lambda}{12}\right)}{1 + \frac{\tau^2\lambda}{12} + \frac{\tau^4\lambda^2}{144}} \le \tau,$ we have that $\|A^{-1/2} R_0\|_{H \to H} \le \tau.$

In the similar way, we can write

$$\left\| \left( I - \frac{\tau^2 A}{12} \right) \left( I \mp \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^{-1} \right\|_{H \to H} \le \sup_{\delta \le \lambda < \infty} \left| \frac{1 - \frac{\tau^2\lambda}{12}}{1 \mp \frac{i\tau\lambda^{1/2}}{2} - \frac{\tau^2\lambda}{12}} \right|.$$

Since

$$\frac{\left(1-\frac{\tau^2\lambda}{12}\right)^2}{\left(1-\frac{\tau^2\lambda}{12}\right)^2+\frac{\tau^2\lambda}{4}}=\frac{1-\frac{\tau^2\lambda}{6}+\frac{\tau^4\lambda^2}{144}}{1+\frac{\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}}\le 1,$$

we have that $\|J_1\|_{H\to H}\le 1, \|J_1\|_{H\to H}\le 1.$

In a similar manner

$$\left\|\tau A^{1/2}\left(i\tau A^{1/2}\left(I\mp\frac{i\tau A^{1/2}}{2}-\frac{\tau^2 A}{12}\right)\right)^{-1}\right\|_{H\to H}\le \sup_{\delta\le\lambda<\infty}\left|\frac{i}{1\mp\frac{i\tau\lambda^{1/2}}{2}-\frac{\tau^2\lambda}{12}}\right|.$$

Since $\frac{1}{\left(1-\frac{\tau^2\lambda}{12}\right)^2+\frac{\tau^2\lambda}{4}}=\frac{1}{1+\frac{\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}}\le 1,$

we have that $\|\tau A^{1/2}J_2\|_{H\to H}\le 1, \|\tau A^{1/2}J_2\|_{H\to H}\le 1.$

In similar ways

$$\left\|A^{-1/2}i\tau A^{1/2}\left(I-\frac{\tau^2 A}{12}\right)\left(I\mp\frac{i\tau A^{1/2}}{2}-\frac{\tau^2 A}{12}\right)^{-2}\right\|_{H\to H}\le \sup_{\delta\le\lambda<\infty}\left|\frac{i\left(1-\frac{\tau^2\lambda}{12}\right)\tau}{\left(1\mp\frac{i\tau\lambda^{1/2}}{2}-\frac{\tau^2\lambda}{12}\right)^2}\right|.$$

Since $\frac{\left(1-\frac{\tau^2\lambda}{12}\right)^2\tau^2}{\left(1-\frac{5\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}\right)^2+\tau^2\lambda\left(1-\frac{\tau^2\lambda}{12}\right)^2}\le\tau^2,$

we have that $\|A^{-1/2}J_3\|_{H\to H}\le\tau, \|A^{-1/2}J_3\|_{H\to H}\le\tau.$

In the same manner, we can write

$$\left\|\tau A^{1/2}\left(I-\frac{\tau^2 A}{12}\right)\left(\pm i\tau A^{1/2}\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)\right)^{-1}\right\|_{H\to H}\le \sup_{\delta\le\lambda<\infty}\left|\frac{i\left(1-\frac{\tau^2\lambda}{12}\right)}{1+\frac{\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}}\right|.$$

Since $\left|\frac{\left(1-\frac{\tau^2\lambda}{12}\right)^2}{\left(1+\frac{\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}\right)^2}\right|\le 1,$ we have that

$$\left\|\tau A^{1/2}J_4\right\|_{H\to H}\le 1,\left\|\tau A^{1/2}J_4\right\|_{H\to H}\le 1.$$

Lemma 2.3 is proved.

**Lemma 2.4** Suppose that (2.30) holds. Then, the operator $I-B_n^{\tau}$ has an inverse $T_\tau=\left(I-B_n^{\tau}\right)^{-1}$. From the symmetry and positivity properties of the operator $A$ the following estimate is satisfied:

$$\left\|T_\tau\right\|_{H\mapsto H}\le M. \tag{2.32}$$

**Proof.** Using the definitions of $B_n^{\tau}, R, \tilde{R},$ estimates (2.31), the triangle inequality, we obtain

$$B_n^{\tau}\le\frac{1}{2}\sum_{k=1}^{n}\left|\alpha_k\right|\times\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\Bigg\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\Bigg\{\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\Bigg\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|\Bigg\{\Bigg[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}R_5\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\frac{1}{2}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^2\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^2\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}R_5\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4$$

$$\left[+\frac{1}{9}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^5A^3\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^5A^3\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}R_5\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6\right]$$

$$\left[+\frac{1}{576}\left[\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^7A^4\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^7A^4\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}R_5\right\|_{H\to H}\right]\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6\right]\Bigg\}\Bigg\}\le q$$

where

$$q=\left\{\sum_{k=1}^{n}|\alpha_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\right\}\right.$$

$$+\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\right\}$$

$$+\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4+\frac{1}{9}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6+\frac{1}{576}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^8\right\}$$

Since $q<1$, the operator $I-B_n^{\tau}$ has a bounded inverse , so

$$\left\|\left(I-B_n^{\tau}\right)^{-1}\right\|_{H\to H}\le\frac{1}{1-q}.$$

Lemma 2.4 is proved.

Now, let us use the following formula for the solution of problem (2.29) which is obtained in [22],

$$u_0=\mu, u_1=\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\times\left(\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\mu+\tau\left(I-\frac{\tau^2A}{12}\right)\omega+\tau^2 f_{2,2}\right),$$

$$u_k=\frac{1}{2}\left[R^k+\tilde{R}^k\right]\mu+\frac{1}{2}\left[\tilde{R}^k-R^k\right]iA^{-1/2}\omega+\frac{1}{2}\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-1/2}\left[\tilde{R}^k-R^k\right]\tau f_{2,2}$$

$$+\frac{1}{2}\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-1/2}\sum_{s=1}^{k-1}\left[\tilde{R}^{k-s}-R^{k-s}\right]\tau f_s. \tag{2.33}$$

Applying formula (2.33) and the nonlocal boundary conditions in (2.29), we obtain

$$\mu=T_\tau\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}$$

$$\times\left\{\left[\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left\{\left[\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)-\frac{7}{6}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\right.\right.\right.$$

$$\times\left[\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\tau f_{2,2}\right.$$

$$+\left(I-\frac{\tau^2 A}{12}\right)^{-1} iA^{-\frac{1}{2}}\left(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5\right]f_s\tau+R_0 f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau\right)\Bigg]\Bigg]$$

$$+\left[1-\frac{3}{2}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]$$

$$\times\left[\left(I-\frac{\tau^2 A}{12}\right)^{-1} iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\tau f_{2,2}+\left(I-\frac{\tau^2 A}{12}\right)^{-1} iA^{-\frac{1}{2}}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right]f_s\tau\right]\Bigg\}$$

$$+\sum_{k=1}^{n}\alpha_k\left\{\frac{3}{2}\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2 f_{\left[\frac{\lambda_k}{\tau}\right]}+\frac{7}{6}\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3 f'_{\left[\frac{\lambda_k}{\tau}\right]}\right.$$

$$\left.+\frac{1}{24}\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4 f''_{\left[\frac{\lambda_k}{\tau}\right]}-\frac{1}{24}A\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4 f_{\left[\frac{\lambda_k}{\tau}\right]}\right\}+\phi\Bigg]$$

$$\times\left[1-\frac{1}{2}\sum_{k=1}^{n}\beta_k\left\{iA^{-\frac{1}{2}}\left[\frac{1}{\tau}-\frac{1}{2}A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]\right.\right.$$

$$\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]+iA^{-\frac{1}{2}}\left[-A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{6}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]$$

$$\left.\left.\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\right]\right\}\Bigg]+\left[\frac{1}{2}\sum_{k=1}^{n}\beta_k\left\{\left[\frac{1}{\tau}-\frac{1}{2}A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]\right.\right.$$

$$\times\left[\left(I-\frac{\tau^2 A}{12}\right)^{-1} iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\tau f_{2,2}\right.$$

$$+\left(I-\frac{\tau^2 A}{12}\right)^{-1} iA^{-\frac{1}{2}}\times\left(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5\right]f_s\tau+R_0 f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau\right)\Bigg]\Bigg]$$

$$+\left[-A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{6}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]$$

$$+\left[\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\tau f_{2,2}+\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right]f_s\tau\right]\Bigg\}$$

$$+\sum_{k=1}^{n}\beta_k\left\{\tau f_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{2}\tau^2 f'_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2\right.$$

$$+\frac{1}{6}\tau^3 f''_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3+\frac{1}{24}\tau^4 f'''_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4$$

$$\left.-\frac{1}{6}A\tau^3 f_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3-\frac{1}{24}A\tau^4 f'_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right\}+\psi\Bigg]$$

$$\times\left[\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left\{\left[\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)-\frac{7}{6}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\right.\right.$$

$$\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]iA^{-\frac{1}{2}}+\left[1-\frac{3}{2}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]$$

$$\left.\left.\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]iA^{-\frac{1}{2}}\right\}\right]\Bigg\}, \tag{2.34}$$

$$\omega=T_\tau\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144}\right)^{-1}$$

$$\times\left\{\left[\frac{1}{2}\sum_{k=1}^{n}\beta_k\left\{\left[\frac{1}{\tau}-\frac{1}{2}A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]\right.\right.\right.$$

$$\times\left[\left(I-\frac{\tau^2 A}{12}\right)^{-1} iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\tau f_{2,2}+\left(I-\frac{\tau^2 A}{12}\right)^{-1} iA^{-\frac{1}{2}}\right.$$

$$\left.\times\left(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5\right]f_s\tau+R_0 f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau\right)\right]$$

$$+\left[-A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{6}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]$$

$$+\left[\left(I-\frac{\tau^2 A}{12}\right)^{-1} iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]f_{2,2}\tau+\left(I-\frac{\tau^2 A}{12}\right)^{-1} iA^{-\frac{1}{2}}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right]f_s\tau\right]\Bigg\}$$

$$+\sum_{k=1}^{n}\beta_k\left\{\tau f_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{2}\tau^2 f'_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2\right.$$

$$+\frac{1}{6}\tau^3 f''_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3+\frac{1}{24}\tau^4 f'''_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4$$

$$\left.-\frac{1}{6}A\tau^3 f_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3-\frac{1}{24}A\tau^4 f'_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right\}+\psi\Bigg]$$

$$\times\left[1-\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left\{\left[\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)-\frac{7}{6}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\right.\right.$$

$$\left.\left.+\left[1-\frac{3}{2}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\right\}\right]$$

$$+\left[\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left\{\left[\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)-\frac{7}{6}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\right.\right.$$

$$\times\left[\left(I-\frac{\tau^2A}{12}\right)^{-1} iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\tau f_{2,2}+\left(I-\frac{\tau^2A}{12}\right)^{-1} iA^{-\frac{1}{2}}\right.$$

$$\left.\times\left(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}R_5\right]f_s\tau+R_0 f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau\right)\right]$$

$$+\left[1-\frac{3}{2}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]$$

$$\left.\times\left[\left(I-\frac{\tau^2A}{12}\right)^{-1} iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\tau f_{2,2}+\left(I-\frac{\tau^2A}{12}\right)^{-1} iA^{-\frac{1}{2}}\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s}-R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right]f_s\tau\right]\right\}$$

$$+\sum_{k=1}^{n}\alpha_k\left\{\frac{3}{2}\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2 f_{\left[\frac{\lambda_k}{\tau}\right]}+\frac{7}{6}\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3 f'_{\left[\frac{\lambda_k}{\tau}\right]}\right.$$

$$\left.+\frac{1}{24}\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4 f''_{\left[\frac{\lambda_k}{\tau}\right]}-\frac{1}{24}A\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4 f_{\left[\frac{\lambda_k}{\tau}\right]}\right\}+\phi\Bigg]$$

$$\times\left[\frac{1}{2}\sum_{k=1}^{n}\beta_k\left\{\left[\frac{1}{\tau}-\frac{1}{2}A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\right.\right.$$

$$\left.\left.\left.+\left[-A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{6}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\right\}\right]\right\}. \qquad (2.35)$$

Thus, formulas (2.33), (2.34) and (2.35) give a solution of problem (2.29).

**Theorem 2.5** Suppose that assumption (2.30) holds when $\phi\in D(A^{1/2}),\ \psi\in D(A^{\frac{1}{2}})$ . Then for the solution of difference scheme (2.29) the following stability estimates hold:

$$\max_{2\le k\le N}\left\|\frac{u_k+u_{k-1}}{2}\right\|_H\le M\left\{\sum_{s=1}^{k-1}\left\|A^{-1/2}f_s\right\|_H\tau+\|\phi\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{2,2}\right\|_H\right\},$$

$$\left\| u_1 \right\|_H \le M \left\{ \sum_{s=1}^{k-1} \left\| A^{-1/2} f_s \right\|_H \tau + \left\| \phi \right\|_H + \left\| A^{-1/2} \psi \right\|_H + \tau \left\| A^{-1/2} f_{2,2} \right\|_H \right\}, \tag{2.36}$$

$$\max_{2\le k\le N-1} \left\| \frac{u_{k+1} - u_{k-1}}{2\tau} \right\|_H + \max_{2\le k\le N} A^{1/2} \left\| \frac{u_k + u_{k-1}}{2} \right\|_H$$

$$\le M \left\{ \sum_{s=1}^{k-1} \left\| f_s \right\|_H \tau + \left\| A^{1/2} \phi \right\|_H + \left\| \psi \right\|_H + \tau \left\| f_{2,2} \right\|_H \right\},$$

$$\left\| A^{1/2} u_1 \right\|_H \le M \left\{ \sum_{s=1}^{k-1} \left\| f_s \right\|_H \tau + \left\| A^{1/2} \phi \right\|_H + \left\| \psi \right\|_H + \tau \left\| f_{2,2} \right\|_H \right\}, \tag{2.37}$$

$$\max_{2\le k\le N-1} \left\| A^{1/2} \frac{u_{k+1} - u_{k-1}}{2\tau} \right\|_H + \max_{1\le k\le N} \left\| A \frac{u_k + u_{k-1}}{2} \right\|_H$$

$$\le M \left\{ \sum_{s=2}^{N-1} \left\| f_s - f_{s-1} \right\|_H + \left\| f_1 \right\|_H + \left\| A\phi \right\|_H + \left\| A^{1/2} \psi \right\|_H + \tau \left\| A^{1/2} f_{2,2} \right\|_H \right\},$$

$$\left\| A u_1 \right\|_H \le M \left\{ \sum_{s=2}^{N-1} \left\| f_s - f_{s-1} \right\|_H + \left\| f_1 \right\|_H + \left\| A\phi \right\|_H + \left\| A^{1/2} \psi \right\|_H + \tau \left\| A^{1/2} f_{2,2} \right\|_H \right\}, \tag{2.38}$$

where $M$ does not depend on $\tau$, $\phi$, $\psi$, $f_{2,2}$ and $f_s$, $1 \le s \le N-1$.

**Proof.** Using formulas (2.34), (2.35) and estimates (2.31), (2.32), we can obtain

$$\left\| \mu \right\|_{H\to H} \le \left\| T_\tau \right\|_{H\to H} \left\{ \left[ \frac{1}{2} \sum_{k=1}^{n} \left| \alpha_k \right| \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \times \left( \left[ \left\| R^{\left[\frac{\lambda_k}{\tau}\right]-1} \right\|_{H\to H} \left\| \left( I - \frac{i\tau A^{1/2}}{2} + \frac{\tau^2 A^3}{12} \right)^{-1} R_5 \right\|_{H\to H} \right.\right.\right.$$

$$\left. + \left\| R^{\left[\frac{\lambda_k}{\tau}\right]-1} \right\|_{H\to H} \left\| \left( I - \frac{i\tau A^{1/2}}{2} + \frac{\tau^2 A^3}{12} \right)^{-1} R_5 \right\|_{H\to H} \right] \left\| A^{-\frac{1}{2}} f_{2,2} \right\|_H \tau$$

$$+ \sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2} \left[ \left\| R^{\left[\frac{\lambda_k}{\tau}\right]-s-1} \right\|_{H\to H} \left\| \left( I - \frac{i\tau A^{1/2}}{2} + \frac{\tau^2 A^3}{12} \right)^{-1} R_5 \right\|_{H\to H} \right.$$

$$\left. + \left\| R^{\left[\frac{\lambda_k}{\tau}\right]-s-1} \right\|_{H\to H} \left\| \left( I - \frac{i\tau A^{1/2}}{2} + \frac{\tau^2 A^3}{12} \right)^{-1} R_5 \right\|_{H\to H} \right] \left\| A^{-\frac{1}{2}} f_s \right\|_H \tau$$

$$+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\right)$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$\left.+\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\right)$$

$$+\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau\right.$$

$$\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\right)$$

$$+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau\right.$$

$$\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\right)$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau\right.$$

$$\left.\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\right)\right\}$$

$$+\sum_{k=1}^{n}|\alpha_k|\left\{\frac{3}{2}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$+\frac{7}{6}\left\|\tau^2A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|A^{-\frac{1}{2}}f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$+\frac{1}{24}\left\|\tau^3A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|A^{-\frac{1}{2}}f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$\left.-\frac{1}{24}\left\|\tau^3A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}$$

$$\left.+\left\|\left(I-\frac{\tau^2A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\|\phi\|_H\right]\left[\left\|\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\right.$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\{\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$\left.\left.+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\right\}\right]$$

$$+\left[\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\{\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$+\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\tau\Bigg)$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg(\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\Bigg)$$

$$+\frac{1}{24}\Bigg(\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$+\left\|\tau^4 A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1} R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}} f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H \tau\right)$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}} f_{2,2}\right\|_H \tau\right.$$

$$\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}} f_s\right\|_H \tau\right)$$

$$+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3 A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}} f_{2,2}\right\|_H \tau\right.$$

$$\left.\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}} f_s\right\|_H \tau\right)\right\}$$

$$+\sum_{k=1}^{n}\beta_k\left\{\left\|\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}} f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H \tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$+\frac{1}{2}\left\|\tau\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}} f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H \tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\frac{1}{6}\left\|\tau^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}} f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H \tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$+\frac{1}{24}\left\|\tau^3\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}} f'''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H \tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4$$

$$+\frac{1}{6}\left\|\tau^{2}A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^{2}A^{3}}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H}\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{3}$$

$$-\frac{1}{24}\left\|\tau^{3}A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^{2}A^{3}}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H}\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{4}\Bigg\}$$

$$+\left\|\left(I-\frac{\tau^{2}A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^{2}A^{3}}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_{H}\Bigg]$$

$$\times\Bigg[\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^{2}A^{3}}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^{2}A^{3}}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{3}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{2}A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^{2}A^{3}}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{2}A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^{2}A^{3}}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^{2}A^{3}}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{3}{2}\left\|\tau^{2}A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^{2}A^{3}}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$\left.+\frac{1}{24}\left\|\tau^{4}A^{2}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^{2}A^{3}}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^{4}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\right\}\Bigg]\Bigg\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\left\|A^{-1/2}f_s\right\|_H\tau+\|\phi\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{2,2}\right\|_H\right\}, \tag{2.39}$$

$$\left\|A^{-\frac{1}{2}}\omega\right\|_{H\to H}\leq\|T_\tau\|_{H\to H}\left\{\left[\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\{\left(\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$\left.+\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\tau\right)$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\Bigg)$$

$$+\frac{1}{24}\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$+\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\Bigg)$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau\right.$$

$$\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\right)$$

$$+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau\right.$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\Bigg)\Bigg\}$$

$$+\sum_{k=1}^{n}\beta_k\left\{\left\|\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$+\frac{1}{2}\left\|\tau\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\frac{1}{6}\left\|\tau^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$+\frac{1}{24}\left\|\tau^3\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f'''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4$$

$$+\frac{1}{6}\left\|\tau^2A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$\left.-\frac{1}{24}\left\|\tau^3A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\right\}$$

$$+\left\|\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_H\Bigg]\left[1+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$\times\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]+\left[\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\right.$$

$$+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|\tau^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}A\right\|_{H\to H}$$

$$\left.+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\right]\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\Bigg\}\Bigg]$$

$$+\left[\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$\left.+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\right)$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau$$

$$+\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\Bigg)$$

$$+\Bigg(\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\Bigg)$$

$$+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\Bigg(\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\Bigg)$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|\tau^4 A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\Bigg(\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\Bigg]\left\|A^{-\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\Bigg)\Bigg\}$$

$$+\sum_{k=1}^{n}|\alpha_k|\left\{\frac{3}{2}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{\tau^2 A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$+\frac{7}{6}\left\|\tau^2 A^{\frac{1}{2}}\left(I-\frac{\tau^2 A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|A^{-\frac{1}{2}}f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$+\frac{1}{24}\left\|\tau^3 A^{\frac{1}{2}}\left(I-\frac{\tau^2 A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|A^{-\frac{1}{2}}f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$\left.-\frac{1}{24}\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{\tau^2 A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}$$

$$+\left\|\left(I-\frac{\tau^2 A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\|\phi\|_H\Bigg]$$

$$\times\left[\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\{\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$\left.\left.\left.+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\right]\right]\right\} \qquad (2.40)$$

Applying $A^{\frac{1}{2}}$ to formulas (2.34) and (2.35), in a similar way, we get

$$\left\|A^{1/2}\mu\right\|_{H\to H}\le\left\|T_\tau\right\|_{H\to H}\left\{\left[\frac{1}{2}\sum_{k=1}^{n}\left|\alpha_k\right|\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|f_{2,2}\right\|_H\tau$$

$$\left.+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\right)$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\|f_s\|_H\,\tau$$

$$\left.+\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\right)$$

$$+\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau\right)$$

$$+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau\right.$$

$$\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau\right)$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau\right.$$

$$\left.\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau\right)\right\}$$

$$+\sum_{k=1}^{n}|\alpha_k|\left\{\frac{3}{2}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$+\frac{7}{6}\left\|\tau^2A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$+\frac{1}{24}\left\|\tau^3A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$\left.-\frac{1}{24}\left\|\tau^3A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}$$

$$+\left\|\left(I-\frac{\tau^2A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}\phi\right\|_H\Bigg]\left[\left\|\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\right.$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\{\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\Bigg\}\Bigg]$$

$$+\left[\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\{\left(\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\right.\right.\right.\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\|f_{2,2}\|_H\,\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\|f_s\|_H\,\tau$$

$$+\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\tau\Bigg)$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left(\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\|f_{2,2}\|_H\,\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\|f_s\|_H\,\tau$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\Bigg)$$

$$+\frac{1}{24}\Bigg(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\|f_s\|_H\,\tau$$

$$+\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\Bigg)$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau\right)$$

$$+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|f_{2,2}\right\|_H\tau+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\left\|f_s\right\|_H\tau\right)\Bigg\}$$

$$+\sum_{k=1}^{n}\beta_k\left\{\left\|\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$+\frac{1}{2}\left\|\tau\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\frac{1}{6}\left\|\tau^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$+\frac{1}{24}\left\|\tau^3\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4$$

$$+\frac{1}{6}\left\|\tau^2A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$\left.-\frac{1}{24}\left\|\tau^3A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\right\}$$

$$+\left\|\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|\psi\right\|_H\Bigg]$$

$$\times\left[\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{3}{2}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{1}{24}\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\Bigg\}\Bigg]\Bigg\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\|f_s\|_H\,\tau+\left\|A^{1/2}\phi\right\|_H+\|\psi\|_H+\tau\left\|f_{2,2}\right\|_H\right\}, \tag{2.41}$$

$$\|\omega\|_H\leq\|T_\tau\|_{H\to H}\left\{\left[\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\{\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\|f_s\|_H\,\tau$$

$$+\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\tau\Bigg)$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg(\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\|f_{2,2}\|_H\,\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\|f_s\|_H\,\tau$$

$$+\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\Bigg)$$

$$+\frac{1}{24}\Bigg(\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\|f_{2,2}\|_H\,\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\|f_s\|_H\,\tau$$

$$+\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\tau\Bigg)$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau\right)$$

$$+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau\right)\Bigg\}$$

$$+\sum_{k=1}^{n}\beta_k\left\{\left\|\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$+\frac{1}{2}\left\|\tau\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\frac{1}{6}\left\|\tau^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$+\frac{1}{24}\left\|\tau^3\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4$$

$$+\frac{1}{6}\left\|\tau^2 A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H \tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$\left.-\frac{1}{24}\left\|\tau^3 A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H \tau\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\right\}$$

$$\left.+\left\|\left(I-\frac{\tau^2 A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\|\psi\|_H\right]$$

$$\times\left[\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.\right.$$

$$\times\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\left[\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|\tau^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}A\right\|_{H\to H}\right.$$

$$\left.\left.\left.+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|\tau^4 A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\right]\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\right\}\right]$$

$$+\left[\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau$$

$$\left.+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\,\tau\right)$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\|f_s\|_H\,\tau$$

$$\left.+\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\,\tau\right)$$

$$+\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau\right)$$

$$+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau\right)$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left.\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\|f_{2,2}\|_H\,\tau+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau\right)\right\}$$

$$+\sum_{k=1}^{n}|\alpha_k|\left\{\frac{3}{2}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right.$$

$$+\frac{7}{6}\left\|\tau^2A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$+\frac{1}{24}\left\|\tau^3A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau$$

$$\left.-\frac{1}{24}\left\|\tau^3A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\tau\right\}$$

$$\left.+\left\|\left(I-\frac{\tau^2A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\phi\right\|_H\right]$$

$$\times\left[\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\{\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$\left.\left.\left.+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\right\}\right]\right\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\|f_s\|_H\tau+\left\|A^{1/2}\phi\right\|_H+\|\psi\|_H+\tau\left\|f_{2,2}\right\|_H\right\}. \tag{2.42}$$

Now, applying Abel's formula to (2.34) and (2.35), we obtain

$$\mu=T_\tau\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{\tau^2A}{4}+\frac{\tau^2A^2}{6}+\frac{\tau^4A^4}{144}\right)^{-1}$$

$$\Bigg\{\Bigg[\frac{1}{2}\sum_{k=1}^{n}\alpha_k\Bigg\{\Bigg[\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)-\frac{7}{6}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\Bigg]$$

$$\times\Bigg[\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\tau f_{2,2}$$

$$+\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-1}\Bigg[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg(\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)(f_s-f_{s-1})$$

$$+\left(\tilde{R}_5R_6-R_5R_6\right)f_{\left[\frac{\lambda_k}{\tau}\right]-2}+\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)f_1\Bigg)\Bigg]+\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}R_0f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau\Bigg]$$

$$+\left[1-\frac{3}{2}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]$$

$$\times\Bigg[\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\tau f_{2,2}$$

$$+\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-1}\sum_{k=1}^{n}\alpha_k\Bigg[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg(\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)(f_s-f_{s-1})+\Bigg(\left(R_6-R_6\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}$$

$$+\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)f_1\Bigg]\Bigg]\Bigg\}+\sum_{k=1}^{n}\alpha_k\Bigg\{\frac{3}{2}\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2f_{\left[\frac{\lambda_k}{\tau}\right]}+\frac{7}{6}\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3f'_{\left[\frac{\lambda_k}{\tau}\right]}$$

$$+\frac{1}{24}\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4f''_{\left[\frac{\lambda_k}{\tau}\right]}-\frac{1}{24}A\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4f_{\left[\frac{\lambda_k}{\tau}\right]}\Bigg\}+\phi\Bigg]$$

$$\times\Bigg[1-\frac{1}{2}\sum_{k=1}^{n}\beta_k\Bigg\{iA^{-\frac{1}{2}}\left[\frac{1}{\tau}-\frac{1}{2}A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]$$

$$\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]+iA^{-\frac{1}{2}}\left[-A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{6}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]$$

$$\times\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\Bigg\}\Bigg]$$

$$+\left[\frac{1}{2}\sum_{k=1}^{n}\beta_k\left\{\left[\frac{1}{\tau}-\frac{1}{2}A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]\right.\right.$$

$$\times\left[\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\tau f_{2,2}\right.$$

$$+\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-1}\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left(\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)(f_s-f_{s-1})\right.\right.$$

$$+\left(\tilde{R}_5R_6-R_5R_6\right)f_{\left[\frac{\lambda_k}{\tau}\right]-2}$$

$$\left.\left.+\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)f_1\right)\right]+\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}R_0f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau\right]$$

$$+\left[-A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{6}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\left[\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\tau f_{2,2}\right.$$

$$+\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-1}\sum_{k=1}^{n}\alpha_k\left[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)(f_s-f_{s-1})+\left(\left(R_6-R_6\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right.\right.\right.$$

$$\left.\left.\left.\left.+\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)f_1\right)\right]\right]\right\}+\sum_{k=1}^{n}\beta_k\left\{\tau f_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{2}\tau^2f'_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2\right.$$

$$+\frac{1}{6}\tau^3f''_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3+\frac{1}{24}\tau^4f'''_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4$$

$$\left.\left.-\frac{1}{6}A\tau^3f_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3-\frac{1}{24}A\tau^4f'_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right\}+\psi\right]$$

$$\times\left[\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left\{\left[\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)-\frac{7}{6}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]iA^{-\frac{1}{2}}\right.\right.$$

$$+\left[1-\frac{3}{2}A\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]iA^{-\frac{1}{2}}\Bigg\}\Bigg]\Bigg\}, \qquad (2.43)$$

$$\omega=T_\tau\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{\tau^2A}{4}+\frac{\tau^2A^2}{6}+\frac{\tau^4A^4}{144}\right)^{-1}\times\Bigg\{\Bigg[\frac{1}{2}\sum_{k=1}^{n}\beta_k\Bigg\{\Bigg[\frac{1}{\tau}-\frac{1}{2}A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2$$

$$+\frac{1}{24}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\Bigg]\Bigg[\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\tau f_{2,2}$$

$$+\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-1}\Bigg[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg(\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)(f_s-f_{s-1})$$

$$+\left(\tilde{R}_5R_6-R_5R_6\right)f_{\left[\frac{\lambda_k}{\tau}\right]-2}$$

$$+\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)f_1\Bigg)\Bigg]+\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}R_0f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau\Bigg]$$

$$+\left[-A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{6}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\left[\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]f_{2,2}\tau\right.$$

$$+\left(I-\frac{\tau^2A}{12}\right)^{-1}iA^{-1}\sum_{k=1}^{n}\alpha_k\Bigg[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg(\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)(f_s-f_{s-1})+\Big(\left(R_6-R_6\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}$$

$$+\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)f_1\Bigg]\Bigg]\Bigg\}+\sum_{k=1}^{n}\beta_k\Bigg\{\tau f_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{2}\tau^2f'_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2$$

$$+\frac{1}{6}\tau^3f''_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3+\frac{1}{24}\tau^4f'''_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4$$

$$-\frac{1}{6}A\tau^{3}f_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^{3}-\frac{1}{24}A\tau^{4}f'_{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^{4}\Bigg\}+\psi\Bigg]$$

$$\times\Bigg[1-\frac{1}{2}\sum_{k=1}^{n}\alpha_k\Bigg\{\Bigg[\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)-\frac{7}{6}A\tau^{2}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^{3}\Bigg]\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]$$

$$+\left[1-\frac{3}{2}A\tau^{2}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^{2}+\frac{1}{24}A^{2}\tau^{4}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^{4}\right]\left[R^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\Bigg\}\Bigg]$$

$$+\Bigg[\frac{1}{2}\sum_{k=1}^{n}\alpha_k\Bigg\{\Bigg[\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)-\frac{7}{6}A\tau^{2}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^{3}\Bigg]$$

$$\times\Bigg[\left(I-\frac{\tau^{2}A}{12}\right)^{-1}iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5-R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\tau f_{2,2}$$

$$+\left(I-\frac{\tau^{2}A}{12}\right)^{-1}iA^{-1}\Bigg[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg(\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right)\left(f_s-f_{s-1}\right)$$

$$+\left(\tilde{R}_5R_6-R_5R_6\right)f_{\left[\frac{\lambda_k}{\tau}\right]-2}$$

$$+\left(\tilde{R}_5R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_5R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)f_1\Bigg)\Bigg]+\left(I-\frac{\tau^{2}A}{12}\right)^{-1}iA^{-\frac{1}{2}}R_0f_{\left[\frac{\lambda_k}{\tau}\right]-1}\tau\Bigg]$$

$$+\left[1-\frac{3}{2}A\tau^{2}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^{2}+\frac{1}{24}A^{2}\tau^{4}\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^{4}\right]$$

$$+\Bigg[\left(I-\frac{\tau^{2}A}{12}\right)^{-1}iA^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\tau f_{2,2}$$

$$+\left(I-\frac{\tau^{2}A}{12}\right)^{-1}iA^{-1}\sum_{k=1}^{n}\alpha_k\Bigg[\sum_{s=2}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg(\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)\left(f_s-f_{s-1}\right)+\Bigg(\left(R_6-R_6\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}$$

$$+\left(R_6\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}-R_6R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)f_1\Big]\Big]\Big\}+\sum_{k=1}^{n}\alpha_k\left\{\frac{3}{2}\tau^2\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2 f_{\left[\frac{\lambda_k}{\tau}\right]}+\frac{7}{6}\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3 f'_{\left[\frac{\lambda_k}{\tau}\right]}\right.$$

$$\left.+\frac{1}{24}\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4 f''_{\left[\frac{\lambda_k}{\tau}\right]}-\frac{1}{24}A\tau^4\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4 f_{\left[\frac{\lambda_k}{\tau}\right]}\right\}+\phi\Big]$$

$$\times\left[\frac{1}{2}\sum_{k=1}^{n}\beta_k\left\{\left[\frac{1}{\tau}-\frac{1}{2}A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^2+\frac{1}{24}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^4\right]\left[R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5+R^{\left[\frac{\lambda_k}{\tau}\right]-1}R_5\right]\right.\right.$$

$$\left.\left.+\left[-A\tau\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)+\frac{1}{6}A^2\tau^3\left(\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right)^3\right]\left[R^{\left[\frac{\lambda_k}{\tau}\right]}+R^{\left[\frac{\lambda_k}{\tau}\right]}\right]\right\}\right]\Big\}. \tag{2.44}$$

Let us obtain the estimates for $\|A\mu\|_H$ and $\|A^{1/2}\omega\|_H$ . First, applying $A$ to formula (2.42), using estimates (2.31), (2.32) and the triangle inequality, we get

$$\|A\mu\|_{H\to H}\le\|T_\tau\|_{H\to H}\left\{\left[\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right]\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_s-f_{s-1}\right\|_H\tau$$

$$+\left[\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\|f_1\|_H$$

$$\left.+\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\right)$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\|f_s-f_{s-1}\|_H$$

$$+\left[\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5 R_6\right\|_{H\to H}\Bigg]\|f_1\|_H$$

$$+\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\Bigg)$$

$$+\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau\right.$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\Bigg]\|f_s-f_{s-1}\|_H$$

$$+\left\|i\tau A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\Bigg]\|f_1\|_H\Bigg)$$

$$+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau\right.$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\Bigg]\|f_s-f_{s-1}\|_H$$

$$+\left\|i\tau A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_1\|_H\Bigg)$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left(\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\Bigg]\|f_s-f_{s-1}\|_H$$

$$+\left\|i\tau^5A^{\frac{5}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\Bigg]\|f_1\|_H\Bigg)\Bigg\}$$

$$+\sum_{k=1}^{n}|\alpha_k|\left\{\frac{3}{2}\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\right.$$

$$+\frac{7}{6}\left\|\tau^3A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H$$

$$+\frac{1}{24}\left\|\tau^4A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H$$

$$\left.+\frac{1}{24}\left\|\tau^5A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\right\}$$

$$+\left\|\left(I-\frac{\tau^2A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\|A\phi\|_H\Bigg]$$

$$\times\Bigg[\left\|\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\{\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\Bigg\}\Bigg]$$

$$+\left[\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\left\{\left(\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\right.\right.\right.\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\Bigg]\left\|f_s-f_{s-1}\right\|_H\tau$$

$$+\left[\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\|f_1\|_H$$

$$\left.+\left\|\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right)$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\|f_s-f_{s-1}\|_H\,\tau$$

$$+\left[\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\Bigg]\|f_1\|_H$$

$$+\left\|\tau^2A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\Bigg)$$

$$+\frac{1}{24}\Bigg(\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\Bigg]\|f_s-f_{s-1}\|_H\tau$$

$$+\Bigg[\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}$$

$$+\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\Bigg]\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\Bigg]\|f_1\|_H$$

$$+\left\|\tau^5 A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1} R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\right)$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau\right.$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1} R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1} R_6\right\|_{H\to H}\right]\left\|f_s-f_{s-1}\right\|_H$$

$$+\left\|i\tau^2 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1} R_6\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1} R_6\right\|_{H\to H}\right]\left\|f_1\right\|_H\right)$$

$$+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left(\left\|\tau^3 A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_s-f_{s-1}\|_H$$

$$+\left\|i\tau^5 A^{\frac{5}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_1\|_H\right)\right\}$$

$$+\sum_{k=1}^{n}\beta_k\left\{\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$+\frac{1}{2}\left\|\tau^2 A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\frac{1}{6}\left\|\tau^3 A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$+\frac{1}{24}\left\|\tau^4 A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4$$

$$+\frac{1}{6}\left\|\tau^3 A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$-\frac{1}{24}\left\|\tau^4 A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\Bigg\}$$

$$+\left\|\left(I-\frac{\tau^2 A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H\Bigg]$$

$$\times\left[\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{3}{2}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$\left.\left.\left.+\frac{1}{24}\left\|\tau^4 A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\right\}\right]\right\}$$

$$\leq M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\phi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\}. \tag{2.45}$$

Next, applying $A^{1/2}$ to (2.44), using estimates (2.31), (2.32) and the triangle inequality, one can obtain

$$\left\|A^{\frac{1}{2}}\omega\right\|_{H\to H} \le \left\|T_\tau\right\|_{H\to H}\left\{\left[\frac{1}{2}\sum_{k=1}^{n}\left|\beta_k\right|\left\{\left(\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_s-f_{s-1}\right\|_H\tau$$

$$+\left[\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_1\right\|_H$$

$$\left.+\left\|\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\right)$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left(\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_s-f_{s-1}\right\|_H\tau$$

$$+\left[\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_1\right\|_H$$

$$\left.+\left\|\tau^2A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\right)$$

$$+\frac{1}{24}\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\Bigg]\|f_s-f_{s-1}\|_H\,\tau$$

$$+\Bigg[\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}$$

$$+\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\Bigg]\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\Bigg]\|f_1\|_H$$

$$+\left\|\tau^5A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\Bigg)$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\,\tau\right.$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\Bigg]\|f_s-f_{s-1}\|_H$$

$$+\left\|i\tau^2A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_1\|_H\Bigg)$$

$$+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left(\left\|\tau^3A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^3A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_s-f_{s-1}\|_H$$

$$+\left\|i\tau^5A^{\frac{5}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_1\|_H\Bigg)\Bigg\}$$

$$+\sum_{k=1}^{n}\beta_k\left\{\left\|\tau A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right.$$

$$+\frac{1}{2}\left\|\tau^2 A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2$$

$$+\frac{1}{6}\left\|\tau^3 A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$+\frac{1}{24}\left\|\tau^4 A^{\frac{1}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4$$

$$+\frac{1}{6}\left\|\tau^3 A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3$$

$$\left.-\frac{1}{24}\left\|\tau^4 A^{\frac{3}{2}}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\right\}$$

$$\left.+\left\|\left(I-\frac{\tau^2 A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H\right]$$

$$\times\left[1+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left\{\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]$$

$$+\left[\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|\tau^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}A\right\|_{H\to H}\right.$$

$$\left.+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|\tau^4A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\right]\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\Bigg\}\Bigg]$$

$$+\left[\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\right]\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_s-f_{s-1}\right\|_H\tau$$

$$+\left[\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}+\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_5R_6\right\|_{H\to H}\right]\left\|f_1\right\|_H$$

$$\left.+\left\|\tau A^{\frac{1}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\right)$$

$$+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5 R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5 R_6\right\|_{H\to H}\right]\left\|f_s-f_{s-1}\right\|_H$$

$$+\left[\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5 R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5 R_6\right\|_{H\to H}\right]\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5 R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5 R_6\right\|_{H\to H}\right]\left\|f_1\right\|_H$$

$$\left.+\left\|\tau^3 A^{\frac{3}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_0\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\right)$$

$$+\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau\right.$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_s-f_{s-1}\|_H$$

$$+\left\|i\tau A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_1\|_H\right)$$

$$+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|\tau^2A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}$$

$$\times\left(\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau\right.$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_s-f_{s-1}\|_H$$

$$+\left\|i\tau A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_1\|_H\Bigg)$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left(\left\|\tau^4 A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\right.$$

$$\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}f_{2,2}\right\|_H\tau$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^4 A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\right\|_{H\to H}\left\|\tau^4 A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_s-f_{s-1}\|_H$$

$$+\left\|i\tau^5 A^{\frac{5}{2}}\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-2}\right\|_H$$

$$+\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4 A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^4 A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_6\right\|_{H\to H}\right]\|f_1\|_H\Bigg)\Bigg\}$$

$$+\sum_{k=1}^{n}|\alpha_k|\left\{\frac{3}{2}\left\|\tau^2 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\right.$$

$$+\frac{7}{6}\left\|\tau^3 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|f'_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H$$

$$+\frac{1}{24}\left\|\tau^4 A\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|f''_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H$$

$$-\frac{1}{24}\left\|\tau^5 A^2\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\Bigg\}$$

$$+\left\|\left(I-\frac{\tau^2 A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}\right\|_{H\to H}\|A\phi\|_H\Bigg]$$

$$\times\Bigg[\frac{1}{2}\sum_{k=1}^{n}|\beta_k|\Bigg\{\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^{-1}\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\Bigg[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}$$

$$+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_{H\to H}\left\|\tau^3 A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2 A^3}{12}\right)^{-1}R_5\right\|_{H\to H}\Bigg]$$

$$+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|\left\|\tau A\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]$$

$$+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\left\|\tau^3A^2\left(I+\frac{i\tau A^{1/2}}{2}+\frac{\tau^2A^3}{12}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right]\Bigg\}\Bigg]\Bigg\}$$

$$\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\phi\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\}. \tag{2.46}$$

Now, Let us prove the estimates (2.36), (2.37), and (2.38) for $k\geq 2$. By using formula (2.33), we obtain

$$\frac{u_k+u_{k-1}}{2}=\frac{1}{2}\left[J_1R^{k-1}+J_1\tilde{R}^{k-1}\right]\phi+\frac{1}{2}\left[J_1\tilde{R}^{k-1}-J_1R^{k-1}\right]iA^{-1/2}\psi$$

$$+\frac{1}{2}\left[J_2\tilde{R}^{k-1}-J_2R^{k-1}\right]\tau^2f_{2,2}+\frac{1}{2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2f_{k-1}$$
$$+\frac{1}{2}\sum_{s=1}^{k-2}\left[J_2\tilde{R}^{k-1-s}-J_2R^{k-1-s}\right]\tau^2f_s. \tag{2.47}$$

Using formula (2.47), estimate (2.31) and the triangle inequality, we get (2.36)

$$\left\|\frac{u_k+u_{k-1}}{2}\right\|_H\leq\frac{1}{2}\left(\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}+\|J_1\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}\right)\|\phi\|_H$$

$$+\frac{1}{2}|i|\left(\|J_1\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\left\|A^{-1/2}\psi\right\|_H$$

$$+\frac{1}{2}\left(\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\tau\left\|A^{-1/2}f_{2,2}\right\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{k-1}\right\|_H$$

$$+\frac{1}{2}\sum_{s=1}^{k-2}\left[\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|\tilde{R}^{k-1-s}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1-s}\right\|_{H\to H}\right]\left\|A^{-1/2}f_s\right\|_H\tau$$

$$\le M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\|\phi\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{2,2}\right\|_H\right\}.$$

Applying $A^{1/2}$ to (2.47), using estimate (2.31) and the triangle inequality, we obtain

$$\left\|A^{1/2}\frac{u_k+u_{k-1}}{2}\right\|_H\le\frac{1}{2}\left(\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}+\|J_1\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}\right)$$

$$\times\left\|A^{1/2}\phi\right\|_H+\frac{1}{2}|i|\left(\|J_1\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\|\psi\|_H$$

$$+\frac{1}{2}\left(\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\tau\left\|f_{2,2}\right\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|f_{k-1}\right\|_H$$

$$+\frac{1}{2}\sum_{s=1}^{k-2}\left[\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|\tilde{R}^{k-1-s}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau$$

$$\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\left\|A^{1/2}\phi\right\|_H+\|\psi\|_H+\tau\left\|f_{2,2}\right\|_H\right\}$$

for $k\ge 2$. Combining these estimates for $\left\|A^{1/2}\frac{u_k+u_{k-1}}{2}\right\|_H$ for any $k\ge 2$, we have

$$\max_{1\le k\le N}\left\|A^{1/2}\frac{u_k+u_{k-1}}{2}\right\|_H\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\left\|A^{1/2}\phi\right\|_H+\|\psi\|_H+\tau\left\|f_{2,2}\right\|_H\right\}.$$

(2.48)

Now, applying Abel's formula to (2.47), we can obtain

$$\frac{u_k+u_{k-1}}{2}=\frac{1}{2}\left[J_1R^{k-1}+J_1\tilde{R}^{k-1}\right]\phi+\frac{1}{2}\left[J_1\tilde{R}^{k-1}-J_1R^{k-1}\right]iA^{-1/2}\psi$$

$$+\frac{1}{2}\left[J_2\tilde{R}^{k-1}-J_2R^{k-1}\right]\tau^2f_{2,2}+\frac{1}{2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2f_{k-1}$$

$$+\frac{1}{2}A^{-1}\left(-\sum_{s=2}^{k-2}\left[\tilde{R}^{k-s}+R^{k-s}\right](f_s-f_{s-1})++\left(R+R\right)f_{k-2}-\left(\tilde{R}^{k-1}+R^{k-1}\right)f_1\right). \tag{2.49}$$

Using estimate (2.31), applying $A$ to formula (2.49), and the triangle inequality, we get

$$\left\|A\frac{u_k+u_{k-1}}{2}\right\|_H \le \frac{1}{2}\left(\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}+\|J_1\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}\right)$$

$$\times\|A\phi\|_H+\frac{1}{2}|i|\left(\|J_1\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\left\|A^{1/2}\psi\right\|_H$$

$$+\frac{1}{2}\left(\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\tau\left\|A^{1/2}f_{2,2}\right\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{k-1}\right\|_H$$

$$+\frac{1}{2}\left(\sum_{s=2}^{k-2}\left(\left\|\tilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right)\|f_s-f_{s-1}\|_H+\right.$$

$$\left.+\left(\left\|\tilde{R}\right\|_{H\to H}+\|R\|_{H\to H}\right)\|f_{k-2}\|_H+\left(\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\left\|R^{k-1}\right\|_{H\to H}\right)\|f_1\|_H\right)$$

$$\le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\phi\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\}.$$

Combining the estimates for $\left\|A\frac{u_k+u_{k-1}}{2}\right\|_H$ for any $k\ge 2$, we obtain

$$\max_{1\le k\le N-1}\left\|A\frac{u_k+u_{k-1}}{2}\right\|_H \le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\left\|A^{1/2}\psi\right\|_H+\|A\phi\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\}. \tag{2.50}$$

Using formula (2.33), we get

$$\frac{u_{k+1}-u_{k-1}}{2\tau}=\frac{1}{2\tau}\left[J_3R^{k-1}+J_3\tilde{R}^{k-1}\right]\phi$$

$$+\frac{1}{2\tau}\left[J_3\tilde{R}^{k-1}-J_3R^{k-1}\right]iA^{-1/2}\psi+\frac{1}{2\tau}\left[\tilde{R}^{k}+R^{k}\right]\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2 f_{2,2}$$

$$+\frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2 f_k+\frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\left[\tilde{R}+R\right]\tau^2 f_{k-1}$$

$$+\frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\sum_{s=1}^{k-2}\left[\tilde{R}^{k-s}+R^{k-s}\right]f_s\tau^2. \tag{2.51}$$

Next, using formula (2.51), estimate (2.31) and the triangle inequality, we obtain

$$\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H\le\frac{1}{2\tau}\left(\left\|A^{-1/2}J_3\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}+\left\|A^{-1/2}J_3\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\left\|A^{1/2}\phi\right\|_H$$

$$+\frac{1}{2}|i|\left(\left\|A^{-1/2}J_3\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\left\|A^{-1/2}J_3\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\|\psi\|_H$$

$$+\frac{1}{2}\left(\left\|\tilde{R}^{k}\right\|_{H\to H}+\left\|R^{k}\right\|_{H\to H}\right)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|f_{2,2}\|_H\,\tau$$

$$+\frac{1}{2}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|f_k\|_H\,\tau$$

$$+\frac{1}{2}\left(\left\|\tilde{R}\right\|_{H\to H}+\left\|R\right\|_{H\to H}\right)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|f_{k-1}\|_H\,\tau+\frac{1}{2}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\sum_{s=1}^{k-2}\left(\left\|\tilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right)\|f_s\|_H\,\tau$$

$$\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\,\tau+\left\|A^{1/2}\phi\right\|_H+\|\psi\|_H+\tau\left\|f_{2,2}\right\|_H\right\}. \tag{2.52}$$

Combining the estimates for $\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H$ for any $k\ge 2,$ we have

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H \le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\,\tau+\left\|A^{1/2}\phi\right\|_H+\|\psi\|_H+\tau\|f_{2,2}\|_H\right\}. \tag{2.53}$$

From estimates (2.48) and (2.53), we obtain (2.37).

Now, applying Abel's formula to (2.51) , we have

$$\frac{u_{k+1}-u_{k-1}}{2\tau}=\frac{1}{2\tau}\left[J_3R^{k-1}+J_3\tilde{R}^{k-1}\right]\phi+\frac{1}{2\tau}\left[J_3\tilde{R}^{k-1}-J_3R^{k-1}\right]iA^{-1/2}\psi$$

$$+\frac{1}{2\tau}\left[\tilde{R}^k+R^k\right]\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2 f_{2,2}+\frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2 f_k$$

$$+\frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\left[\tilde{R}+R\right]\tau^2 f_{k-1}+\frac{1}{2}\tau\sum_{s=2}^{k-2}\left[J_2\tilde{R}^{k-s}+J_2R^{k-s}\right](f_s-f_{s-1})$$

$$-\frac{1}{2}\tau\left[J_2R+J_2R\right]f_{k-2}+\frac{1}{2}\tau\left[J_2\tilde{R}^{k-1}+J_2R^{k-1}\right]f_1\Big\},2\le k\le N. \tag{2.54}$$

Next, applying $A^{1/2}$ to formula (2.54), using estimate (2.31) and the triangle inequality, we get

$$\left\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H\le\frac{1}{2\tau}\left(\left\|A^{-1/2}J_3\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}+\left\|A^{-1/2}J_3\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\|A\phi\|_H$$

$$+\frac{1}{2}|i|\left(\left\|A^{-1/2}J_3\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\left\|A^{-1/2}J_3\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\left\|A^{1/2}\psi\right\|_H$$

$$+\frac{1}{2}\left(\left\|\tilde{R}^k\right\|_{H\to H}+\left\|R^k\right\|_{H\to H}\right)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}f_{2,2}\right\|_H\tau$$

$$+\frac{1}{2}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}f_k\right\|_H\tau$$

$$+\frac{1}{2}\left(\left\|\tilde{R}\right\|_{H\to H}+\|R\|_{H\to H}\right)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|A^{1/2} f_{k-1}\right\|_H \tau+\frac{1}{2}\sum_{s=2}^{k-2}\left(\left\|\tau A^{1/2} J_2\right\|_{H\to H}\left\|\tilde{R}^{k-s}\right\|_{H\to H}+\left\|\tau A^{1/2} J_2\right\|_{H\to H}\left\|R^{k-s}\right\|_{H\to H}\right)\left\| f_s-f_{s-1}\right\|_H$$

$$+\frac{1}{2}\left[\left\|\tau A^{1/2} J_2\right\|_{H\to H}\left\|R\right\|_{H\to H}+\left\|\tau A^{1/2} J_2\right\|_{H\to H}\left\|R\right\|_{H\to H}\right]\left\| f_{k-2}\right\|_H$$

$$+\frac{1}{2}\left[\left\|\tau A^{1/2} J_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}+\left\|\tau A^{1/2} J_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right]\left\| f_1\right\|_H$$

$$\le M\left\{\sum_{s=2}^{N-1}\left\| f_s-f_{s-1}\right\|_H+\left\| f_1\right\|_H+\left\|A\phi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2} f_{2,2}\right\|_H\right\}.$$

Combining the estimates for $\left\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H$ for any $k$, we get

$$\max_{1\le k\le N-1}\left\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H\le M\left\{\sum_{s=2}^{N-1}\left\| f_s-f_{s-1}\right\|_H+\left\| f_1\right\|_H+\left\|A^{1/2}\psi\right\|_H+\left\|A\phi\right\|_H+\tau\left\|A^{1/2} f_{2,2}\right\|_H\right\}. \tag{2.55}$$

From estimates (2.50) and (2.55) it follows estimate (2.38). Theorem 2.5 is proved.

The stability estimates for the solutions of these difference schemes are obtained. We denote the mesh function of approximation by $a^\tau=\{a_k\}_{k=0}^N$. Then $\left\|(I+\frac{\tau^2A^2}{4}+\frac{\tau^2A^3}{6}+\frac{\tau^4A^6}{144})^{-1}a_1\right\|_H\sim\left\|a_1\right\|_H=o(\tau)$ if we assume that $\tau\left\|Aa_1\right\|_H$ tends to $0$ as $\tau\to 0$ not slower than $\left\|a_1\right\|_H$. It takes place in applications by supplementary restriction of the smooth property of the data of space variables.

Let us give a number of corollaries of Theorem 2.5.

First, we consider the mixed problem for hyperbolic equation (2.21).The discretization of problem (2.21) is carried out in two steps. First step is exactly the same with Section 2.2. In the second step, we replace problem (2.23) by difference scheme (2.56)

$$\begin{cases}
\tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\frac{5}{6}A_h^x u_k^h(x)+\frac{1}{12}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)\\
-\frac{1}{72}\tau^2(A_h^x)^2u_k^h(x)+\frac{\tau^2}{144}(A_h^x)^2\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)=f_k^h(x),\\
f_k^h(x)=\frac{5}{6}f^h(t_k,x)+\frac{1}{12}(f^h(t_{k+1},x)+f^h(t_{k-1},x))\\
+\frac{\tau^2}{72}\left(-A_h^xf^h(t_k,x)+f_{tt}^h(t_k,x)\right)-\frac{\tau^2}{144}(-A_h^x(f^h(t_{k+1},x)+f^h(t_{k-1},x))\\
+f_{tt}^h(t_{k+1},x)+f_{tt}^h(t_{k-1},x)\Big),x\in[0,1]_h,t_k=k\tau,N\tau=1,1\le k\le N-1,\\
u_0^h(x)=\left(I-\frac{i\tau}{2}(A_h^x)^{1/2}+\frac{\tau^2}{12}(A_h^x)^3\right)^{-1}\\
\sum_{k=1}^n\alpha_k\left\{\left(\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)-\frac{7\tau^2}{6}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3\right)\left(u_{[\lambda_j/\tau]}^h(x)-u_{[\lambda_j/\tau]-1}^h(x)\right)\right.\\
+\left(1-\frac{3\tau^2}{2}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2+\frac{\tau^4}{24}(A_h^x)^2\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\right)u_{[\lambda_j/\tau]}^h\\
+\frac{3\tau^2}{2}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2f_{[\lambda_j/\tau]}+\frac{7\tau^3}{6}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3f'_{[\lambda_j/\tau]}\\
\left.+\frac{\tau^4}{24}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4f''_{[\lambda_j/\tau]}-\frac{\tau^4}{24}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4f_{[\lambda_j/\tau]}\right\}+\varphi^h(x),x\in[0,1]_h,\\
\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right)=\left(I-\frac{\tau^2}{12}A_h^x\right)\left(I+\frac{i\tau}{2}(A_h^x)^{1/2}+\frac{\tau^2}{12}(A_h^x)^3\right)^{-1}\\
\times\sum_{k=1}^n\beta_k\left\{\left[\frac{1}{\tau}-\frac{\tau}{2}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2+\frac{\tau^3}{24}(A_h^x)^2\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\right]\right.\\
\times(u_{[\lambda_j/\tau]}-u_{[\lambda_j/\tau]-1})+\left(-A_h^x\tau\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)+\frac{\tau^3}{6}(A_h^x)^2\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3\right)u_{[\lambda_j/\tau]}\\
+\tau f_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)+\frac{\tau^2}{2}f'_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2\\
+\frac{\tau^3}{6}f''_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3+\frac{\tau^4}{24}f'''_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\\
\left.-\frac{\tau^3}{6}A_h^xf_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3-\frac{\tau^4}{24}A_h^xf'_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\right\}+\psi^h(x),x\in[0,1]_h.
\end{cases}$$

(2.56)

**Theorem 2.6** Let $\tau$ and $h$ be sufficiently small numbers. Then, the solution of difference scheme (2.56) satisfies the following stability estimates:

$$\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{L_{2h}}+\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^1}+\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{L_{2h}}$$

$$\le M_1\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\phi^h\right\|_{W_{2h}^1}+\tau\left\|f_{2,2}^h\right\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{W_{2h}^1}+\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}$$

$$\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\phi^h\right\|_{W_{2h}^2}+\tau\left\|f_{2,2}^h\right\|_{W_{2h}^1}\right].$$

Here $M_1$ does not depend on $\tau$, $h$, $\phi^h(x)$, $\psi^h(x)$, $f_{2,2}^h$ and $f_k^h, 1 \le k < N$.

The proof of Theorem 2.6 is based on the abstract Theorem 2.5 and the symmetry property of the operator $A_h^x$ defined by (2.22).

Next, we consider the mixed boundary value problem for the multi-dimensional hyperbolic equation (2.25).

The discretization of problem (2.25) is carried out in two steps and first step is exactly the same with Section 2.2

In the second step, we replace problem (2.27) by the difference scheme (2.57)

$$\begin{cases}
\tau^{-2}\left(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right) + \frac{5}{6}A_h^x u_k^h(x) + \frac{1}{12}A_h^x\left(u_{k+1}^h(x) + u_{k-1}^h(x)\right) \\
-\frac{1}{72}\tau^2(A_h^x)^2 u_k^h(x) + \frac{\tau^2}{144}(A_h^x)^2\left(u_{k+1}^h(x) + u_{k-1}^h(x)\right) = f_k^h(x) \\
f_k^h(x) = \frac{5}{6}f^h(t_k,x) + \frac{1}{12}(f^h(t_{k+1},x) + f^h(t_{k-1},x)) + \frac{\tau^2}{72}\left(-A_h^x f^h(t_k,x) + f_{tt}^h(t_k,x)\right) \\
-\frac{\tau^2}{144}\left(-A_h^x(f^h(t_{k+1},x) + f^h(t_{k-1},x)) + f_{tt}^h(t_{k+1},x) + f_{tt}^h(t_{k-1},x)\right), x \in \Omega_h, \\
t_k = k\tau, 1 \le k \le N-1, N\tau = 1, \\
u_0^h(x) = \left(I - \frac{i\tau}{2}(A_h^x)^{1/2} + \frac{\tau^2}{12}(A_h^x)^3\right)^{-1} \\
\sum_{k=1}^{n}\alpha_k\left\{\left(\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right) - \frac{7\tau^2}{6}A_h^x\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^3\right)\left(u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)\right)\right. \\
+\left(1 - \frac{3\tau^2}{2}A_h^x\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^2 + \frac{\tau^4}{24}(A_h^x)^2\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^4\right)u_{[\lambda_j/\tau]}^h \\
+\frac{3\tau^2}{2}\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^2 f_{[\lambda_j/\tau]} + \frac{7\tau^3}{6}\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^3 f'_{[\lambda_j/\tau]} \\
\left.+\frac{\tau^4}{24}\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^4 f''_{[\lambda_j/\tau]} - \frac{\tau^4}{24}A_h^x\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^4 f_{[\lambda_j/\tau]}\right\} + \varphi^h(x), x \in \widetilde{\Omega}_h, \\
\tau^{-1}\left(u_1^h(x) - u_0^h(x)\right) = \left(I - \frac{\tau^2}{12}A_h^x\right)\left(I + \frac{i\tau}{2}(A_h^x)^{1/2} + \frac{\tau^2}{12}(A_h^x)^3\right)^{-1} \\
\times\sum_{k=1}^{n}\beta_k\left\{\left[\frac{1}{\tau} - \frac{\tau}{2}A_h^x\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^2 + \frac{\tau^3}{24}(A_h^x)^2\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^4\right]\right. \\
\times(u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}) + \left(-A_h^x\tau\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right) + \frac{\tau^3}{6}(A_h^x)^2\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^3\right)u_{[\lambda_j/\tau]} \\
+\tau f_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right) + \frac{\tau^2}{2}f'_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^2 \\
+\frac{\tau^3}{6}f''_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^3 + \frac{\tau^4}{24}f'''_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^4 \\
\left.-\frac{\tau^3}{6}A_h^x f_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^3 - \frac{\tau^4}{24}A_h^x f'_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau} - [\lambda_j/\tau]\right)^4\right\} + \psi^h(x), x \in \widetilde{\Omega}_h.
\end{cases} \tag{2.57}$$

**Theorem 2.7** Let $\tau$ and $|h|$ be sufficiently small numbers. Then, the solution of difference scheme (2.57) satisfies the following stability estimates:

$$\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^1}+\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{L_{2h}}$$
$$\le M_1\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\phi^h\right\|_{W_{2h}^1}+\tau\left\|f_{2,2}^h\right\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{W_{2h}^1}+\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}$$

$$\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\phi^h\right\|_{W_{2h}^2}+\tau\left\|f_{2,2}^h\right\|_{W_{2h}^1}\right].$$

Here $M_1$ does not depend on $\tau$, $h$, $\phi^h(x)$, $\psi^h(x)$, $f_{2,2}^h$ and $f_k^h, 1\le k<N$.

The proof of Theorem 2.7 is based on the abstract Theorem 2.5, the symmetry property of the operator $A_h^x$ defined by formula (2.26) and Theorem 2.4, on the coercivity inequality for the solution of the elliptic difference problem in $L_{2h}$.

## CHAPTER 3

# NUMERICAL RESULTS

### 3.1 The Problem

In this section we consider the multipoint nonlocal boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = (t^2 - 2t + 3)\sin x, \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \frac{1}{8}u(1,x) + \frac{1}{8}u(\frac{1}{2},x) + \frac{31}{32}\sin x, 0 \le x \le \pi, \\ u_t(0,x) = \frac{1}{8}u_t(1,x) + \frac{1}{8}u_t(\frac{1}{2},x) - \frac{15}{8}\sin x, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \ \ 0 \le t \le 1. \end{cases} \tag{3.1}$$

for one dimensional hyperbolic equation. The exact solution of (3.1) is

$u(t,x) = (1-t)^2 \sin x.$

For approximate solution of problem (3.1) we consider the set $[0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters $\tau$ and $h$ : $[0,1]_\tau \times [0,\pi]_h = \{(t_k, x_n) : \ t_k = k\tau, 0 \le k \le N, N\tau = 1, x_n = nh, \ \ 0 \le n \le M, \quad Mh = \pi\}.$ and we will use the third and fourth order of accuracy difference schemes respectively. We will use a procedure of modified Gauss elimination method with respect to $n$ with matrix coefficients.Here, theoretical results are supported by this numerical experiments. Results are presented in the last chapter.

### 3.2 Third Order of Accuracy Difference Scheme Solution for Multipoint Nonlocal BVP of the Hyperbolic Type

Using these simple formulas

$$\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) = O(h^2), \tag{3.2}$$

$$\frac{u(x_{n+2})-4u(x_{n+1})+6u(x_n)-4u(x_{n-1})+u(x_{n-2})}{h^4}-u^{(iv)}(x_n)=O(h^2),$$

(3.3)

$$\frac{2u(0)-5u(h)+4u(2h)-u(3h)}{h^2}-u''(0)=O(h^2), \tag{3.4}$$

$$\frac{2u(1)-5u(1-h)+4u(1-2h)-u(1-3h)}{h^2}-u''(1)=O(h^2), \tag{3.5}$$

and difference scheme (2.2), we obtain the third order of accuracy in $t$ and fourth order of accuracy in $x$ difference scheme for the approximate solutions of the nonlocal bvp (3.1)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{2}{3}\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\right)-\frac{1}{6}\left(\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\ +\frac{\tau^2}{12}\left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4}\right)=\phi_n^k, \\ \phi_n^k=\left\{\frac{2}{3}(t_k^2-2t_k+3)+\frac{\tau^2}{12}(t_k-1)\right. \\ \left.+\frac{1}{6}\left(\left(t_{k+1}^2-2t_{k+1}+3\right)+\left(t_{k-1}^2-2t_{k-1}+3\right)\right)\right\}\sin x_n, \\ t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ x_n=nh, 2\le n\le M-2, Mh=\pi, \\ u_n^0-\frac{1}{8}u_n^{(N/2)}-\frac{1}{8}u_n^N=\frac{31}{32}\sin(x_n), 0\le n\le M, \\ (u_n^1-u_n^0)-\frac{\tau^2}{12}\left(\frac{\left(u_{n+1}^1-u_{n+1}^0\right)-2\left(u_n^1-u_n^0\right)+\left(u_{n-1}^1-u_{n-1}^0\right)}{h^2}\right) \\ +\frac{\tau^4}{144}\frac{(u_{n+2}^1-u_{n+2}^0)-4\left(u_{n+1}^1-u_{n+1}^0\right)+6\left(u_n^1-u_n^0\right)-4\left(u_{n-1}^1-u_{n-1}^0\right)+(u_{n-2}^1-u_{n-2}^0)}{h^4}=\phi_n^N, \\ \phi_n^N=(-\tau+\tau^2-\frac{\tau^3}{6}+\frac{5}{24}\tau^4-\frac{7}{36}\tau^5+\frac{\tau^6}{288}+\frac{\tau^7}{216})\sin x+\frac{\tau}{8}\sin x \\ 2\le n\le M-2, \\ u_0=u_M=0, 0\le k\le N, \\ u_3=4u_2-5u_1, u_{M-3}=4u_{M-2}-5u_{M-1}, 0\le k\le N. \end{cases} \tag{3.6}$$

We have an $(N+1)\times(N+1)$ system of linear equations. We resort the system (3.6) and write in the matrix form

$$
\begin{cases}
\left(\frac{\tau^2}{12h^4}\right)u_{n+2}^{k+1} + \left(-\frac{1}{6h^2} - \frac{\tau^2}{3h^4}\right)u_{n+1}^{k+1} + \left(-\frac{2}{3h^2}\right)u_{n+1}^{k} \\
+\left(-\frac{1}{6h^2}\right)u_{n+1}^{k-1} + \left(\frac{1}{\tau^2} + \frac{1}{3h^2} + \frac{\tau^2}{2h^4}\right)u_{n}^{k+1} + \left(-\frac{2}{\tau^2} + \frac{4}{3h^2}\right)u_{n}^{k} \\
+\left(\frac{1}{\tau^2} + \frac{1}{3h^2}\right)u_{n}^{k-1} + \left(-\frac{1}{6h^2} - \frac{\tau^2}{3h^4}\right)u_{n-1}^{k+1} + \left(-\frac{2}{3h^2}\right)u_{n-1}^{k} \\
+\left(-\frac{1}{6h^2}\right)u_{n-1}^{k-1} + \left(\frac{\tau^2}{12h^4}\right)u_{n-2}^{k+1} = \varphi_n^k, 1 \le k \le N-1, 2 \le n \le M-2, \\
\varphi_n^k = \left\{\frac{2}{3}(t_k^2 - 2t_k + 3) + \frac{\tau^2}{12}(t_k - 1)^2 + \right. \\
\left.\frac{1}{6}[(t_{k+1}^2 - 2t_{k+1} + 3) + (t_{k-1}^2 - 2t_{k-1} + 3)]\right\} \sin x_n, \\
t_k = k\tau, 1 \le k \le N-1, N\tau = 1, x_n = nh, 2 \le n \le M-2, Mh = \pi, \\
u_n^0 - \frac{1}{8}u_n^{(N/2)} - \frac{1}{8}u_n^N = \frac{31}{32}\sin(x_n), 0 \le n \le M, \\
\left(\frac{\tau^4}{144h^4}\right)u_{n+2}^1 + \left(-\frac{\tau^4}{144h^4}\right)u_{n+2}^0 + \left(-\frac{\tau^4}{36h^4} - \frac{\tau^2}{12h^2}\right)u_{n+1}^1 \\
+\left(\frac{\tau^4}{36h^4} + \frac{\tau^2}{12h^2}\right)u_{n+1}^0 + \left(1 + \frac{\tau^2}{6h^2} + \frac{\tau^4}{24h^4}\right)u_n^1 + \left(-1 - \frac{\tau^2}{6h^2} - \frac{\tau^4}{24h^4}\right)u_n^0 \\
+\left(-\frac{\tau^4}{36h^4} - \frac{\tau^2}{12h^2}\right)u_{n-1}^1 + \left(\frac{\tau^4}{36h^4} + \frac{\tau^2}{12h^2}\right)u_{n-1}^0 + \left(\frac{\tau^4}{144h^4}\right)u_{n-2}^1 + \left(-\frac{\tau^4}{144h^4}\right)u_{n-2}^0 = \varphi_n^N, \\
\varphi_n^N = \left(-2\tau + \tau^2 - \frac{\tau^3}{6} + \frac{5}{24}\tau^4 - \frac{7}{36}\tau^5 + \frac{\tau^6}{288} + \frac{\tau^7}{216}\right)\sin(x_n) \\
+\frac{\tau}{8}\sin(x_n), 2 \le n \le M-2, u_0 = u_M = 0, 0 \le k \le N, \\
u_3 = 4u_2 - 5u_1, u_{M-3} = 4u_{M-2} - 5u_{M-1}, 0 \le k \le N.
\end{cases}
\tag{3.7}
$$

After resorting the system we have the following linear system

$$
\begin{cases}
AU_{n+2} + BU_{n+1} + CU_n + DU_{n-1} + EU_{n-2} = R\phi_n, 2 \le n \le M-2, \\
U_0 = U_M = \widehat{0}, U_3^k = 4U_2^k - 5U_1^k, U_{M-3}^k = 4U_{M-2}^k - 5U_{M-1}^k, 0 \le k \le N.
\end{cases}
\tag{3.8}
$$

Here,

$$
A = \begin{bmatrix}
0 & 0 & 0 & . & 0 & 0 & 0 \\
0 & 0 & x & . & 0 & 0 & 0 \\
0 & 0 & 0 & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . \\
0 & 0 & 0 & . & 0 & x & 0 \\
-r & r & 0 & . & 0 & 0 & 0
\end{bmatrix}_{(N+1)\times(N+1)},
$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ y & w & v & 0 & . & 0 & 0 & 0 & 0 \\ 0 & y & w & v & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & y & w & v \\ s & -s & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$C=\begin{bmatrix} 1 & 0 & 0 & . & -\frac{1}{8} & . & 0 & 0 & -\frac{1}{8} \\ l & n & m & 0 & . & 0 & 0 & 0 & 0 \\ 0 & l & n & m & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & l & n & m \\ -t & t & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$D=B,\ E=A,$$

where

$$x=\frac{\tau^2}{12h^4},v=-\frac{1}{6h^2}-\frac{\tau^2}{3h^4},w=-\frac{2}{3h^2},$$

$$y=-\frac{1}{6h^2},m=\frac{1}{\tau^2}+\frac{1}{3h^2}+\frac{\tau^2}{2h^4},$$

$$n=-\frac{2}{\tau^2}+\frac{4}{3h^2},l=\frac{1}{\tau^2}+\frac{1}{3h^2},$$

$$r=\frac{\tau^4}{144h^4},s=\frac{\tau^2}{12h^2}+\frac{\tau^4}{36h^4}$$

$$t=1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4}.$$

$$R=\begin{bmatrix}1 & . & 0\\ . & . & .\\ 0 & . & 1\end{bmatrix}_{(N+1)\times(N+1)}, \quad \phi_n^k=\begin{bmatrix}\phi_n^0\\ \phi_n^1\\ .\\ \phi_n^N\end{bmatrix}_{(N+1)\times 1}, 0\le k\le N,$$

$$\phi_n^0=\frac{31}{32}\sin\left(x_n\right),$$

$$\phi_n^N=\left(-2\tau+\tau^2-\frac{\tau^3}{6}+\frac{5}{24}\tau^4-\frac{7}{36}\tau^5+\frac{\tau^6}{288}+\frac{\tau^7}{216}\right)\sin\left(x_n\right)+\frac{\tau}{8}\sin\left(x_n\right),$$

$$\phi_n^k=\left\{\frac{2}{3}(t_k^2-2t_k+3)+\frac{\tau^2}{12}(t_{k+1}-1)^2\right.$$

$$\left.+\frac{1}{6}\left[\left(t_{k+1}^2-2t_{k+1}+3\right)+\left(t_{k-1}^2-2t_{k-1}+3\right)\right]\right\}\sin\left(x_n\right), 1\le k\le N-1,$$

and

$$U_s^k=\begin{bmatrix}u_s^0\\ u_s^1\\ .\\ u_s^N\end{bmatrix}_{(N+1)\times 1}, 0\le k\le N, s=n-2,n-1,n,n+1,n+2.$$

For solution of difference equation (3.6), we applied the modified Gauss elimination method. Therefore, we are looking for a solution of the matrix equation by the following form

$$U_n=\alpha_{n+1}U_{n+1}+\beta_{n+1}U_{n+2}+\gamma_{n+1}, n=M-2,...2,1,0 \tag{3.9}$$

where $\alpha_j,\beta_j$ $(j=1,\cdots M$ are $(N+1)\times(N+1)$ square matrices and $\gamma_j$ are $(N+1)\times 1$ column matrices. Then, we obtain formula of $\alpha_{n+1},\beta_{n+1},\gamma_{n+1}$,

$$\begin{cases} \beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A), \\ \alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \gamma_{n+1} = +(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1} \\ \times(R\phi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}), \end{cases} \tag{3.10}$$

where $n = 2 : M-2$ and $\alpha_1,\ \beta_1,\ \gamma_1,\ \alpha_2,\ \beta_2,\ \gamma_2$

$$\alpha_1 = \begin{bmatrix} 0 & . & 0 \\ . & . & . \\ 0 & . & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,\beta_1 = \begin{bmatrix} 0 & . & 0 \\ . & . & . \\ 0 & . & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$\gamma_1 = \gamma_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(N+1)\times 1} ,\quad \alpha_2 = \begin{bmatrix} \frac{4}{5} & 0 & . & 0 \\ 0 & \frac{4}{5} & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \frac{4}{5} \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$\beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & . & 0 \\ 0 & -\frac{1}{5} & . & 0 \\ . & . & . & . \\ 0 & 0 & . & -\frac{1}{5} \end{bmatrix}_{(N+1)\times(N+1)} ,$$

and $U_M = \widehat{0}$. Applying formulas

$$\begin{cases} U_{M-2} = \alpha_{M-1}U_{M-1} + \gamma_{M-1}, \\ U_{M-3} = \alpha_{M-2}U_{M-2} + \beta_{M-2}U_{M-1} + \gamma_{M-2}, \\ U_{M-3} = 4U_{M-2} - 5U_{M-1}, \end{cases}$$

we obtain

$$\left(4I - \alpha_{M-2}\right)U_{M-2} + \left(-5I - \beta_{M-2}\right)U_{M-1} = \gamma_{M-2}$$

or

$$\left(4I - \alpha_{M-2}\right)\left(\alpha_{M-1}U_{M-1} + \gamma_{M-1}\right) - \left(5I + \beta_{M-1}\right)U_{M-1} = \gamma_{M-2}.$$

Thus, we get

$$U_{M-1} = [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}].$$

Finally, using the formula and matrices presented above we obtain the fourth order of accuracy difference scheme solution of nonlocal bvp (3.1). By matlab implementation, we get the numerical results for the solution of (3.10) for different $M$ and $N$ values.

### 3.3 Fourth Order of Accuracy Difference Scheme Solution for Multipoint Nonlocal Boundary Value Problem of the Hyperbolic Type

Applying formulas (3.2), (3.3), (3.4), (3.5) and using difference scheme (2.29), we obtain the fourth order of accuracy difference scheme for the approximate solution of nonlocal bvp (3.1)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{5}{6}\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\right) - \frac{1}{12}\left(\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} + \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\ -\frac{\tau^2}{72}\left(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{h^4}\right) \\ +\frac{\tau^2}{144}\left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4} + \frac{u_{n+2}^{k-1}-4u_{n+1}^{k-1}+6u_n^{k-1}-4u_{n-1}^{k-1}+u_{n-2}^{k-1}}{h^4}\right) = \varphi_n^k, \\ \varphi_n^k = \left\{\frac{5}{6}(t_k^2-2t_k+3) + \right. \\ \left. \frac{1}{12}[(t_{k+1}^2-2t_{k+1}+3)+(t_{k-1}^2-2t_{k-1}+3)]\right\}\sin(x_n), \\ \\ t_k = k\tau,\ 1 \le k \le N-1, N\tau = 1, x_n = nh, 1 \le n \le M-1, Mh = \pi, \\ \varphi_n^0 = \frac{31}{32}\sin(x_n), 0 \le n \le M, \\ (u_n^1-u_n^0) - \frac{\tau^2}{12}\left(\frac{\left(u_{n+1}^1-u_{n+1}^0\right)-2\left(u_n^1-u_n^0\right)+\left(u_{n-1}^1-u_{n-1}^0\right)}{h^2}\right) \\ +\frac{\tau^4}{144}\frac{(u_{n+2}^1-u_{n+2}^0)-4\left(u_{n+1}^1-u_{n+1}^0\right)+6\left(u_n^1-u_n^0\right)-4\left(u_{n-1}^1-u_{n-1}^0\right)+(u_{n-2}^1-u_{n-2}^0)}{h^4} = \varphi_n^N, \\ \varphi_n^N = \left(-2\tau+\tau^2-\frac{\tau^3}{6}+\frac{\tau^4}{12}+\frac{\tau^5}{18}+\frac{\tau^6}{18}-\frac{\tau^7}{432}-\frac{\tau^8}{1152}\right)\sin(x_n) \\ +\frac{\tau}{8}\sin(x_n), 2 \le n \le M-2, u_0 = u_M = 0, 0 \le k \le N, \\ u_3 = 4u_2-5u_1, u_{M-3} = 4u_{M-2}-5u_{M-1}, 0 \le k \le N. \end{cases} \tag{3.11}$$

We have an $(N+1)\times(N+1)$ system of linear equations.Resorting the system (3.11) and writing in the matrix form we have again the same $(N+1)\times(N+1)$ system of linear equations (3.8).

In (3.8), for this problem

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ x & y & x & 0 & . & 0 & 0 & 0 & 0 \\ 0 & x & y & x & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & x & y & x \\ -r & r & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ v & w & v & 0 & . & 0 & 0 & 0 & 0 \\ 0 & v & w & v & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & v & w & v \\ s & -s & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$C=\begin{bmatrix} 1 & 0 & 0 & . & -\frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} \\ m & n & m & 0 & . & 0 & 0 & 0 & 0 \\ 0 & m & n & m & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & m & n & m \\ -t & t & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$D=B,\ E=A,$$

where

$$x=\frac{\tau^2}{144h^4}, y=-\frac{\tau^2}{72h^4} v=-\frac{1}{12h^2}-\frac{\tau^2}{36h^4}, w=-\frac{5}{6h^2}+\frac{\tau^2}{18h^4},$$

$$m=\frac{1}{\tau^2}+\frac{1}{6h^2}+\frac{\tau^2}{24h^4}, n=-\frac{2}{\tau^2}+\frac{5}{3h^2}-\frac{\tau^2}{12h^4},$$

$$r=\frac{\tau^4}{144h^4}, s=\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2},$$

$$t=1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4},$$

$$R=\begin{bmatrix}1 & . & 0\\ . & . & .\\ 0 & . & 1\end{bmatrix}_{(N+1)\times(N+1)},$$

and

$$U_s^k=\begin{bmatrix}u_s^0\\ u_s^1\\ .\\ u_s^N\end{bmatrix}_{(N+1)\times 1},0\le k\le N, s=n-2,n-1,n,n+1,n+2.$$

$$\phi_n^k=\begin{bmatrix}\phi_n^0\\ \phi_n^1\\ .\\ \phi_n^N\end{bmatrix}_{(N+1)\times 1},0\le k\le N,$$

$$\phi_n^0=\frac{31}{32}\sin\left(x_n\right),$$

$$\phi_n^N=\left(-2\tau+\tau^2-\frac{\tau^3}{6}+\frac{\tau^4}{12}+\frac{\tau^{\mathbf{5}}}{18}+\frac{\tau^{\mathbf{6}}}{18}-\frac{\tau^{\mathbf{7}}}{432}-\frac{\tau^{\mathbf{8}}}{1152}\right)\sin\left(x_n\right)+\frac{\tau}{8}\sin\left(x_n\right),$$

$$\varphi_n^k=\left\{\frac{5}{6}(t_k^2-2t_k+3)+\frac{1}{12}[(t_{k+1}^2-2t_{k+1}+3)+(t_{k-1}^2-2t_{k-1}+3)]\right.$$

$$\left.-\frac{\tau^2}{72}(t_k-1)^2+\frac{\tau^2}{144}\left[\left(t_{k+1}^2-1\right)^2+\left(t_{k-1}^2-1\right)^2\right]\right\}\sin\left(x_n\right),1\le k\le N-1,$$

For solution of difference equation (3.11) we use the same method of the previous section. Using the formulas and matrices above we get the fourth order of accuracy difference scheme of problem (3.1) and using matlab we obtain numerical results for the solution of (3.11) for various $M$ and $N$ values.

## 3.4 Error Analysis

We consider the nonlocal boundary value problem (3.1). For the approximate solutions of problem (3.1) we used the third and the fourth order of accuracy difference schemes defined by the self adjoint posistive definite operator A. The graphs of exact and numerical solutions are presented below:

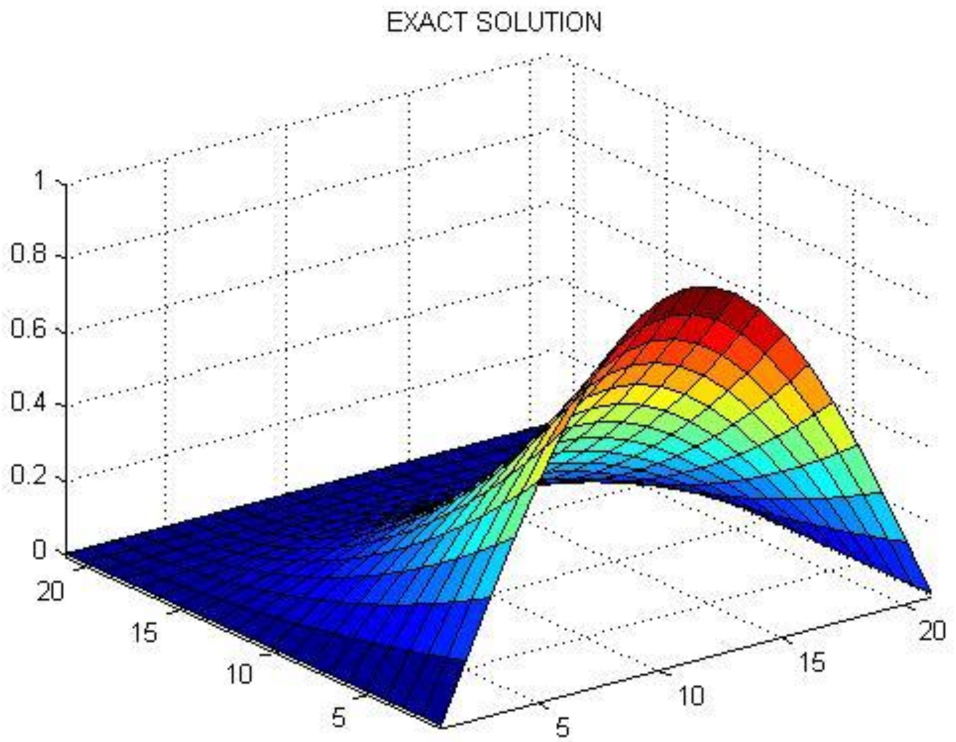


Figure 3.1. The exact solution of problem (3.1)

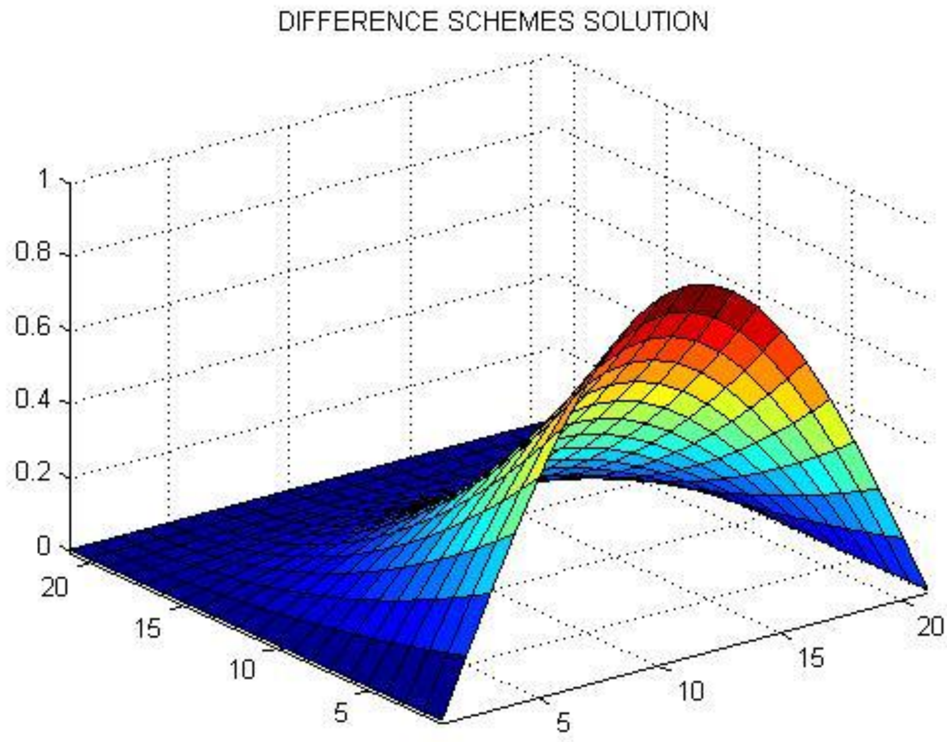


Figure 3.2. The third order of accuracy difference scheme solution (2.2)

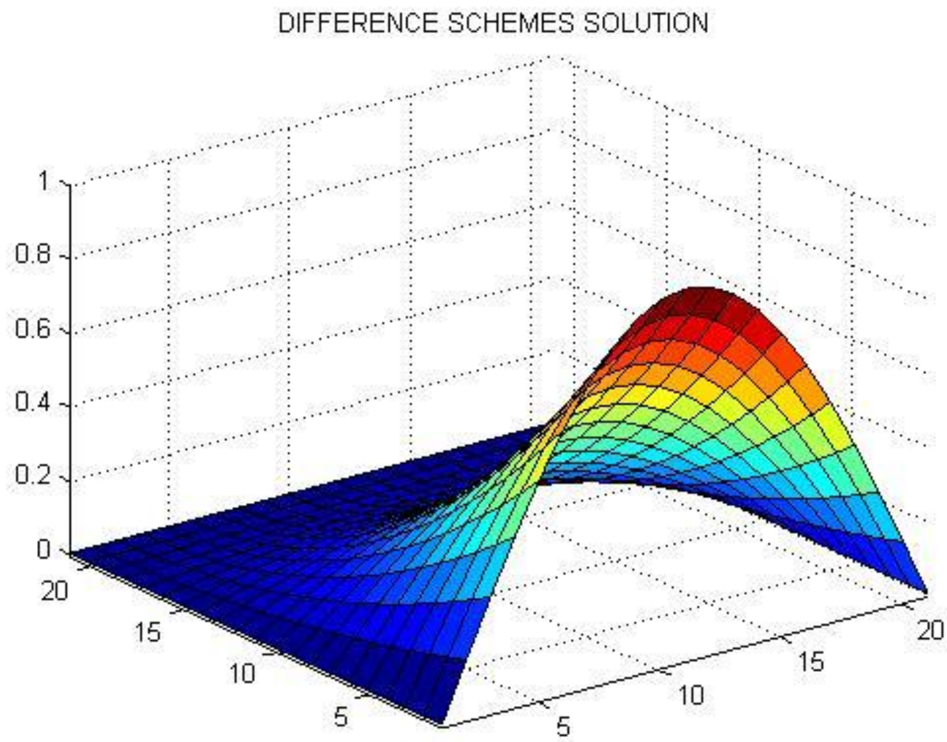


Figure 3.3. The fourth order of accuracy difference scheme solution (2.29)

Now, we will show the results of the numerical analysis. The errors are computed by the following formula

$$E_M^N = \max_{1 \le k \le N-1} \left( \sum_{n=1}^{M-1} \left| u\left(t_k, x_n\right) - U_n^k \right|^2 h \right)^{\frac{1}{2}},$$

where $u\left(t_k, x_n\right)$ represents the exact solution and $U_n^k$ represents the numerical solution at $\left(t_k, x_n\right)$.

The result of numerical experiments are shown in the following table. We denote the third order of accuracy difference scheme (2.2) by TO and the fourth order of accuracy difference scheme (2.29) by FO.

**Table 3.1**. Errors for the approximate solutions of problem (3.1).

| Method | M=20 , N=20 | M=40 , N=60 | M=80 , N=180 |
|---|---|---|---|
| TO | $0,5962.10^{-3}$ | $0,7029.10^{-4}$ | $0,8009.10^{-5}$ |
| FO | $0,1143.10^{-3}$ | $0,4799.10^{-4}$ | $0,2120.10^{-4}$ |

Thus, the fourth order of accuracy difference scheme is more accurate comparing with the third order of accuracy difference scheme.

# CHAPTER 4

## CONCLUSIONS

This work is devoted to the study of the stability of difference schemes for the solution of the multipoint nonlocal boundary value problem for hyperbolic equations. The following original results are obtained:

- The nonlocal boundary value problem for hyperbolic equation

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad 0 \le t \le 1, \\ u(0) = \sum_{r=1}^{n} \alpha_r u(\lambda_r) + \phi, \\ u_t(0) = \sum_{r=1}^{n} \beta_r u_t(\lambda_r) + \psi, \\ 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k \le 1, \end{cases}$$

in a Hilbert space $H$ with self -adjoint positive definite operator $A$ is studied.

- The third order of accuracy difference scheme (2.2) generated by the integer power of $A$ for approximately solving the abstract nonlocal boundary value problem (4.1) is presented. Stability estimates for the solution of this difference scheme are established.

- The fourth order of accuracy difference scheme generated by the integer power of $A$ (2.29) for approximately solving abstract nonlocal boundary value problem (4.1) is presented. Stability estimates for the solution of this difference scheme are established.

Theoretical statements for the solution of these difference schemes are supported by the results of numerical experiments

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