

**REPUBLIC OF TURKEY  
YILDIZ TECHNICAL UNIVERSITY  
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES**

**ROBUST MOVING HORIZON  $\mathcal{H}_\infty$  CONTROL OF DISCRETE  
TIME STATE-DELAYED SYSTEMS**

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**PhD. THESIS  
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PROGRAM OF CONTROL AND AUTOMATION**

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A thesis submitted by Fatma YILDIZ TAŞCIKARAOĞLU in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY** is approved by the committee on 21.06.2013 in Department of Electrical Engineering, Control and Automation Program.

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## LIST OF SYMBOLS

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$K$	State feedback gain
$V$	Lyapunov function
$d$	Delay coefficient
$d_{max}$	Upper bound of time-delay
$d_{min}$	Lower bound of time-delay
$diag$	Diagonal
$p$	Dissipation level
$q$	Delay based dissipation level
$\mathbb{R}$	Set of real numbers
$u$	Control input
$x$	State-vector
$w$	Disturbance input
$y$	Measured output
$z$	Exogenous controlled output
$\alpha$	Disturbance bound coefficient
$\gamma$	Performance index from disturbance to performance output
$\gamma_{opt}$	Performance index from disturbance to performance output without constraints
$\  \cdot \ $	Vector 2-norm

## LIST OF ABBREVIATIONS

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L-K	Lyapunov-Krasovskii
LMI	Linear Matrix Inequality
LPV	Linear Parameter Varying
MHC	Moving Horizon Control
MHHC	Moving Horizon $\mathcal{H}_\infty$ Control
MPC	Model Predictive Control
RHC	Receding Horizon Control
RHHC	Receding Horizon $H_\infty$ Control
BRL	Bounded Real Lemma
TDS	Time Delay System

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Ph.D Thesis

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This thesis is concerned with design of a delay-dependent type moving horizon state-feedback control (MHHC) for a class of linear discrete-time system subject to time-varying state delays, norm bounded uncertainties and disturbances with bounded energies. The closed-loop robust stability and robust performance problems are considered to overcome the instability and poor disturbance rejection performance due to the existence of parametric uncertainties and time-delay appeared in the system dynamics. The main objective is the development of possible low conservative stability condition for time-delay systems and of a novel control method for such systems using dissipation constraint together with on-line adapting of the dissipation level. For that purpose, the constrained minimax optimization problem is dealt with in the framework of Linear Matrix Inequality (LMI), in which the maximization is realized over a set of uncertainties and bounded disturbances. Utilizing a discrete-time Lyapunov-Krasovskii (L-K) functional, some delay-dependent, LMI based conditions are provided. It is shown that if one can find a feasible solution set for these LMI conditions iteratively in each step of run-time, then we can construct a control law which guarantees the closed-loop asymptotic stability, maximum disturbance rejection performance and closed-loop dissipativity in view of the actuator limitations. Two numerical examples and simulations on a nominal and uncertain discrete-time, time-delayed systems, are presented at the end, in order to demonstrate the efficiency of the proposed method.

**Key words:** Moving horizon control,  $\mathcal{H}_\infty$  control, model predictive control, time-delay system, delay-dependent control, time-varying delay, robust stability.

**DAYANIKLI MODEL ÖNGÖRÜLÜ  $\mathcal{H}_\infty$  KONTROL İLE  
AYRIK ZAMANLI DURUM GECİKMELİ SİSTEMLERİN KONTROLÜ**

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Bu tez çalışmasında, zamanla değişebilen durum gecikmesi, normu sınırlandırılmış belirsizlikler ve sınırlı enerjiye sahip bozucu etkiler içeren doğrusal ayrık zamanlı sistemler için zaman gecikmesine bağlı geri-beslemeli model öngörülü  $\mathcal{H}_\infty$  kontrol yöntemi (MHHC) tasarımı üzerinde durulmuştur. Sistem dinamiklerinde bulunan belirsizliklerden ve zaman gecikmesinden kaynaklanan kararsızlığı ve kötü performansı önlemek amacıyla kapalı çevrim dayanıklı kararlılık ve dayanıklı performans problemleri ele alınmıştır. Çalışmanın temel hedefi, zaman gecikmeli sistemler için tutuculuğu düşük kararlılık şartı ve bu sistemler için çevrim içi değişen kayıplılık seviyesi ile kayıplılık kısıtını kullanarak yeni bir kontrol yöntemi geliştirmektir. Bu amaçla, en kötü durumun bir takım belirsizlikler ve sınırlı bozucu dahilinde bulunduğu kısıtlı minimax optimizasyon problemi Lineer Matris Eşitsizlikleri (LME) çerçevesinde ele alınmıştır. Kapalı çevrim sistemin asimptotik kararlılığını,  $\mathcal{H}_\infty$  performansını ve yitirgenliğini sağlamak amacıyla ayrık zamanlı Lyapunov-Krasovskii (L-K) fonksiyoneliinden faydalanılarak her bir adımda yinelemeli olarak çözülen gecikmeye bağlı LME tabanlı şartlar sunulmuştur. Son olarak, önerilen yöntemin etkinliğini gösterebilmek amacıyla, nominal ve belirsizlik içeren ayrık zamanlı zaman gecikmeli sistemler için çeşitli sayısal örnekler sunulmuştur.



**Anahtar Kelimeler:** Model öngörölü kontrol,  $\mathcal{H}_\infty$  kontrol, zaman gecikmeli sistemler, gecikmeye bağı kontrol, zamanla değişen gecikme, dayanıklı kararlılık.

**INTRODUCTION****1.1 Literature Review**

Time-delay systems have drawn considerable attention for the last few decades due to the fact that these systems represent the behaviour of processes more close to real world situations. The primary effects of delay in the behaviour of physical plants and systems are a deterioration of performance, and exhibition of an unstable respond. There are many sources of delay existence in a system which can be exemplified as due to long transmission lines, or intensive communication channels, approximations in the identification or modelling of real systems and finite rate or capability of computing power for control and communication purposes in remote systems. The theoretical and practical investigation of time-delay systems and negative effects of delay phenomenon on the stability and performance of feedback systems have been comprehensively investigated in the literature by a great deal of studies such as [1–3]. Besides, a review article on definitions and classifications of time-delay systems has been made together with stability analysis in [4].

The stability analysis of time delay systems can be divided into two groups regarding whether the derived stability condition depends on the delay size or not. Namely, the former one is called the delay-dependent stability and the latter one is called the delay-independent stability. Recently, the focus has been initiated to concentrate on delay-dependent controller because of its less conservative structure [3, 5–7] .

$\mathcal{H}_\infty$  control mainly deals with the synthesis of feedback or feed-forward controller for dynamical systems to establish a stable behaviour or respond in a robust sense when the controlled system is subjected to an external disturbance effect.  $\mathcal{H}_\infty$  control problem received a considerable amount of significance in the last decade under the existence of time-delay which might be constant or even time-varying. The main reason arises from

the fact that the so-called Bounded Real Lemma (BRL) signifying the  $\mathcal{H}_\infty$  performance of a given plant or system devoid of any delay effect proves to yield a necessary condition while its counterpart given for the case of an existence of time-delay can afford to achieve only a sufficient condition. This observation implies that the  $\mathcal{H}_\infty$  results concerning time-delay systems inherently involve potential conservativeness which leads the researchers in the time-delay community to construct further investigation to achieve better  $\mathcal{H}_\infty$  performance. Therefore, for continuous time case, one can refer to references [5, 8–11] and the references therein for the  $\mathcal{H}_\infty$  control of time-delay systems (TDSs) having zero lower delay bound and [12–16] for the  $\mathcal{H}_\infty$  control of TDSs with interval time-delays. For the discrete-time counterpart, the reader can refer to [17, 18] and the references therein.

Moving Horizon Control (MHC), also called as Receding Horizon Control (RHC) or Model Predictive Control (MPC) in the literature, is a widely used method of control of industrial processes, especially having large number of variables and constraints, due to its superior capabilities in handling constraints on control and states, operating with less human intervention and reacting dynamically to system changes rapidly. The main approach in MPC is to solve a constraint optimization problem on-line at each time instant by utilizing the past and current feedback information and apply only the very recent control solution to the system. For a good survey and recent approaches on the concept, one can refer to [19–25] and the references therein. The general classification and areas of application of these systems have been comprehensively investigated in references [26] and [27].

In [26], Mayne clusters the papers in the literature regarding the subjects of stability and optimality of MPC. This research deals with MPC of constrained linear and nonlinear dynamic systems. In this note, the stability techniques, state and output feedback forms, tracking and  $\mathcal{H}_\infty$  MPC control are briefly mentioned. Moreover, the basic formulations of MPCs are presented.

The MPC design methods typically have been based on the concept of Lyapunov Theory. At the beginning of the 1990s, stability of MPC was achieved for linear, unconstrained systems. The first general stability theory of constrained MPC for time-varying, constrained, nonlinear discrete-time systems was described and the objective function was first employed as a Lyapunov function in the study of [28]. For the other

important literature examples, the main studies concerning the development of MPC stability is presented as follows:

In [29], Chen and Allgöwer studied guaranteed asymptotic closed-loop stability for MPC that can be applied to both stable and unstable systems with input constraints. The quadratic objective function to be minimized consists of an integral-square-error and a quadratic terminal cost. Their controller employs both a terminal cost and a terminal constraint. Terminal cost term has a terminal state penalty matrix which is chosen as a Lyapunov function. The terminal inequality constraint drives the states to a terminal region at the end of the finite prediction horizon. It has been proven that this off-line determination of penalty term and terminal region with respect to Lyapunov stability guarantees asymptotic stability of the closed loop system if a feasible solution of optimization problem is present at time  $t = 0$ .

For robust MPC, in [30] Zheng and Morari put forward that the stability is not guaranteed in the original minimax MPC. A new robust MPC technique was proposed for asymptotic stability for control of uncertain linear systems as well. In [31] it is addressed LMI based minimax formulation that has difficult implementation with quadratic criteria. The formulation is conservative due to the upper bound of the cost function for maximization, a constant feedback gain over the prediction horizon and time-domain constraints by keeping the state trajectory in a fixed ellipsoid.

Recently, [32] put forward a novel LMI based infinite-horizon MPC scheme for constrained linear parameter varying (LPV) systems. Since, the proposed method is parameter dependent, less conservative conditions are obtained for stability. In a more recent study, [33] presents a robust MPC scheme for nonlinear systems with state and input constraints and bounded disturbances. First, a standard MPC problem is solved online that includes the initial state of the model as a decision variable. The latter control law is designed to keep the state trajectory within a prescribed ellipsoid in the presence of bounded disturbances. The nominal control action drives these ellipsoids to a desired reference state.

It is well-known that uncertainties and time-delays cannot be avoided in real life, especially, in many processes such as chemical, biological and network problems where the MPCs have been widely used. The term of robustness is used for the systems whose stability is maintained and performance specifications are met over a specific

uncertainty range. Because of its importance, a large number of studies have dealt with the constructing of robust MPC schemes such as [22, 31] and [34]. However, it is apparently seen from the literature that only a very few results exists concerning the MPC of time-delay systems. Based on the Lyapunov-Krasovskii functional approach and the use of some relaxation matrices [35] and [36] have considered the problem of receding horizon control for state-delayed systems where the delay is fixed. Also they have presented an eigenvalue search algorithm to check the closed-loop stability of the system. However their control strategy is delay-independent, therefore highly conservative. Jeong introduced a novel optimization method for MPC of uncertain time-delay systems having constant state delays and constrained input [37]. But, again, their method does not depend on the size of delay. In [38] a robust one-step LMI based MPC scheme is developed for discrete time-delay systems having fixed delay and polytopic-type uncertainties. Their proposed MPC method is obtained by minimizing a new cost function that includes multi-terminal weighting terms, subject to constraints on input.

The deficiency of MPC systems in disturbance attenuation has drawn the attention towards the Moving Horizon  $\mathcal{H}_\infty$  Control (MHHC) scheme in recent years, such as [39] and [25]. In this perspective, a Riccati-based MHHC algorithm was developed for a nominal time-delay system in [35] and [40]. Then, a similar approach was proposed in [41], based on LMIs. However, all the above-mentioned approaches are based on criteria which do not depend on the size of delay, and therefore they provide highly conservative results. While taking the time-delay bounds into consideration, a delay-dependent MPC is proposed in [38], which guarantees only the closed-loop stability. Mei and Huihe have dealt with the receding horizon  $\mathcal{H}_\infty$  control for constraint time-delay systems having fixed delays in [41]. However, their method is delay-independent as the ones indicated above. Finally, proposing a new cost function for a finite horizon dynamic game problem which includes two terminal weighting terms and a delay-independent LMI condition, Lee et al. have studied the receding horizon  $\mathcal{H}_\infty$  control problem for time-delay systems with fixed state delays in [42].

Among these aforementioned work, it is apparently seen that only a very few results exist, concerning the MPC of TDSs which gives us a motivation to study this problem. Furthermore, to the best of author's knowledge, there does not any other reference in literature which deals with the  $\mathcal{H}_\infty$  MPC of uncertain time-delay systems having time-

varying delays in delay-dependent fashion. Therefore, combining these two sources of inspiration leads us to study that particular subject thoroughly.

In this study, we investigate the design of a stabilizing moving horizon  $\mathcal{H}_\infty$  state-feedback controller for linear state-delayed interval TDS having norm-bounded uncertainties and constrained inputs. First, based on the selection of an L-K functional, a stabilizing delay-dependent  $\mathcal{H}_\infty$  controller is introduced for nominal TDS having delay in the state. Then, the existing result is extended to cover TDS having norm-bounded uncertainties. Finally, the proposed approach is adapted to the so-called moving horizon scheme introduced in [43] to obtain a robust, delay-dependent moving horizon  $\mathcal{H}_\infty$  control technique which ensures a dissipative closed-loop system. The proposed technique is practically implementable since it takes the input constraints into consideration and it does not employ any linearization techniques such as cone-complementary methods which are widely used in delay-dependent control approaches such as those used in [7].

The rest of this study is organized as follows: Chapter 2 gives the mathematical background in order to provide a priori knowledge about the methodology followed. Chapter 3 gives an insight of the time-delay systems in which Lyapunov based stability methods are given in order to show asymptotic stability of time-delay systems. Chapter 4 provides an overview of MHHC method for nominal systems. Chapter 5 is devoted to the proposed method, MHHC for time delay systems and addresses the problem of the robust stability of the method in presence of uncertainty. Three different illustrative examples are presented in Chapter 6 to demonstrate the effectiveness of the approach. Finally, Chapter 7 concludes the thesis.

## 1.2 Purpose of Thesis

In this study, we are aiming to develop a novel control method, named as MHHC, for uncertain TDSs having time-varying delays and uncertainties. To this end, first the methodology of LMI-based MHHC method is comprehensively presented for both nominal and uncertain systems. Briefly, this method has the advantages of guaranteeing the asymptotic stability, even in the presence of uncertainties, handling with the challenging constraints and attenuating disturbance under the bounded disturbance condition. In addition to the existing approaches in the literature of MHHC control, time-varying delay-dependent state-feedback controller is developed for the first time in

order to minimize the conservatism. With the aim of satisfying the required performance level, robustness and related constraint in each step, the most suitable state feedback controller guaranteeing the dissipativeness of the system is determined. The main advantage of the proposed method lies in its algorithm which takes disturbance level and present states of the system into account in each step of run-time, when compared with classical  $\mathcal{H}_\infty$  control algorithms whose feedback gains are calculated offline. This enables us to obtain much efficient real-time controllers.

## PRELIMINARIES

This chapter presents basic knowledge on the main definitions used in this study. To this end, we start with the definition of *dissipativity* which plays an important role during the implementation of moving horizon control. Then, the formulation of Schur complement is summarized and finally, the uncertainty definition, which is one of the key points in robustness analysis and robust controller design, is presented.

**Definition 2.1 Dissipativity** Consider the following linear discrete-time system

$$\begin{aligned} x(k+1) &= f(k, x(k), u(k)) \\ z(k) &= h(k, x(k), u(k)) \end{aligned} \quad (2.1)$$

where  $x(k)$  is the state vector, with supply function  $s(w(k), z(k)) := \gamma^2 \|w(k)\|^2 - \|z(k)\|^2$  is said to be dissipative if there exist a quadratic storage function  $V(x(k))$  such that

$$V(x(k)) + (s(w(k), z(k))) \geq V(x(k+1)) \quad \text{for all } k \geq 0. \quad (2.2)$$

This implies that the change of storage function from step  $k$  to step  $(k+1)$  is always less than the supplied rate to the system. Therefore, the inequality (2.2) is named as *dissipation inequality*.

A system is dissipative if there exists an associated storage that is non-negative when supply is bounded. Dissipativeness of dynamical systems has been introduced in [44]. Dissipativity theory is very useful in the stability analysis of non-autonomous dynamical systems and control systems like adaptive control, nonlinear  $\mathcal{H}_\infty$  and inverse optimal control, which uses an input-output description based on energy-related notion [44–46].



**Definition 2.2 Vector norm** Let  $X$  be an linear vector space associated with  $\mathbb{F}$ . A *vector norm* on  $X$  is a function  $\|\cdot\| : X \rightarrow \mathbb{R}$  that satisfies the following conditions:

- i.  $\|x\| \geq 0$  for all  $x \in X$
- ii.  $\|x\| = 0$  if and only if  $x = 0$
- iii.  $\|ax\| = |a|\|x\|$  for all  $a \in \mathbb{F}, x \in X$
- iv.  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in X$

The most commonly used  $p$  - norm,

$$\|x\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \quad (2.3)$$

is defined for  $p \in [1, \infty)$ . As  $p \rightarrow \infty$ , we have

$$\|x\|_\infty := \max_{i=1:n} |x_i|.$$

The 2 - norm,

$$\|x\|_2 = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} = \sqrt{x * x}.$$

is called as *Euclidean norm*.

**Definition 2.3 Congruence Transformation** Two matrices  $A, B \in \mathbb{C}^{n \times n}$  are said to be *congruent* to each other if there exists a nonsingular  $T \in \mathbb{C}^{n \times n}$  such that

$$B = T^* A T.$$

The transformation  $A \rightarrow T^* A T$  is called a *congruence transformation* of  $A$  under  $T$ .

**Definition 2.4 Inertia** Given a matrix  $A = A^* \in \mathbb{C}^{n \times n}$ , the inertia of  $A$  is the triplet

$$in(A) = [n_+, n_-, n_0]$$

where  $n_+, n_-$  and  $n_0$  denote the number of positive, negative and zero eigenvalues, respectively.

**Lemma 2.5** [47] Given two Hermitian matrices  $A, B \in \mathbb{C}^{n \times n}$ ,

$$A \text{ and } B \text{ are congruent} \iff in(A) = in(B).$$

**Proof:** Let  $A$  and  $B$  be congruent and  $\text{in}(A) = \{n_+(A), n_-(A), n_0(A)\}$ . First, observe that  $\text{rank}(A) = \text{rank}(B)$  by Sylvester's rank inequality, since there exists a nonsingular  $T \in \mathbb{C}^{n \times n}$  such that  $B = T^*AT$ . It then follows that  $n_0(A) = n_0(B)$ . So, we have to show that  $n_+(A) = n_+(B)$ .

Let  $v_1, \dots, v_{n_+(A)}$  be the orthogonal eigenvectors of  $A$  associated with the positive eigenvalues  $\lambda_1, \dots, \lambda_{n_+}$ . Define  $S_+(A) := \text{span}\{v_1, \dots, v_{n_+(A)}\}$ . Then,  $\dim(S_+(A)) = n_+(A)$ . Now, for any nonzero  $x = \sum_{i=1}^{n_+(A)} \alpha_i v_i$ , we have  $x^*Ax = \sum_{i=1}^{n_+(A)} \lambda_i |\alpha_i|^2 > 0$ . In this case, the fact that

$$x^*T^{-*}BT^{-1}x = (T^{-1}x)^*B(T^{-1}x) > 0$$

implies  $y^*By > 0$  for all nonzero  $y \in \text{span}\{T^{-1}v_1, \dots, T^{-1}v_{n_+(A)}\}$ , which also has dimension  $n_+(A)$ . Then,  $n_+(B) \geq n_+(A)$ . By reversing the roles of  $A$  and  $B$  in the argument above, we obtain  $n_+(A) \geq n_+(B)$ , implying  $n_+(A) = n_+(B)$ . Together with  $n_0(A) = n_0(B)$ , this leads to the conclusion that  $\text{in}(A) = \text{in}(B)$ . Hence,  $A$  and  $B$  are congruent.

**Definition 2.6 Positive/Negative (semi) definiteness** A matrix  $A = A^* \in \mathbb{C}^{n \times n}$  is said to be

- i. Positive (semi) definite if

$$x^T Ax > 0 \ (\geq 0) \quad \forall x \in \mathbb{C}^n, \quad x \neq 0$$

- ii. Negative (semi) definite if

$$x^T Ax < 0 \ (\leq 0) \quad \forall x \in \mathbb{C}^n, \quad x \neq 0$$

**Lemma 2.7 Schur Complement** Let  $F : X \rightarrow \mathbb{H}$  be an affine function which is partitioned according to

$$F(x) = \begin{bmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{bmatrix} \quad (2.4)$$

where  $F_{11}(x)$  is square. Then  $F(x) \prec 0$  if and only if

$$\begin{cases} F_{11}(x) \prec 0 \\ F_{22}(x) - F_{21}(x)[F_{11}(x)]^{-1}F_{12}(x) \prec 0. \end{cases} \quad (2.5)$$

or if and only if

$$\begin{cases} F_{22}(x) \prec 0 \\ F_{11}(x) - F_{12}(x)[F_{22}(x)]^{-1}F_{21}(x) \prec 0. \end{cases} \quad (2.6)$$

Hence, the nonlinear structures in inequalities (2.5) and (2.6) can be transformed into the form of a matrix inequality,  $F(x) < 0$ .

**Definition 2.8 Norm-Bounded Uncertainties** The parameter uncertainty matrix  $\Delta A$  is in the form of

$$\Delta A = GFE, \quad \|F\| \leq I \quad (2.7)$$

where  $G, E_1, E_2$  are known time-invariant matrices and  $F(k)$  is an unknown time-varying matrix function of uncertain parameters.

**Definition 2.9 Ellipsoid** Given a matrix  $P \in \mathbb{R}^{n \times n}$ , where  $P = P^T \geq 0$ , the set

$$\varepsilon_P := \{x \in \mathbb{R}^n : x^T P x \leq 1\} \quad (2.8)$$

is called the ellipsoid associated with  $P$ .

**Proposition 2.10** [47] Given two matrices  $A \geq 0$  and  $B \geq 0$ ,

$$\varepsilon_A \subseteq \varepsilon_B \iff B \leq A.$$

**Proof:** Suppose  $B \leq A$  and let  $x \in \varepsilon_A$ . Then,  $x^T B x \leq x^T A x \leq 1$ . Therefore,  $x \in \varepsilon_B$ , so that  $\varepsilon_A \subseteq \varepsilon_B$ .

## TIME-DELAY SYSTEMS

Time delay is a phenomenon which generally appears in control systems, particularly in the sciences of engineering, biology, chemistry, mechanics and physics. These systems, also called as systems with after effect or dead-time, have a capability of representing the behaviour of processes more close to real world situations. A great deal of studies has been devoted to the analysis of time-delay control systems over the past decades since time-delay factors perturb the system behavior and cause poor performance as well as instability of the system. Next, we will briefly introduce time-delay systems and present some useful stability concepts about the subject.

### 3.1 System Description

This study focuses mainly on Delay Difference Equations (DDEs) among the related mathematical representations of TDS due to its popularity in representation of discrete-time [48, 49]. Thus, the linear discrete-time system with constant delay acting on the state is given as follows:

$$x(k+1) = Ax(k) + A_dx(k-d) \quad (3.1)$$

where  $x(k) \in \mathbb{R}^n$  is the state vector,  $A \in \mathbb{R}^{n \times n}$ ,  $A_d \in \mathbb{R}^{n \times n}$  are constant system matrices and  $d$  is an unknown integer representing the amount of delay in the state. If the delay shown in (3.1) is time-varying, the system can be stated as follows:

$$x(k+1) = Ax(k) + A_dx(k-d(k)) \quad (3.2)$$

where  $d(k)$  is a time-varying delay that satisfies

$$0 \leq d_{min} \leq d(k) \leq d_{max} \quad \forall k \geq 0. \quad (3.3)$$

In the presence of input-delay the system can be formed as:

$$x(k+1) = Ax(k) + A_dx(k-d) + Bu(k) + B_du(k-d). \quad (3.4)$$

If there are parametric uncertainties in the system, the following representation is used.

$$\begin{aligned} x(k+1) = & (A + \Delta A(k))x(k) + (A_d + \Delta A_d(k))x(k-d) \\ & + (B + \Delta B(k))u(k) + (B_d + \Delta B_d(k))u(k-d) \end{aligned} \quad (3.5)$$

Herein,  $\Delta A(k), \Delta B(k), \Delta A_d(k), \Delta B_d(k)$  are time-varying matrices of norm-bounded uncertain parameters, in appropriate dimensions.

### 3.2 Stability of Time-Delay Systems

Time delays in control systems generally give rise to unsynchronization in the application of the control forces, which in turn deteriorate the system performance and affect the stability of the system response. Since the difficulty increases with time-varying delays, nonlinearity or uncertainties, various approaches have been employed for the stability analysis of time delay systems. Among these, Lyapunov-Razumikhin and Lyapunov-Krasovskii are the most widely preferred theorems.

As stated before, stability of TDS are classified into two groups depending on delay size: delay-independent stability and delay-dependent stability, which will be discussed in subsections 2.2.3 and 2.2.4 respectively.

#### 3.2.1 Lyapunov-Razumikhin Theorem

Consider the functional differential equation

$$\begin{aligned} x(k+1) &= f(t, x(t)), \quad t \geq t_0 \\ x_{t_0}(\theta) &= \Phi(k+\theta), \quad \forall \theta \in [-\tau, 0] \end{aligned} \quad (3.6)$$

where  $x_t(\theta) = \Phi(t+\theta), \forall \theta \in [-\tau, 0]$  with  $\Phi \in \mathcal{C}_{n,\tau}$ .

**Theorem 3.1** [1] The zero solution of the system in (3.6) is asymptotically stable (i) if there exists a positive definite and decreasing continuous functional  $V(k, x) : \mathbb{R} \times \mathbb{D} \rightarrow \mathbb{R}^+$ , which has an upper and lower limits as follows:

$$u(\|x\|) \leq V(k, x) \leq v(\|x\|)$$

where  $\exists u(x), v(x), \gamma(x) : \mathbb{R}^n \rightarrow \mathbb{R}^+$  and (ii) if the difference  $\Delta V(k, x) \leq -\gamma(\|x\|)$  supposing that  $V(k+\eta, x(k+\eta)) < \delta(V(k, x(k))), \quad \forall \eta \in [-\tau, 0]$ .

### 3.2.2 Lyapunov-Krasovskii Theorem

Consider the following differential equation

$$\begin{aligned} x(k+1) &= f(k, x(k)) \\ x(\theta) &= \Phi(k+\theta), \forall \theta \in [-d, 0] \end{aligned} \quad (3.7)$$

where  $d \in \mathbb{Z}^+$  is unknown time delay.

**Theorem 3.2** [1, 4] The zero solution of the system in (3.7) is asymptotically stable (i) if there exists a positive definite and decreasing continuous functional  $V(k, x) : \mathbb{R} \times \mathbb{D} \rightarrow \mathbb{R}^+$ , which has an upper and lower limits as follows:

$$u(\|\Phi(0)\|) \leq V(k, x) \leq v(\|\Phi\|_{\mathbb{D}})$$

where  $\exists u(x), v(x) : \mathbb{R}^n \rightarrow \mathbb{R}^+$  and (ii) if the difference  $\Delta V(k, \Phi)$  is negative definite over a neighborhood of zero, that is,  $V(x(k+1)) - V(x(k)) \leq -\gamma(\|\Phi(0)\|)$ .

A large number of L-K functionals have been presented in the literature. Selection of  $P$  is generally standard; however,  $Q$  is determined with respect to the problem. It can be noticed that all of them are in sums of quadratic forms that depend on the delayed states. The most common form is given as follows:

$$V(x(k)) = x(k)^T P x(k) + \sum_{s=k-d}^{k-1} x^T(s) Q x(s), \quad (3.8)$$

where  $P = P^T \in \mathbb{R}^{n \times n}$  and  $Q = Q^T \in \mathbb{R}^{n \times n}$  are weighting matrices. Herein, the first term of the functional represent the present state and the second term aggregates the effect of the delayed state. Different forms of  $Q$  can be given as follows

$$\begin{aligned} V_1(k) &= x^T(k) \sum_{s=k-d}^{k-1} x^T(s) Q x(s), \\ V_2(k) &= \sum_{s=k-d}^{k-1} x^T(s) Q x(s), \\ V_3(k) &= \sum_{s=-d_{max}+2}^{-d_{min}+1} \sum_{s=k+j-1}^{k-1} x^T(s) Q x(s), \\ V_4(k) &= \sum_{s=-d_{max}+1}^0 \sum_{s=k+j-1}^{k-1} y^T(s) Z y(s), \end{aligned} \quad (3.9)$$

In (3.9),  $V_1(k)$  and  $V_2(k)$  are used for the delay-independent stability criteria and the delay-dependent stability is acquired with the including of  $V_3(k)$ . For instance,

$V(k) = V_2(k) + V_3(k)$  is a general candidate L-K function. Besides,  $V_4(k)$  leads reduction of conservatism with the cost of increasing computational burden.

### 3.2.3 Delay-Independent Stability

**Theorem 3.3** The time-delay system (3.1) is asymptotically stable for any time-delay,  $d$  if the following condition holds:

$$\begin{bmatrix} Q - P & 0 & A^T \\ 0 & -Q & A_d^T \\ A & A_d & -P^{-1} \end{bmatrix} < 0 \quad (3.10)$$

where  $P = P^T > 0$  and  $Q = Q^T > 0$ .

**Proof:** Let us consider the L-K functional given by (3.8). Then the difference of the energy function  $V(x(k), k)$  is obtained as follows:

$$\begin{aligned} \Delta V &= V(x(k+1)) - V(x(k)) \\ &= x^T(k)(A^T P A - P + Q)x(k) + x^T(k)A^T P A_d x(k-d) \\ &\quad + x^T(k-d)A_d^T P A x(k) + x^T(k-d)A_d^T P A_d x(k-d) \end{aligned} \quad (3.11)$$

Then, (3.11) can be rewritten as

$$\begin{bmatrix} x(k) \\ x(k-d) \end{bmatrix}^T \begin{bmatrix} A^T P A - P + Q & A^T P A_d \\ A_d^T P A & A_d^T P A_d - Q \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-d) \end{bmatrix} < 0. \quad (3.12)$$

Applying Schur Complement Formula on the multiplier of (3.12) gives (3.10). This ends the proof.

### 3.2.4 Delay-Dependent Stability

**Theorem 3.4** Given the scalar constants  $d_{min}$  and  $d_{max}$ , the time-delay system (3.2) is asymptotically stable for  $d(k)$  satisfying (3.3), if the following condition holds:

$$\begin{bmatrix} (d_{max} - d_{min} + 1)Q - P & 0 & A^T \\ 0 & -Q & A_d^T \\ A & A_d & -P^{-1} \end{bmatrix} < 0 \quad (3.13)$$

for some  $P = P^T > 0$  and  $Q = Q^T > 0$ .

**Proof:** Let us choose a candidate L-K functional as follows:

$$V(x(k), k) = V_1(k) + V_2(k) + V_3(k) \quad (3.14)$$

where

$$\begin{aligned} V_1(k) &= x^T(k)Px(k), \\ V_2(k) &= \sum_{s=k-d(k)}^{k-1} x^T(s)Qx(s), \\ V_3(k) &= \sum_{i=-d_{max}+2}^{-d_{min}+1} \sum_{s=k+i-1}^{k-1} x^T(s)Qx(s). \end{aligned}$$

In view of time-varying delay system (3.2), we obtain the first difference of the energy function  $V(x(k), k)$  as follows:

$$\Delta V(k) \triangleq V(k+1) - V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) \quad (3.15)$$

where

$$\begin{aligned} \Delta V_1(k) &= x^T(k)(A^T PA - P)x(k) + x^T(k)A^T PA_d x(k-d) \\ &\quad + x^T(k-d)A_d^T PAx(k) + x^T(k-d)A_d^T PA_d x(k-d) \\ \Delta V_2(k) &\leq x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) + \sum_{s=k-d_{max}+1}^{k-d_{min}} x^T(s)Qx(s) \\ \Delta V_3(k) &= (d_{max} - d_{min})x^T(k)Qx(k) - \sum_{i=k-d_{max}+1}^{k-d_{min}} x^T(s)Qx(s) \end{aligned}$$

The detailed derivations of  $\Delta V_1(k), \Delta V_2(k), \Delta V_3(k)$  will be given in Section 4.2.

Thus, a bound on  $\Delta V(k)$  can be obtained as follows:

$$\Delta V(k) \leq \begin{bmatrix} x(k) \\ x(k-d) \end{bmatrix}^T \begin{bmatrix} A^T PA - P + (d_{max} - d_{min} + 1)Q & A^T PA_d \\ A_d^T PA & A_d^T PA_d - Q \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-d) \end{bmatrix}. \quad (3.16)$$

Applying Schur Complement Formula on the multiplier of the right hand side of (3.16) gives (3.13). If the condition given in (3.13) is satisfied, then it is guaranteed that  $\Delta V(k) < 0$ . This ends the proof. ■



## MOVING HORIZON $\mathcal{H}_\infty$ CONTROL

In the literature, Moving Horizon Control (MHC) method has a wide application area, especially in the fields of electrical engineering and mechanics. The method can be briefly described as the online solution of the optimization problem which keeps the stability of the system as overcoming the challenging control constraints. Conventional MHC can be defined as a recursive procedure which consists of two steps; (1) solving of a pre-defined optimization problem over a finite control horizon using the mathematical model and present state of system in the presence of constraints and (2) the first value of the control series which is in each step calculated is applied to the system. For more details on MHC, the reader might refer to references [23] and [24].

MHHC approach is introduced in [25] and [50] to guarantee disturbance attenuation in MHC. The aim is to obtain a less conservative solution of the  $l_2$ -gain disturbance attenuation problem for linear systems with saturated actuators by adding a dissipation constraint to the online optimization problem. In MHHC, there exists no explicit prediction as in the MHC. Instead, it performs an infinite horizon feedback prediction.

### 4.1 Problem Formulation

Consider a linear discrete-time system governed by the equations,

$$\begin{aligned} x(k+1) &= Ax(k) + B_u u(k) + B_w w(k) \\ z(k) &= Cx(k) + D_u u(k) + D_w w(k). \end{aligned} \quad (4.1)$$

On the other hand the system is subject to control constraints

$$\|u_i(k)\|_{max} \leq u_{i,max}^2, \quad i = 1, 2, \dots, m. \quad (4.2)$$

where  $x(k) \in \mathbb{R}^n$  is the state vector;  $u(k), w(k) \in \mathbb{R}^m$  are the constrained control input and the disturbance vectors, respectively.  $z(k) \in \mathbb{R}^p$  is the controlled output  $k$  is the  $k$ -th

sample-time,  $x(k) \in \mathbb{R}^n$  is the state vector,  $u(k) \in \mathbb{R}^m$  represents the control input,  $\Phi(k)$  is a given initial condition sequence and  $w(k) \in \mathbb{R}^q$  is a disturbance signal in  $l_2$ , which satisfies

$$\sum_{i=0}^{\infty} \|w(i)\|^2 \leq \alpha^2 \quad (4.3)$$

The objective of MHC problem is to find the state-feedback control law in the form of  $u(k) = K_k x(k)$  at each instant  $k$ , so that the following conditions hold for the closed-loop system:

I. The closed-loop system

$$\begin{aligned} x(k+1) &= A_{cl}x(k) + B_w w(k) \\ z(k) &= C_{cl}x(k) + D_{1w}w(k) \end{aligned} \quad (4.4)$$

where  $A_{cl} \triangleq A(k) + B_u K_k$  and  $C_{cl} \triangleq C(k) + D_u K_k$ , is globally uniformly asymptotically stable under the conditions,  $w(k) = 0$  for all  $k \geq 0$  and (4.2).

II.

$$\|z_1(k)\|^2 < \gamma^2 \|w(k)\|^2 \quad \forall k \geq 0 \quad (4.5)$$

and for all non-zero disturbance signal  $w(k) \in l_2[0, \infty)$  and a given scalar  $\gamma > 0$  under the condition  $x(0) = 0$ .

III. The state trajectory remains in the ellipsoid

$$\Omega(P, r) := \{x \in \mathbb{R}^n : V(x(k)) \leq r \quad \forall k \geq 0\} \quad (4.6)$$

when the system is subject to a bounded-energy disturbance (4.3).

IV. The constraint on the size of control effort (4.2), is satisfied for all  $k \geq 0$ .

## 4.2 Overview of Moving-Horizon $\mathcal{H}_\infty$ Control

At each step  $k$  for  $k > 0$ , the optimization problem of constrained  $\mathcal{H}_\infty$  control is presented in (4.7).

$$\min_{\gamma, L, Y, X} \gamma \quad (4.7)$$

$$\begin{bmatrix} Y & * & * & * \\ 0 & \gamma I & * & * \\ AY + B_u L & B_w & Y & * \\ C_1 Y + D_{1u} L & D_{1w} & 0 & \gamma I \end{bmatrix} \geq 0$$

$$\begin{bmatrix} \frac{1}{r} X & L \\ L^T & Y \end{bmatrix} \geq 0 \quad X_{ii} \leq u_{i,max}^2$$

$$\begin{bmatrix} r - \gamma^2 \alpha^2 & x(k)^T \\ x(k) & Y \end{bmatrix} \geq 0$$

$$\begin{bmatrix} p_0 - p_{k-1} + x(k)^T P_{k-1} x(k) & x(k)^T \\ x(k) & Y \end{bmatrix} \geq 0$$

where  $Y \succ 0$ ,  $L = K P^{-1}$  and  $p(0)$  and  $p(k)$  are defined as follows:

$$\begin{aligned} p_0 &= x(0)^T P_0 x(0) \\ p_k &:= p_{k-1} - [x(k)^T P_{k-1} x(k) - x(k)^T P_k x(k)] \end{aligned} \quad (4.8)$$

The terms  $\gamma_k, L_k, Y_k, X_k$  are obtained by solving the optimization problem (4.7) at each time instant,  $k$ . Then the feedback control gain for each sampling time  $k$  can be obtained as  $K_k = L_k Y_k^{-1}$ .

LMIs given in (4.7) are used for the purpose of guaranteeing the  $\mathcal{H}_\infty$  performance, constraining the control input, keeping the states in a trajectory meeting the dissipativity condition, respectively. Next, the derivation of the algorithm will be introduced.

#### 4.2.1 Stability of the Moving-Horizon $\mathcal{H}_\infty$ Control

The following theorem describes a criterion for synthesizing of an  $\mathcal{H}_\infty$  state-feedback controller,  $u(k) = Kx(k)$  such that the nominal closed-loop system (4.4) is globally asymptotically stable. Although the proofs of theorems are given in references, we will introduce the proofs of theorem for the completeness of the note.

**Theorem 4.1:** [25] The state-feedback control law  $u(k) = LY^{-1}x(k)$  globally asymptotically stabilizes the system (4.1) with an  $\mathcal{H}_\infty$  disturbance attenuation level,  $\gamma$ , if there exist matrices  $Y = Y^T > 0$  and  $W = W^T > 0$ ,  $L$  with appropriate dimensions satisfying

$$\begin{bmatrix} Y & * & * & * \\ 0 & \gamma I & * & * \\ AY + B_u L & B_w & Y & * \\ CY + D_u L & D_w & 0 & \gamma I \end{bmatrix} > 0 \quad (4.9)$$

**Proof:** Let us choose a candidate Lyapunov functional as follows:

$$V(x(k)) = x^T(k)Px(k) \quad (4.10)$$

In view of (4.10), if

$$V(x(k+1)) - V(x(k)) + \|z_1(k)\| - \gamma^2 \|\omega(k)\| < 0 \quad (4.11)$$

is satisfied for all  $k > 0$ , then the closed-loop system (4.4) is guaranteed to be asymptotically stable when  $w(k) = 0$  and  $k \geq 0$  and  $\|z\|_2^2 \leq \gamma^2 \|w\|_2^2$  for all  $w(k) \in l_2[0, \infty)$ . Thus, (4.11) can be written as follows:

$$\begin{aligned} \Delta V(x(k)) = & x^T(k)[A_{cl}^T P A_{cl} - P + C_{cl}^T C_{cl}]x(k) + x^T(k)[A_{cl}^T P B_w + C_{cl}^T D_w]w(k) \\ & + x^T(k)[A_{cl}^T P B_w + C_{cl}^T D_w]w(k) + w^T(k)[B_w^T P A_{cl} + D_w^T C_{cl}]x(k) \\ & + w^T(k)[B_w^T P B_w + D_w^T D_w]w(k) + \gamma^2 w(k)^T w(k) < 0 \end{aligned} \quad (4.12)$$

Equivalently, (4.12) can be rewritten as

$$\begin{bmatrix} A_{cl}^T P A_{cl} - P + C_{cl}^T C_{cl} & A_{cl}^T P B_w + C_{cl}^T D_w \\ * & B_w^T P B_w + D_w^T D_w - \gamma^2 I \end{bmatrix} < 0. \quad (4.13)$$

Using the definitions of  $A_{cl}$ ,  $C_{cl}$  and applying immediately lead to

$$\begin{bmatrix} -P & 0 & A_{cl}^T P & C_{cl}^T \\ * & -\gamma^2 I & B_w^T P & D_w \\ * & * & -P & 0 \\ * & * & * & -I \end{bmatrix} < 0. \quad (4.14)$$

Pre- and post-multiplying (4.14) by  $\text{diag}\{P^{-1}, I, P^{-1}, I\}$  and substituting  $A_{cl} = A + B_u K$  and  $C_{cl} = C + D_u K$ , applying the Schur complement formula and defining  $Y = P^{-1}$  and  $L = K P^{-1}$  lead to the stability LMI.

#### 4.2.2 Dissipativity of Moving-Horizon $\mathcal{H}_\infty$ Control

The system has to satisfy the dissipation inequality in (2.1) at each  $k$  time. The following theorem provides a convex condition which guarantees the closed-loop dissipativity.

**Theorem 4.2:** [25, 51] Suppose that there exist an optimal solution  $(\gamma_0, Y_0, X_0)$  that meets the LMIs given in (4.7) for system (4.1) that is controlled with  $u = K_0 x$ , where  $K_0 = L_0 Y_0^{-1}$ . Then, the dissipation inequality

$$\sum_{i=0}^k (\|z_1(i)\|^2 - \gamma^2 \|w(i)\|^2) \leq x(0)^T P_0 x(0) \quad (4.15)$$

also holds if the LMI condition

$$\begin{bmatrix} p_0 - p_{k-1} + x(k)^T P_{k-1} x(k) & x(k)^T \\ x(k) & Y \end{bmatrix} \geq 0 \quad (4.16)$$

where,

$$p_k = p_{k-1} - x(k)^T [P_{k-1} - P_k] x(k) \quad (4.17)$$

is satisfied.

**Proof:** Let us consider the predetermined Lyapunov candidate functional for the  $i$ -th energy level as follows:

$$V_i = x^T(i) P x(i) \quad (4.18)$$

By rearranging the inequality (2.2) at each step  $i$ , and assuming that the applied control at  $i$ -th instant is kept constant up to the  $(i+1)$ -th instant, one can write the dissipation inequality as

$$V_{i+1} + \|z(i)\|^2 \leq \gamma^2 \|w(i)\|^2 + V_i \quad (4.19)$$

and the dissipation level memorized online computations as

$$p_{i-1} = p_0 - \sum_{k=1}^{i-1} [x(i)^T P_{i-1} x(i) - x(i)^T P_i x(i)]. \quad (4.20)$$

Considering (4.19) for  $i = 0, 1, \dots, k$

$$\begin{aligned} \|z(0)\|^2 - \gamma_0^2 \|w(0)\|^2 &\leq x(0)^T P_0 x(0) - x(1)^T P_0 x(1) \\ \|z(1)\|^2 - \gamma_1^2 \|w(1)\|^2 &\leq x(1)^T P_1 x(1) - x(2)^T P_1 x(2) \\ &\vdots \\ &\vdots \\ \|z(k)\|^2 - \gamma_k^2 \|w(k)\|^2 &\leq x(k)^T P_k x(k) - x(k+1)^T P_k x(k+1) \end{aligned} \quad (4.21)$$

yields

$$\begin{aligned} \sum_{i=0}^k \|z(i)\|^2 - \gamma_i^2 \|w(i)\|^2 &\leq x(0)^T P_0 x(0) - \sum_{i=1}^k x(i)^T [P_{i-1} - P_i] x(i) \\ &\quad + x(k+1)^T P_k x(k+1) \end{aligned} \quad (4.22)$$

and (4.20) for  $i = 0, 1, \dots, k$  yields

$$\begin{aligned} p_0 &= x(0)^T P_0 x(0) \\ p_1 &= p_0 - [x(1)^T P_0 x(1) - x(1)^T P_1 x(1)] \\ p_2 &= p_1 - [x(2)^T P_1 x(2) - x(2)^T P_2 x(2)] \\ &\vdots \\ p_k &= p_{k-1} - [x(k)^T P_{k-1} x(k) - x(k)^T P_k x(k)] \end{aligned} \quad (4.23)$$

Note that if

$$\sum_{i=1}^k x_i^T (P_{i-1} - P_i) x_i \geq 0 \quad (4.24)$$

then the closed-loop system behaves dissipatively, that is satisfies (4.15). In view of (4.22) and (4.24), we yield

$$\begin{aligned} \sum_{i=0}^k \|z(i)\|^2 - \max\{\gamma_0, \gamma_1, \dots, \gamma_k\}^2 \|w(i)\|^2 &\leq x(0)^T P_0 x(0) \\ &\quad - \sum_{i=1}^k x(i)^T [P_{i-1} - P_i] x(i) + x(k+1)^T P_k x(k+1). \end{aligned} \quad (4.25)$$

We infer from (4.17) that

$$p_{k-1} = p_0 - \sum_{i=1}^{k-1} x(i)^T [P_{i-1} - P_i] x(i) \quad (4.26)$$

Applying Schur complement formula to LMI (4.9) yields

$$p_0 - p_{k-1} + x(k)^T P_{k-1} x(k) - x(k)^T P_k x(k) \geq 0 \quad (4.27)$$

Therefore, using the definitions given in (4.26), in (4.27), yields (4.15). This concludes the proof. In order to avoid repetition, the derivation and proofs of the LMIs for input and state constraints will be given in detail in the following section.

## MOVING-HORIZON CONTROL FOR TIME-DELAY SYSTEMS

### 5.1 Problem Formulation

Let us consider a class of linear discrete-time, time-delay system with norm-bounded uncertainties of the form

$$\begin{aligned} x(k+1) &= A(k)x(k) + A_d(k)x(k-d) + B_w w(k) + B_u(k)u(k) \\ z(k) &= Cx(k) + C_d x(k-d) + D_w w(k) + D_u u(k) \\ x(k) &= \Phi(k), \quad k \in [-d_{max}, 0] \end{aligned} \quad (5.1)$$

subject to control constraints

$$\|u_i(k)\|_{max} \leq u_{i,max}^2, \quad i = 1, 2, \dots, m. \quad (5.2)$$

where  $k$  is the  $k$ -th sample-time,  $x(k) \in \mathbb{R}^n$  is the state vector,  $u(k) \in \mathbb{R}^m$  represents the control input,  $\Phi(k)$  is a given initial condition sequence and  $w(k) \in \mathbb{R}^q$  is a disturbance signal in  $l_2$ , which satisfies

$$\sum_{i=0}^{\infty} \|w(i)\|^2 \leq \alpha^2 \quad (5.3)$$

$z(k) \in \mathbb{R}^p$  is the controlled output and  $d(k) \in \mathbb{R}$  represents a time-varying delay which satisfies

$$d_{min} \leq d(k) \leq d_{max} \quad \forall k \geq 0 \quad (5.4)$$

where the nonnegative integers  $d_{min}$  and  $d_{max}$  stand for the lower and upper bounds of the delay,  $d(k)$ , respectively and all are assumed to be known. Note that the time-varying delay  $d(k)$  reduces to a constant delay  $d$  when  $d_{min} = d_{max} = d$ . On the other hand, the uncertain system matrices are assumed to be in the form of  $A(k) = A + \Delta A(k)$ ,  $A_d(k) = A + \Delta A_d(k)$ ,  $B_u(k) = B_u + \Delta B_u(k)$ . Here, the unknown

matrices  $\Delta A(k), \Delta A_d(k), \Delta B_u(k)$ , are real-valued, time-varying matrix functions representing the parameter uncertainties of the system, and are in the form of

$$\begin{bmatrix} \Delta A(k) & \Delta A_d(k) & \Delta B_u(k) \end{bmatrix} = GF(k) \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix} \quad (5.5)$$

where  $G, E_1, E_2$  are known time-invariant matrices and  $F(k)$  is an unknown time-varying matrix function with Lebesgue measurable elements satisfying

$$F(k)^T F(k) \leq I, \quad \forall k \geq 0.$$

The objective of MHC problem is to obtain the state-feedback control law in the form of  $u(k) = K_k x(k)$  at each instant  $k$ , so that the following conditions hold for the closed-loop system:

I. The closed-loop system

$$\begin{aligned} x(k+1) &= A_{cl}(k)x(k) + A_d(k)x(k-d(k)) + B_w w(k) \\ z(k) &= C_{cl}x(k) + C_d x(k-d(k)) + D_w w(k) \\ x(k) &= 0, \quad k \in [-d_{max}, 0] \end{aligned} \quad (5.6)$$

where  $A_{cl}(k) \triangleq A(k) + B_u(k)K_k$  and  $C_{cl} \triangleq C(k) + D_u(k)K_k$ , is globally uniformly asymptotically stable under the conditions,  $w(k) = 0$  for all  $k \geq 0$  and (5.2)

II.

$$\|z(k)\|^2 \leq \gamma^2 \|w(k)\|^2 \quad \forall k \geq 0 \quad (5.7)$$

and for all non-zero disturbance signal  $w(k) \in l_2[0, \infty)$  and a given scalar  $\gamma > 0$  under the condition  $x(k) = 0$  for all  $k \in [-d_{max}, 0]$ .

III. Let us define two nested ellipsoids as follows:

$$\begin{aligned} \varepsilon_1(P, r) &:= x \in \mathfrak{R}^n : V(x(k)) \leq r \quad \forall k \geq 0 \\ \varepsilon_2(P, r, \alpha) &:= x \in \mathfrak{R}^n : \gamma^2 \alpha^2 + V(x(k)) \leq r \quad \forall k \geq 0 \end{aligned} \quad (5.8)$$

If the initial state  $x(0)$  satisfies  $\gamma^2 \alpha^2 + V(x(0)) \leq r$ , i.e.,  $x(0)$  belongs to  $\varepsilon_2$ , then all perturbed state trajectories remain in the ellipsoid  $\varepsilon_1$ .

IV. The constraint on the size of control effort (5.2), is satisfied for all  $k \geq 0$ .



## 5.2 Stability of Delay Dependent Moving-Horizon $\mathcal{H}_\infty$ Control

Taking  $A(k) = A$ ,  $A_d(k) = A_d$ ,  $B_u(k) = B_u$ , the following theorem describes the criterion for constructing an  $\mathcal{H}_\infty$  state-feedback controller in the form of  $u(k) = Kx(k)$  such that the closed loop nominal system (5.6) with (5.4) is globally asymptotically stable.

**Theorem 5.1:** Given the scalar constants  $d_{min}$ ,  $d_{max}$  and  $\gamma$ , the state-feedback control law  $u(k) = LY^{-1}x(k)$  globally asymptotically stabilizes the time-delayed system (5.1) with (5.4) with an  $\mathcal{H}_\infty$  disturbance attenuation level,  $\gamma$ , if there exist matrices  $Y = Y^T > 0$  and  $W = W^T > 0$ ,  $L$  with appropriate dimensions satisfying

$$\begin{bmatrix} Y & * & * & * & * & * \\ 0 & W & * & * & * & * \\ 0 & 0 & \gamma^2 I & * & * & * \\ AY + B_u L & A_d W & B_w & Y & * & * \\ CY + D_u L & C_d W & D_w & 0 & I & * \\ Y & 0 & 0 & 0 & 0 & d_m^{-1} W \end{bmatrix} > 0 \quad (5.9)$$

where  $d_m \triangleq d_{max} - d_{min} + 1$ .

**Proof:** Let us choose a candidate L-K functional as follows:

$$V(x(k)) = V_1(k) + V_2(k) + V_3(k) \quad (5.10)$$

where

$$\begin{aligned} V_1(x(k)) &= x^T(k) P x(k), \\ V_2(x(k)) &= \sum_{s=k-d(k)}^{k-1} x^T(s) Q x(s), \\ V_3(x(k)) &= \sum_{i=-d_{max}+2}^{-d_{min}+1} \sum_{s=k+i-1}^{k-1} x^T(s) Q x(s). \end{aligned}$$

In view of closed-loop system trajectory (5.6), one can define a first difference of the energy function  $V(x(k))$  as follows:

$$\Delta V(k) \triangleq V(k+1) - V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) \quad (5.11)$$

where

$$\begin{aligned}
\Delta V_1(k) &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\
&= x^T(k)(A_{cl}^T P A_{cl} - P)x(k) + 2x^T(k)A_{cl}^T P A_d x(k-d(k)) \\
&\quad + 2x^T(k)A_{cl}^T P B_w w(k) + x^T(k-d(k))A_d^T P A_d x(k-d(k)) \\
&\quad + 2x^T(k-d(k))A_d^T P B_w w(k) + w^T(k)B_w^T P B_w w(k)
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\Delta V_2(k) &= \sum_{s=k-d(k+1)+1}^k x^T(s)Qx(s) - \sum_{s=k-d(k)}^{k-1} x^T(s)Qx(s) \\
&= x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) \\
&\quad + \sum_{s=k-d(k+1)+1}^{k-1} x^T(s)Qx(s) - \sum_{s=k-d(k)+1}^{k-1} x^T(s)Qx(s)
\end{aligned} \tag{5.13}$$

Since

$$\begin{aligned}
\sum_{s=k-d(k+1)+1}^k x^T(s)Qx(s) &= \sum_{s=k-d_{min}+1}^k x^T(s)Qx(s) + \sum_{s=k-d(k+1)+1}^{k-d_{min}} x^T(s)Qx(s) \\
&\leq \sum_{s=k-d(k)+1}^k x^T(s)Qx(s) + \sum_{s=k-d_{max}+1}^{k-d_{min}} x^T(s)Qx(s)
\end{aligned}$$

we can write

$$\Delta V_2(k) \leq x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) + \sum_{s=k-d_{max}+1}^{k-d_{min}} x^T(s)Qx(s). \tag{5.14}$$

Finally,

$$\begin{aligned}
\Delta V_3(k) &= \sum_{i=-d_{max}+2}^{-d_{min}+1} \left( \sum_{s=k+i}^k x^T(s)Qx(s) - \sum_{s=k+i-1}^{k-1} x^T(s)Qx(s) \right) \\
&= \sum_{i=-d_{max}+2}^{-d_{min}+1} (x^T(k)Qx(k) - x^T(k+i-1)Qx(k+i-1)) \\
&= (d_{max} - d_{min})x^T(k)Qx(k) - \sum_{i=k-d_{max}+1}^{k-d_{min}} x^T(s)Qx(s)
\end{aligned} \tag{5.15}$$

Then, in the light of (5.12), (5.14) and (5.15), a bound on  $\Delta V(k)$  can be obtained as follows:

$$\begin{aligned}
\Delta V(k) \leq & x^T(k)(A_{cl}^T P A_{cl} - P)x(k) + 2x^T(k)A_{cl}^T P A_d x(k-d(k)) \\
& + 2x^T(k)A_{cl}^T P B_w w(k) + x^T(k-d(k))A_d^T P A_d x(k-d(k)) \\
& + 2x^T(k-d(k))A_d^T P B_w w(k) + w^T(k)B_w^T P B_w w(k) \\
& + x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) \\
& + \sum_{s=k-d_{max}+1}^{k-d_{min}} x^T(s)Qx(s) + (d_{max} - d_{min})x^T(k)Qx(k) \\
& - \sum_{i=k-d_{max}+1}^{k-d_{min}} x^T(s)Qx(s)
\end{aligned} \tag{5.16}$$

Then, if we define the right-hand side of (5.16) by  $\bar{\Delta}V(k)$ , it is obvious that

$$\Delta V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \leq \bar{\Delta}V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k)$$

Therefore, if

$$\bar{\Delta}V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) < 0, \tag{5.17}$$

is satisfied for all  $k \geq 0$ , then the nominal closed-loop system (5.6) with (5.4) is guaranteed to be globally, asymptotically stable when  $w(k) = 0$  for all  $k \geq 0$ , and we get  $\|z\|_2^2 \leq \gamma^2 \|w\|_2^2$  for all  $w(k) \in l_2[0, \infty)$ . Equivalently, a straight forward application of the Schur complement formula on (5.17) allows to write

$$\begin{bmatrix} d_m Q - P + C_{cl}^T C_{cl} & * & * & * \\ C_d^T C_{cl} & -Q + C_d^T C_d & * & * \\ D_w^T C_{cl} & 0 & D_w^T D_w - \gamma^2 I & * \\ A_{cl} & A_d & B_w & -P^{-1} \end{bmatrix} < 0. \tag{5.18}$$

where  $d_m \triangleq d_{max} - d_{min} + 1$ . Applying the Schur complement formula on (5.18) for the terms,  $C_{cl}^T C_{cl}$ ,  $C_d^T C_d$  and  $D_1^T D_1$  yields

$$\begin{bmatrix} d_m Q - P & * & * & * & * \\ 0 & -Q & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * \\ A_{cl} & A_d & B_w & -P^{-1} & * \\ C_{cl} & C_d & D_w & 0 & -I \end{bmatrix} < 0. \tag{5.19}$$

Then, substituting  $A_{cl} = A + B_u K$  and  $C_{cl} = C + D_u K$  into (5.19) and pre- and post-multiplying (5.19) by  $\text{diag}\{P^{-1}, Q^{-1}, I, I, I\}$  allows to get

$$\begin{bmatrix} d_m P^{-1} Q P^{-1} - P^{-1} & * & * & * & * \\ 0 & -Q^{-1} & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * \\ AP^{-1} + B_u K P^{-1} & A_d Q^{-1} & B_w & -P^{-1} & * \\ CP^{-1} + D_u K P^{-1} & C_d Q^{-1} & D_w & 0 & -I \end{bmatrix} < 0 \quad (5.20)$$

Finally, defining  $Y := P^{-1}$ ,  $W := Q^{-1}$ ,  $L := K P^{-1}$  and applying Schur complement formulae, immediately lead to the LMI condition given in (5.9). This concludes our proof. ■

### 5.3 Robust Moving-Horizon $\mathcal{H}_\infty$ Control For Time Delay Systems

Now, let us consider (5.1) with uncertainties. The following corollary considers the robust state-feedback  $\mathcal{H}_\infty$  controller design problem for the uncertain system given in (5.1) and (5.5).

#### 5.3.1 Robust Stability Condition

**Corollary 5.2:** The control law  $u(k) = LY^{-1}x(k)$  robustly asymptotically stabilizes the uncertain system (5.1) and guarantees the  $\mathcal{H}_\infty$  performance index,  $\gamma$  for any time-varying delay,  $d(k)$  satisfying (5.4), if there exist matrices  $Y = Y^T > 0$ ,  $W = W^T > 0$ , matrix  $L$  in appropriate dimensions and a scalar  $\epsilon > 0$  which satisfy the following LMI condition:

$$\begin{bmatrix} Y & * & * & * & * & * & * \\ 0 & W & * & * & * & * & * \\ 0 & 0 & \gamma^2 I & * & * & * & * \\ AY + B_u L & A_d W & B_w & Y - \epsilon G G^T & * & * & * \\ CY + D_u L & C_d W & D_w & 0 & I & * & * \\ \theta Y & 0 & 0 & 0 & 0 & \theta W & * \\ E_2 L + E_1 Y & E_3 W & 0 & 0 & 0 & 0 & \epsilon I \end{bmatrix} > 0 \quad (5.21)$$

**Proof:** Replacing  $A$ ,  $A_d$  and  $B_u$  with  $A(k) = A + \Delta A$ ,  $A_d(k) = A_d + \Delta A_d$  and  $B_u(k) = B_u + \Delta B_u$ , in (5.9) and utilizing the definitions given in (5.5), together with the definitions

$$\Pi \triangleq \begin{bmatrix} E_2 L + E_1 Y & E_3 W & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.22)$$

and

$$\Theta \triangleq \begin{bmatrix} 0 & 0 & 0 & G^T & 0 & 0 \end{bmatrix}^T \quad (5.23)$$

yield

$$\begin{bmatrix} -Y & * & * & * & * & * \\ 0 & -W & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ AY + B_u L & A_d W & B_w & -Y & * & * \\ CY + D_u L & C_d W & D_w & 0 & -I & * \\ d_m Y & 0 & 0 & 0 & 0 & -d_m W \end{bmatrix} + \Pi^T F(k) \Theta^T + \Theta F(k) \Pi < 0 \quad (5.24)$$

Note that for any matrices  $\Sigma_1$  and  $\Sigma_2$  of appropriate dimensions, there always exists an  $\epsilon > 0$  such that

$$\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_1 < \frac{1}{\epsilon} \Sigma_1^T \Sigma_1 + \epsilon \Sigma_2^T \Sigma_2$$

Therefore

$$\Pi^T F(k) \Theta^T + \Theta F(k) \Pi \leq \frac{1}{\epsilon} \Pi^T \Pi + \epsilon \Theta \Theta^T \quad (5.25)$$

Then, if

$$\begin{bmatrix} -Y & * & * & * & * & * \\ 0 & -W & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ AY + B_u L & A_d W & B_w & -Y & * & * \\ CY + D_u L & C_d W & D_w & 0 & -I & * \\ d_m Y & 0 & 0 & 0 & 0 & -d_m W \end{bmatrix} + \frac{1}{\epsilon} \Pi^T \Pi + \epsilon \Theta \Theta^T < 0 \quad (5.26)$$

is satisfied, (5.24) is also satisfied. Finally, applying Schur complement formulae to (5.26) yields (5.21). This completes the proof. ■

Also, we need to take the constrained input into consideration to fulfill the requirement given in item IV of the problem statement. The next subsection is devoted to the derivation of the requirement which guarantees an unsaturated signal for all times.

### 5.3.2 Input Constraint

If the dissipation inequality (2.2) holds true for any system trajectory of (5.1) and (5.4), then for any disturbance signal satisfying (5.3), the energy of the controlled output is also bounded as

$$\sum_{i=0}^{\infty} \|z(i)\|^2 \leq r \quad (5.27)$$

and  $l_2$  gain of the closed-loop system (5.6) from the disturbance  $w(k)$  to the performance output  $z(k)$  is bounded by  $\gamma$ .

Regarding the state trajectories belonging to  $\varepsilon_1$ , the control constraint condition given in (5.2) is satisfied as follows [52]:

$$\begin{aligned} \max_{k \geq 0} \|u_i(k)\|_{max} &= \max_{k \geq 0} \|(LY^{-1})_i x(k)\|_{max} \\ &\leq \max_{x \in \varepsilon_1(P, r)} \|(LY^{-1})_i x\|_{max} \\ &\leq r \|(LY^{-1/2})_i x\|_2^2 = r(LY^{-1}L^T)_{ii} \end{aligned} \quad (5.28)$$

Therefore, the taking the Schur complement of (5.28) gives the following LMI condition which is equivalent to the constraint (5.2):

$$\begin{bmatrix} 1/rX & L \\ L^T & Y \end{bmatrix} \geq 0, \quad X_{ii} \leq u_{i,max}^2 \quad (5.29)$$

where  $Y = P^{-1}$  and  $L = KP^{-1}$ .

Next we'll introduce some LMI conditions which fulfills the requirement given in item III.

### 5.3.3 State Constraint

**Lemma 5.3:** Suppose that there exist a solution  $(\gamma, Y, W, X)$  that meets the stability condition (5.21) with input constraint (5.2) for system (5.1) and (5.4), then state trajectory starting from  $x(0)$  remains in  $\varepsilon_1(P, r)$ , if there exist matrices  $Y = Y^T > 0$  and  $W = W^T > 0$  for given positive scalar constants  $\alpha$  and  $r$ , such that

$$\begin{bmatrix} r - \gamma^2 \alpha^2 & * & * & \cdots & * & \cdots & * & * & * \\ x(k) & Y & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ x(k-1) & 0 & d_m^{-1}W & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x(k-d_{min}) & 0 & 0 & \cdots & d_m^{-1}W & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x(k-d_{max}+2) & 0 & 0 & \cdots & 0 & \cdots & 3^{-1}W & 0 & 0 \\ x(k-d_{max}+1) & 0 & 0 & \cdots & 0 & \cdots & 0 & 2^{-1}W & 0 \\ x(k-d_{max}) & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & W \end{bmatrix} \geq 0 \quad (5.30)$$

where  $d_m = (d_{max} - d_{min} + 1)$ .

**Proof:** The LMI condition (5.30) corresponds to the state constraint that is defined in (5.8). Utilizing the energy bound of disturbance given in (5.3), the dissipation inequality (2.2) implies

$$V(x(k)) + \sum_{i=0}^{k-1} \|z(i)\|^2 \leq V(x(0)) + \gamma^2 \alpha^2 \quad \forall k \geq 0. \quad (5.31)$$

Therefore, the output energy is bounded as (5.27).

As explained in [43],

- (i). if  $r > 0$ ,  $x(k) \in \varepsilon_1(P, r)$  for  $\forall k \geq 0$ .
- (ii). if  $r > \gamma^2 \alpha^2$ ,  $x(k) \in \varepsilon_1(P, r)$  for  $x(0) \in \varepsilon_2(P, r, \alpha)$ .

Here, by choosing  $r > \gamma^2 \alpha^2$ , the state trajectory starting from  $x(0)$  meets  $\sum_{i=0}^{\infty} \|z(i)\|^2 \leq r$  and  $V(x(k)) \leq r$  if  $V(x(k)) \leq r - \gamma^2 \alpha^2$  with respect to (5.31) as mentioned in (5.7). Note that it is easy to see that the LMI condition (5.30) is congruent to

$$x^T(k)Px(k) + \sum_{s=k-d(k)}^{k-1} x^T(s)Qx(s) + \sum_{i=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=k+i-1}^{k-1} x^T(s)Qx(s) \leq r - \gamma^2 \alpha^2 \quad (5.32)$$

in view of the variable changes  $Y := P^{-1}$  and  $W := Q^{-1}$  and the successive implementations of Schur complement formulae together with the bounding

$$\sum_{s=k-d(k)}^{k-1} x^T(s)Qx(s) \leq \sum_{s=k-d_{\min}}^{k-1} x^T(s)Qx(s) + \sum_{s=k-d_{\max}}^{k-d_{\min}-1} x^T(s)Qx(s) \quad (5.33)$$

which has been used to eliminate the requirement on the knowledge of  $d(k)$ .

Therefore, for given scalar constants  $\alpha > 0$  and  $r > 0$ , the optimization problem can be stated as follows:

$$\begin{aligned} & \min_{L, Y, W, X} \gamma^2 \\ & \text{subject to} \quad (5.21) \text{ and } (5.29), (5.30). \end{aligned} \quad (5.34)$$

Next, we will extend the  $\mathcal{H}_{\infty}$  controller design approach to the moving horizon scheme. Exploiting the moving horizon strategy, one would solve the optimization problem

given in (5.34) utilizing the actual state measurements provided at each time instant  $k$ . Unfortunately, this simple implementation of the moving horizon strategy might fail to guarantee the dissipation requirement given in the statement of the problem (*see* [51] for details). For this reason, in order to extend the idea of  $\mathcal{H}_\infty$  control approach provided for the TDSs to the moving horizon platform, one needs to introduce some additional convex requirement so that the dissipativity of the closed-loop system is ensured. Next theorem provides an LMI condition for the optimization problem so that the closed-loop system is guaranteed to be dissipative.

### 5.3.4 Dissipativity Condition

**Theorem 5.4:** Suppose that there exist an optimal solution  $(\gamma_0, Y_0, W_0, X_0)$  that meets the LMIs, (5.21), (5.29) and (5.30) for system (5.1) that is controlled with  $u = K_0 x$ , where  $K_0 = L_0 Y_0^{-1}$ . Then the dissipation inequality

$$\sum_{i=0}^k (\|z(i)\|^2 - \gamma^2 \|w(i)\|^2) \leq x(0)^T P_0 x(0) + \sum_{s=-d_0}^{-1} x(s)^T Q_0 x(s) + \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j-1}^{-1} x(s)^T Q_0 x(s) \quad (5.35)$$

also holds if the LMI condition

$$\begin{bmatrix} \Omega & * & * & \cdots & * & \cdots & * & * \\ x(k) & Y_k & 0 & \cdots & 0 & \cdots & 0 & 0 \\ x(k-1) & 0 & d_m^{-1} W_k & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ x(k-d_{min}) & 0 & 0 & \cdots & d_m^{-1} W_k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ x(k-d_{max}+2) & 0 & 0 & \cdots & 0 & \cdots & 2^{-1} W_k & 0 \\ x(k-d_{max}+1) & 0 & 0 & \cdots & 0 & \cdots & 0 & W_k \end{bmatrix} \geq 0 \quad (5.36)$$

where  $d_m = (d_{max} - d_{min} + 1)$  and

$$\begin{aligned} \Omega = & p_0 + q_0 - p_{k-1} - q_{k-1} + x^T(k) P_{k-1} x(k) \\ & + \sum_{s=k-d_{max}}^{k-1} x^T(s) Q_{k-1} x(s) + \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j+k-1}^{k-1} x^T(s) Q_{k-1} x(s). \end{aligned}$$



with

$$\begin{aligned}
p_k &= p_{k-1} - x(k)^T [P_{k-1} - P_k] x(k) \\
q_k &= q_{k-1} - \sum_{s=k-d_{\min}}^{k-1} x(s)^T Q_{k-1} x(s) - \sum_{k-d_{\min}}^{k-1} x(s)^T Q_k x(s) \\
&\quad - \sum_{j=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=j+k-1}^{k-1} x(s)^T [Q_{k-1} - Q_k] x(s)
\end{aligned} \tag{5.37}$$

is satisfied when  $Y_k$  and  $W_k$  are replaced with  $Y_0 = Y_0^T$  and  $W_0 = W_0^T$ , respectively.

**Proof:** Assume there exists a feasible solution set which holds (5.36). Let us consider the predetermined L-K candidate functional for the  $i$ -th energy level as follows:

$$V_i = x^T(i) P x(i) + \sum_{s=i-d(i)}^{i-1} x^T(s) Q x(s) + \sum_{j=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=i+j-1}^{i-1} x^T(s) Q x(s) \tag{5.38}$$

By rearranging the inequality (2.2) at each step  $i$ , and assuming that the applied control at  $i$ -th instant is kept constant up to the  $(i+1)$ -th instant, one can write the dissipation inequality as

$$V_{i+1} + \|z(i)\|^2 \leq \gamma^2 \|w(i)\|^2 + V_i.$$

Re-arranging the dissipation inequality together with using the L-K functional given in (5.38) leads to

$$\begin{aligned}
&x^T(i+1) P x(i+1) + \sum_{s=i-d(i+1)+1}^i x^T(s) Q x(s) + \sum_{j=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=i+j}^i x^T(s) Q x(s) + \|z(i)\|^2 \\
&\leq \gamma^2 \|w(i)\|^2 + x^T(i) P x(i) + \sum_{s=i-d(i)}^{i-1} x^T(s) Q x(s) + \sum_{j=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=i+j-1}^{i-1} x^T(s) Q x(s)
\end{aligned} \tag{5.39}$$

In order to get rid of the term  $d(i+1)$  in (5.39), we need to find an upper bound on the second term on the left side of (5.39). Note that

$$\begin{aligned}
\sum_{s=i-d(i+1)+1}^i x^T(s) Q x(s) &= \sum_{s=i-d_{\min}+1}^i x^T(s) Q x(s) + \sum_{s=i-d(i+1)+1}^{i-d_{\min}} x^T(s) Q x(s) \\
&\leq \sum_{s=i-d(i)+1}^i x^T(s) Q x(s) + \sum_{s=i-d_{\max}+1}^{i-d_{\min}} x^T(s) Q x(s)
\end{aligned}$$

for  $i = 0, 1, \dots, k$  where  $V_{i+1}$  is the energy of the closed-loop system attained by using the Lyapunov matrices obtained at the  $i$ -th step. Then we get

$$\begin{aligned}
& x^T(1)P_0x(1) + \sum_{s=-d(1)+1}^0 x^T(s)Q_0x(s) + \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j}^0 x^T(s)Q_0x(s) + \|z(0)\| \\
& \leq \gamma^2\|w(0)\| + x^T(0)P_0x(0) + \sum_{s=-d(0)}^{-1} x^T(s)Q_0x(s) + \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j-1}^{-1} x^T(s)Q_0x(s) \\
& x^T(2)P_1x(2) + \sum_{s=-d(2)+2}^1 x^T(s)Q_1x(s) + \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j+1}^1 x^T(s)Q_1x(s) + \|z(1)\| \\
& \leq \gamma^2\|w(1)\| + x^T(1)P_1x(1) + \sum_{s=1-d(1)}^0 x^T(s)Q_1x(s) + \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j}^0 x^T(s)Q_1x(s) \\
& \quad \vdots \\
& \quad \vdots \\
& x^T(k+1)P_kx(k+1) + \sum_{s=k-d(k)+1}^k x^T(s)Q_kx(s) + \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=k+j}^k x^T(s)Q_kx(s) + \|z(k)\| \\
& \leq \gamma^2\|w(k)\| + x^T(k)P_kx(k) + \sum_{s=k-d(k)}^{k-1} x^T(s)Q_kx(s) + \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=k+j-1}^{k-1} x^T(s)Q_kx(s)
\end{aligned} \tag{5.40}$$

Note that (5.40) can be rewritten in a compact representation as follows:

$$\begin{aligned}
& \sum_{i=0}^k (\|z(i)\|^2 - \gamma^2\|w(i)\|^2) \leq x(0)^T P_0 x(0) - x(k+1)^T P_k x(k+1) - \sum_{i=1}^k x(i)^T [P_{i-1} - P_i] x(i) \\
& + \sum_{s=-d_0}^{-1} x(s)^T Q_0 x(s) - \sum_{s=-d_{k+1}+k+1}^k x(s)^T Q_k x(s) - \sum_{i=1}^k \sum_{s=i-d_i}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s) \\
& + \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j-1}^{-1} x(s)^T Q_0 x(s) - \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j+k}^k x(s)^T Q_k x(s) \\
& - \sum_{i=1}^k \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j+i-1}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s)
\end{aligned} \tag{5.41}$$

Assuming that  $P_k \geq 0$ ,  $Q_k \geq 0$  for all  $k \geq 0$ , one can write

$$\begin{aligned}
\sum_{i=0}^k (\|z(i)\|^2 - \gamma^2 \|w(i)\|^2) &\leq x(0)^T P_0 x(0) - \sum_{i=1}^k x(i)^T [P_{i-1} - P_i] x(i) \\
&\quad + \sum_{s=-d_0}^{-1} x(s)^T Q_0 x(s) - \sum_{i=1}^k \sum_{s=i-d_i}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s) \\
&\quad + \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j-1}^{-1} x(s)^T Q_0 x(s) - \sum_{i=1}^k \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j+i-1}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s) \quad (5.42)
\end{aligned}$$

Note that if

$$\begin{aligned}
\sum_{i=1}^k x(i)^T (P_{i-1} - P_i) x(i) &+ \sum_{i=1}^k \sum_{s=i-d_i}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s) \\
&\quad + \sum_{i=1}^k \sum_{j=-d_{max}+2}^{-d_{min}+1} \sum_{s=j+i-1}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s) \geq 0 \quad (5.43)
\end{aligned}$$

then the closed-loop system behaves dissipatively. However, the inequality

(5.43) involves the term

$$\sum_{i=1}^k \sum_{s=i-d_i}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s)$$

which is dependent on the knowledge of  $d(i)$ . Therefore, if one can find a lower bound,  $B$  such that

$$\sum_{i=1}^k \sum_{s=i-d_i}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s) \geq B \geq 0$$

then we guarantee the dissipativity of the overall closed-loop system. Note that

$$\sum_{i=1}^k \sum_{s=i-d_i}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s) \geq \sum_{i=1}^k \sum_{s=i-d_{min}}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s)$$

Hence, we need to satisfy

$$\begin{aligned}
\sum_{i=1}^k x(i)^T (P_{i-1} - P_i) x(i) &+ \sum_{i=1}^k \sum_{s=i-d_{\min}}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s) \\
&+ \sum_{i=1}^k \sum_{j=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=j+i-1}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s) \geq 0
\end{aligned} \tag{5.44}$$

We infer from (5.37) that

$$\begin{aligned}
p_{k-1} &= p_0 - \sum_{i=1}^{k-1} x(i)^T [P_{i-1} - P_i] x(i) \\
q_{k-1} &= q_0 - \sum_{i=1}^k \sum_{s=i-d_i}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s) \\
&\quad - \sum_{i=1}^k \sum_{j=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=j+i-1}^{i-1} x(s)^T [Q_{i-1} - Q_i] x(s)
\end{aligned} \tag{5.45}$$

Applying Schur complement formulae to LMI (5.36) yields

$$\begin{aligned}
p_0 - p_{k-1} &+ x(k)^T [P_{k-1} - P_k] x(k) \\
&+ q_0 - q_{k-1} + \sum_{s=k-d_{\min}}^{k-1} x(s)^T [Q_{k-1} - Q_k] x(s) \\
&+ \sum_{j=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=j+k-1}^{k-1} x(s)^T [Q_{k-1} - Q_k] x(s) \geq 0
\end{aligned} \tag{5.46}$$

Then, using the definitions given in (5.45), in (5.46), yields (5.44). This concludes the proof. ■

Therefore, in view of (5.29) together with Corollary 5.2, Lemma 5.3 and Theorem 5.4, we present a new convex optimization problem as follows:

**Theorem 5.5:** Given scalar constants  $\alpha_0 > 0$  and  $r_0 > 0$ , at each step  $k$ , if one can find a common feasible solution set  $\{\gamma_k, L_k, Y_k, W_k, X_k\}$  to the convex optimization problem

$$\begin{aligned}
&\min_{L, Y, W, X} \gamma^2 \\
&\text{subject to} \quad (5.21), (5.29), (5.30) \text{ and } (5.36)
\end{aligned} \tag{5.47}$$

then system (5.1) controlled with state-feedback control law  $u(k) = K_k x(k)$  where  $K_k = L_k Y_k^{-1}$  has the following properties:

- (i). The closed-loop system is  $\mathcal{H}_\infty$  stable,
- (ii). The  $\mathcal{H}_\infty$  gain  $\gamma$  from disturbance input  $w$  to controlled output  $z$  is minimum.
- (iii). The control input constraint is satisfied.
- (iv). The closed-loop system is dissipative.

The following receding-horizon  $\mathcal{H}_\infty$  algorithm can be used to find the optimal control law for system (5.1) with (5.4) which repeats at each time instant  $k$ .

**Algorithm1:**

**Step1.** According to given  $\alpha_0$ ,  $r_0$ ,  $d_{max}$  and  $d_{min}$  find the optimal  $\gamma$  by solving the optimization problem (5.34). First, fix  $\gamma$  to a sufficiently large value and solve the optimization problem (5.34). If there exist a feasible solution to the problem, set  $\gamma = \gamma - d\gamma$  until infeasibility. Then, assign  $\gamma_{opt} = \gamma + d\gamma$ . Compute  $P_0, Q_0$ . Then  $K_0 = L_0 Y_0^{-1}$  and the initial dissipation level  $p(0) = x(0)^T P_0 x(0)$  and  $q(0) = \sum_{s=-d_{min}}^{-1} x^T(s) Q_0 x(s) + \sum_{s=-d_{max}+2}^{-d_{min}+1} \sum_{s=j-1}^{-1} x^T(s) Q_0 x(s)$ . Go to Step3.

**Step2.** According to given  $\alpha_0$ ,  $r_0$ ,  $d_{max}$  and  $d_{min}$  and obtained dissipation level  $p(k)$  and  $q(k)$ , solve the optimization problem (5.47). If there is a feasible solution, set  $\gamma(k) = \gamma(k) - d\gamma$  until infeasibility. Then, assign  $\gamma(k) = \gamma(k) + d\gamma$ . If there is no possible solution, increase  $r$  until a feasible solution is found and repeat  $\gamma(k)$  minimization procedure. Then, for the optimal value of  $\gamma_{min}$ , compute  $P_k, Q_k$  and  $K_k = L_k Y_k^{-1}$  and the dissipation level  $p_k$  and  $q_k$ . Go to Step3.

**Step3.** Apply the control signal  $u(k) = K_k x(k)$  to the system. Change the step from  $k$  to  $k + 1$  and continue with Step2.

## NUMERICAL EXAMPLES

**Example 1:**

In the first example, moving horizon  $\mathcal{H}_\infty$  control used for tracking a desired methane gas output from an anaerobic wastewater treatment reactor is compared with classical  $\mathcal{H}_\infty$  control and PID. The nonlinear system is first linearized around the set of the equilibrium points and the operating points selected, then LMI conditions required for comply with the constraints and guarantee the stability are constructed. It is assumed that time-delay,  $d = 0$ . Finally, the experimental results are demonstrated by implementing the proposed control method on Matlab environment.

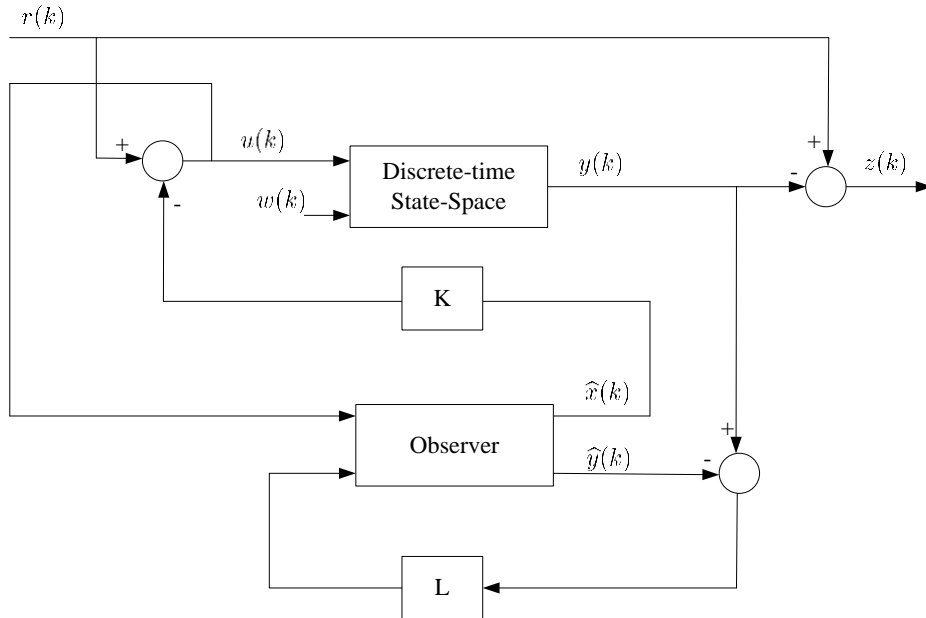


Figure 6.1 Linear state-space model (linmod) of the system with reference

The mathematical model of the system is described by the following ordinary differential equations:

$$\begin{aligned}
\frac{dX_1}{dt} &= (\mu_1 - D)X_1 \\
\frac{dX_2}{dt} &= (\mu_2 - D)X_2 \\
\frac{dS_1}{dt} &= D(S_1^{in} - S_1) - \mu_1 k_1 X_1 \\
\frac{dS_2}{dt} &= D(S_1 - S_2) + \mu_1 k_2 X_1 - \mu_2 k_3 X_2
\end{aligned} \tag{6.1}$$

Here, the states of the system,  $X_1$  and  $X_2$  denote biomass concentrations and  $S_1$  and  $S_2$  denote substrate concentrations. The disturbance, which is the influent concentration, is denoted with  $S_1^{in}$ . The dilution rate,  $D$ , is chosen as the control input. The specific growth rate of the microorganisms,  $\mu$ , is defined to be as

$$\mu = \frac{\mu_{max} S}{K_S + S} \tag{6.2}$$

The methane flow rate,  $Q$  is the measurable output and defined to be as:

$$Q = k_4 \mu_2 X_2 \tag{6.3}$$

The parameters of the anaerobic system to be controlled are presented in Table 6.1. The values at the operating points, where the linearization process is performed are selected as follows,  $D = 0.03 \text{ (day}^{-1}\text{)}$  and  $S_1^{in} = 1 \text{ (g/l)}$ .

Table 6.1 The parameters of the anaerobic system

$\mu_{1max}$ (day) <sup>-1</sup>	$\mu_{2max}$ (day) <sup>-1</sup>	$K_{S1}$ (g/l)	$K_{S2}$ (g/l)	$k_1$	$k_2$	$k_3$	$k_4$
0.2	0.25	0.3	0.87	6.7	4.2	5	4.35

The linearization procedure is given in detail in the Ref. [53]. After some straight forward calculations the linearized state-space model can be obtained as follows:

$$\begin{aligned}
A &= \begin{bmatrix} 0.9999 & 0 & 0.0066 & 0 \\ 0 & 1 & 0 & 0.0023 \\ -0.0196 & 0 & 0.9525 & 0 \\ 0.0122 & -0.0149 & 0.0307 & 0.9853 \end{bmatrix} \\
B_w &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.0029 \\ 0 & 0 \end{bmatrix} & B_u &= \begin{bmatrix} -0.0138 \\ -0.0106 \\ 0.0926 \\ -0.0051 \end{bmatrix} \\
C_y &= [0 \ 0.132 \ 0 \ 1.79] & C_z &= [0 \ -0.132 \ 0 \ -1.79]
\end{aligned} \tag{6.4}$$

As explained with a flow diagram shown in Figure 6.2, first, the minimum  $\gamma_{opt}$  is computed for  $\mathcal{H}_\infty$  problem with no constraint. We obtain  $\gamma_{opt} = 1.2247$ . The parameter for bounding the disturbance was selected as  $\alpha = 1$  and consequently  $r_0 = 10\gamma_{opt}^2\alpha^2$  is obtained. In this case, the control input is constrained as  $\|u(k)\|_{max} \leq 1$ .

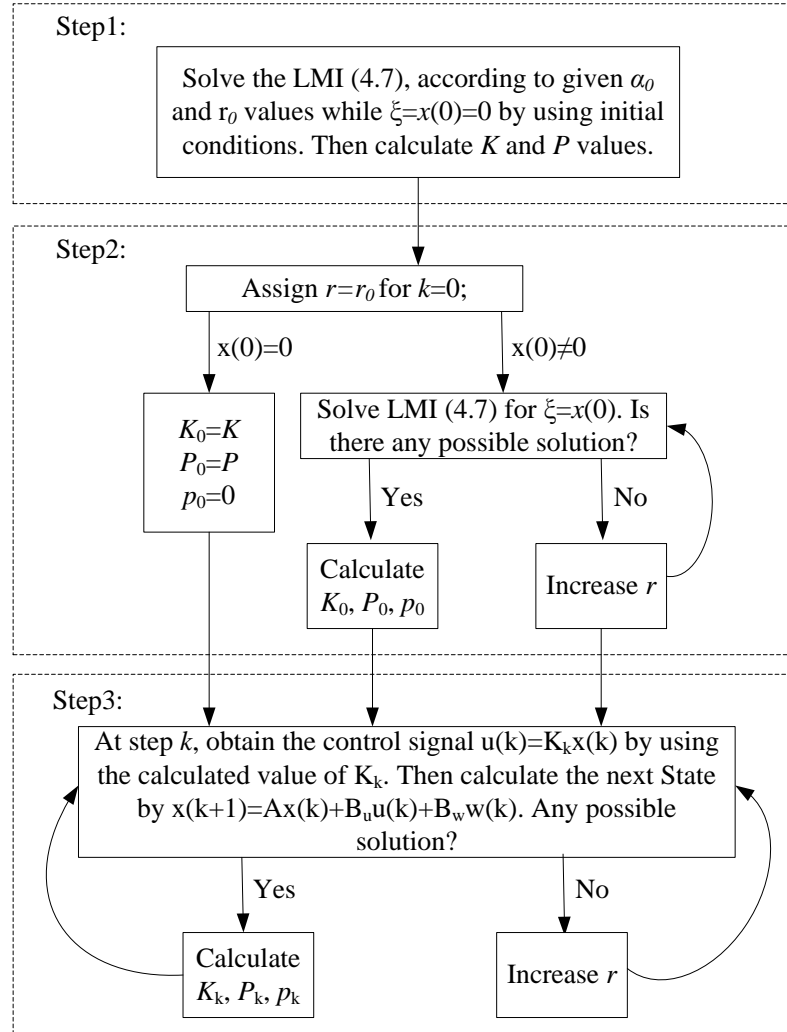


Figure 6.2 The flowchart of the algorithm for the moving horizon  $\mathcal{H}_\infty$  control



During the control experiments, first, a PID control is applied to the system. The PID coefficients that make the controller performance optimum were obtained by means of the Ziegler-Nichols method given in [54] and provided in Table 6.2.

Table 6.2 The PID coefficients computed by means of Ziegler-Nichols method.

$K_P$	$K_I$	$K_D$
2.23	0.98	1.32

A step type disturbance input, as shown in Figure 6.3 (a), is applied to the system controlled with the PID controller. The variation of the states of the system under the PID control is presented in Fig. 6.3 (b).

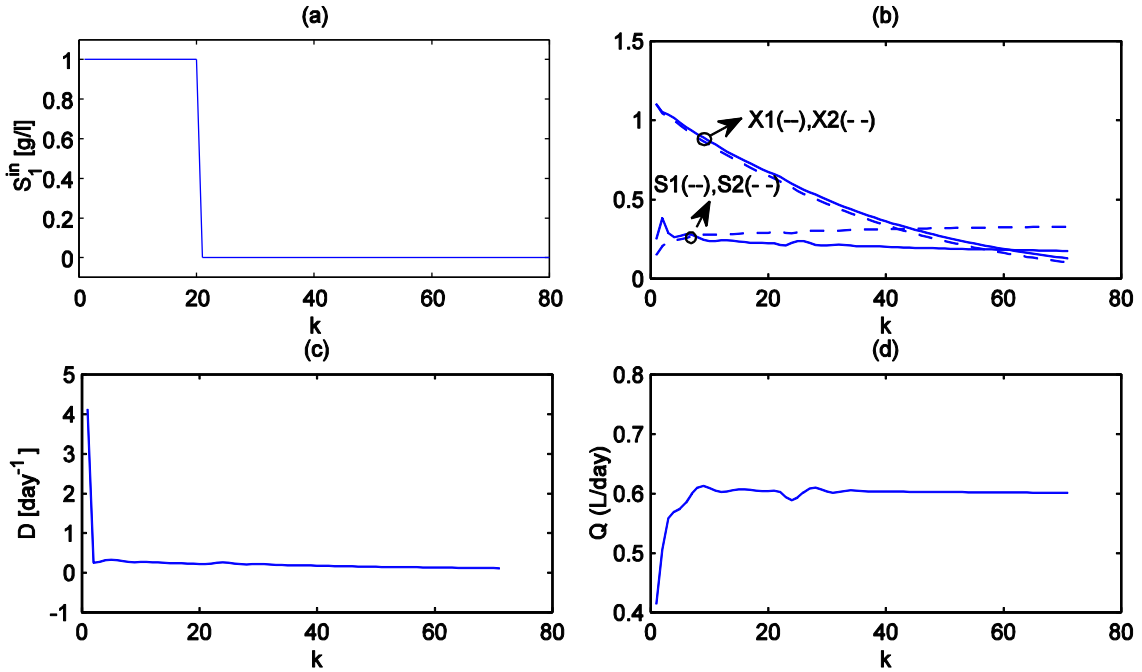


Figure 6.3 PID control of the anaerobic system (a)Disturbance  $w(k)$ , (b) bacteria growth curves in the reactors,  $X_1$ ,  $X_2$ , and substrate levels  $S_1$ ,  $S_2$ , (c) control input,  $u(k)$ (d) methane gas output  $y(k)$ .

Figure 6.3 (c) shows the time-history of the control signal. It is practically obtained through an input valve, which manipulates the dilution rate,  $D$ . Due to the effect of integral term of the PID controller, the methane gas output reaches the desired level in a reasonably short time as seen in Figure 6.3 (d). However, it should be noted that the PID coefficients computed are valid only for the operating point taken into consideration.

Then, as a second case, the system of interest is controlled first with the classical  $\mathcal{H}_\infty$  controller followed by the moving horizon  $\mathcal{H}_\infty$  controller. The simulation results obtained from these two control approaches are plotted in the same graph as shown in Figure 6.4. For this case the same disturbance signal is used as in the previous case. The variation of the in the state variables of the first and the second reactor are shown in Figure 6.4 (a) and Figure 6.4 (b), respectively. The energy of control signals are shown in Figure 6.4 (c) and the variation of the methane gas outputs are depicted in Figure 6.4 (d). The methane gas outputs produced by applying the moving horizon  $\mathcal{H}_\infty$  and classical  $\mathcal{H}_\infty$  controls to the system are plotted in the same graph for a fair comparison as shown in Figure 6.4 (d). It is easy to see that, MHHC method allows us to obtain much better control performance when compared to those obtained by using classical  $\mathcal{H}_\infty$  method. This is reasonable because the optimization is performed at each step of run-time instead of an off-line design. Furthermore, the energy of the control signal used for the moving horizon control is found to be smaller compared to the classical  $H_\infty$  control, as clearly indicated in Figure 6.4 (c). Thus, it is concluded that moving horizon  $\mathcal{H}_\infty$  control has yielded satisfactory results.

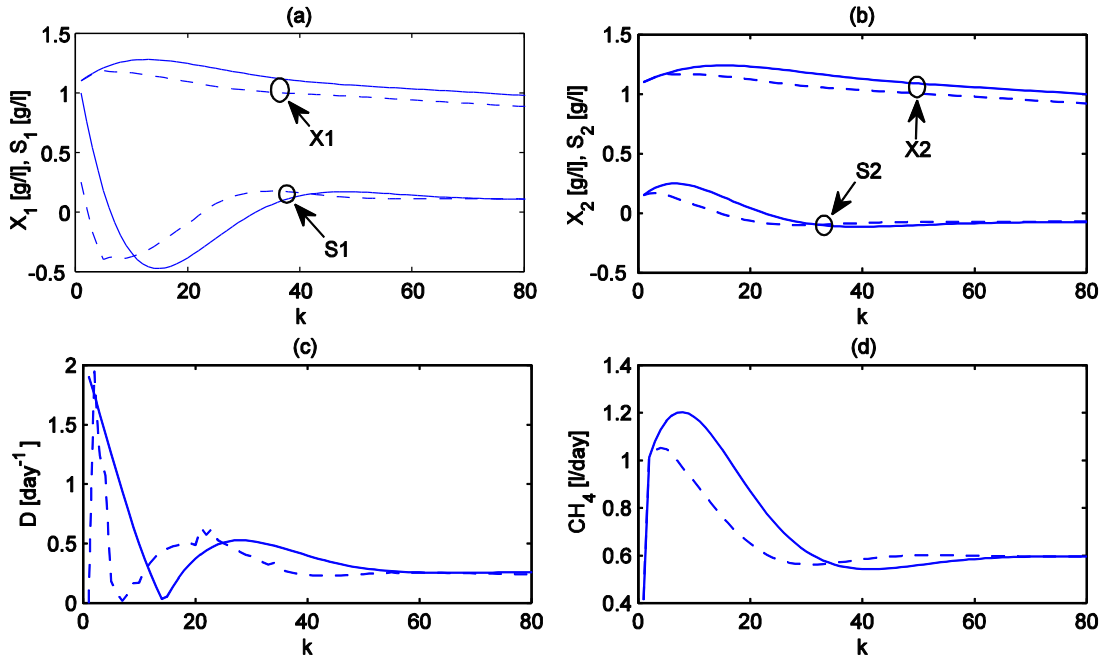


Figure 6.4 Moving horizon  $\mathcal{H}_\infty$  control (—) vs.  $\mathcal{H}_\infty$  control (—) on

- (a) State variables of the first reactor, (b) state variables of the second reactor, (c) energy of inputs  $\|u(k)\|$  and, (d) controlled outputs,  $z(k)$ .

**Example 2:**

The following example is borrowed from [41] in which a nominal  $\mathcal{H}_\infty$  control problem has been considered. Consider a state-delayed system (5.1) whose model parameters are given by

$$A = \begin{bmatrix} 0 & 1 \\ -0.14 & 0.9 \end{bmatrix} \quad A_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix} \quad B_w = B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \quad C_d = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad D_w = \begin{bmatrix} 0 \end{bmatrix} \quad D_u = \begin{bmatrix} 0.1 \end{bmatrix}$$

$$x(i) = [0 \ -1.5]^T \quad i = 0, -1, \dots, d_{max} \quad \|u(k)\| \leq 1, \forall k \geq 0.$$

Choosing  $r_0 = 0.3$ , and using the energy bounded disturbance whose time-history is as shown in Figure 6.5, we compute  $\gamma_{opt} = 1.18$  for  $d_{max} = d_{min} = 2$  at the initial step of Algorithm1. Applying the proposed algorithm on the system we obtain satisfactory results in  $\gamma$  minimization which also leads us to obtain a significant improvement in disturbance rejection performance. Fig. 6.6 shows the changes in  $\gamma$  with respect to delay bounds. On the other hand, Table 6.3 provides the  $r$  values at each step.

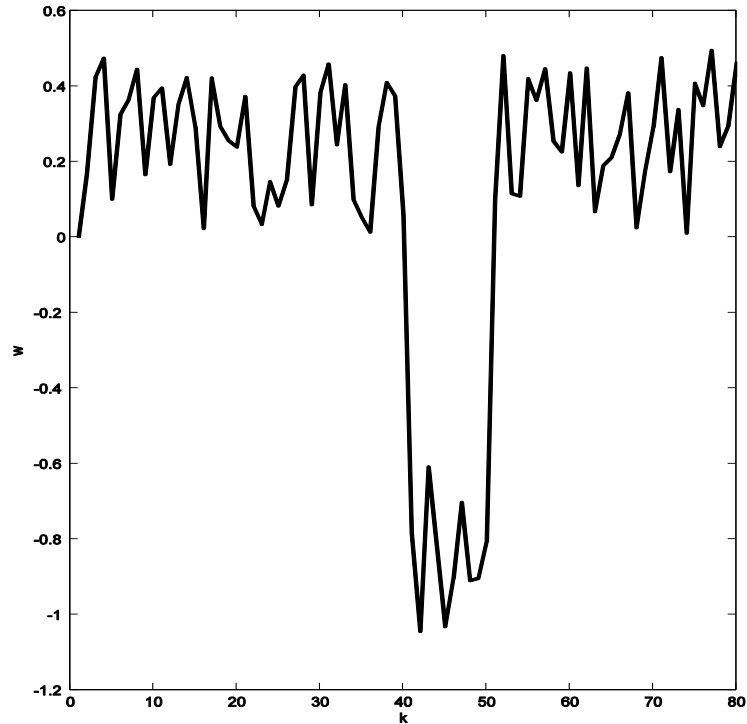


Figure 6.5 Disturbance signal applied to the system

Table 6.3  $r$  values at each step in Example 2

$k$	1	2-40	41	42-48	49-50	51-58
$r$	0.3	0.4	0.6	0.7	0.75	0.45

For different delay bounds,  $d_{max}$  and  $d_{min}$  the variation of the controlled output,  $z$  and the control signal,  $u$  are shown in Figure 6.7, Figure 6.8, Figure 6.9 and Figure 6.10.

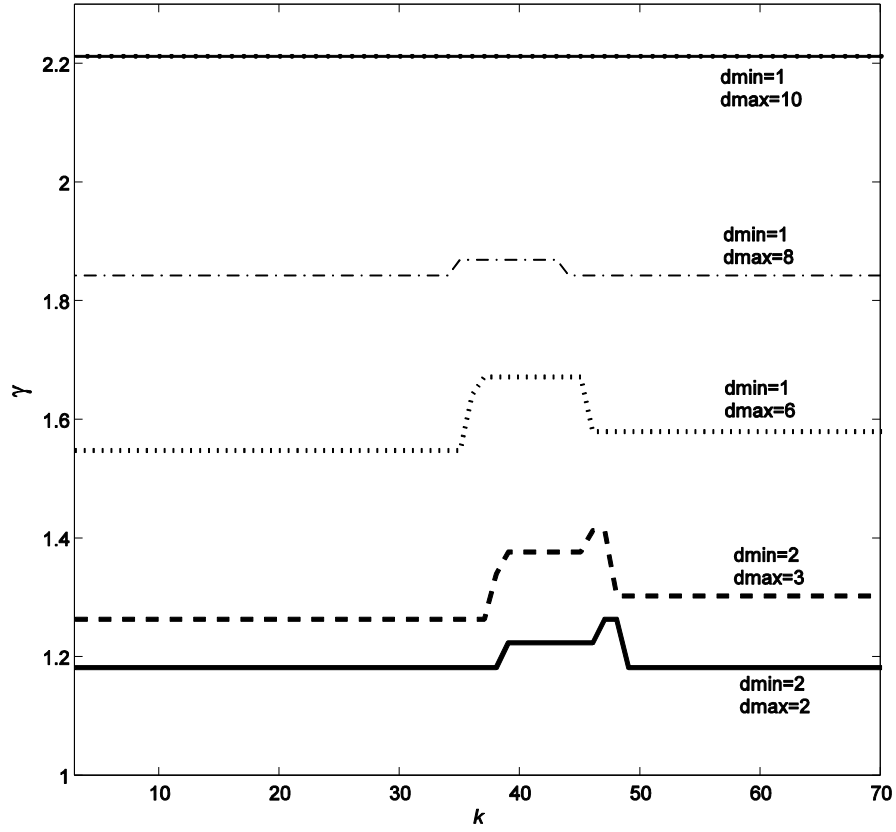


Figure 6.6 The variation of disturbance attenuation level,  $\gamma$ , for different delay sizes.

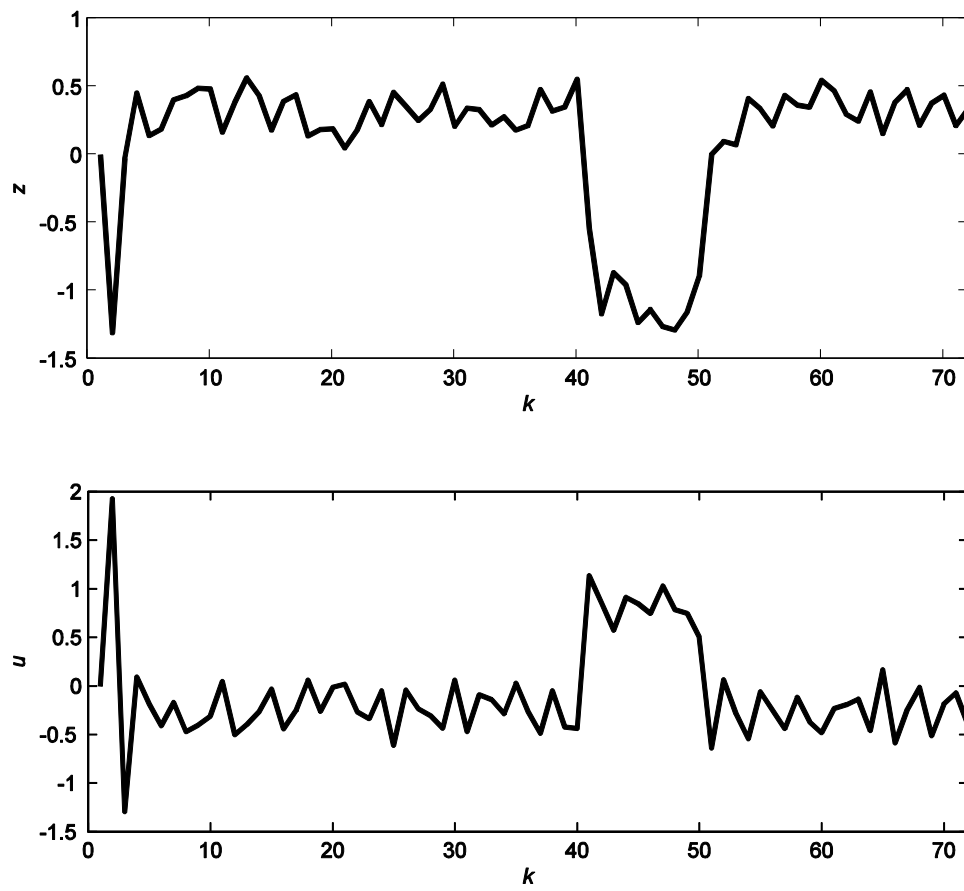


Figure 6.7 The time-history of the controlled output,  $z$ , and control input,  $u$ , for  $d_{max} = d_{min} = 2$ .

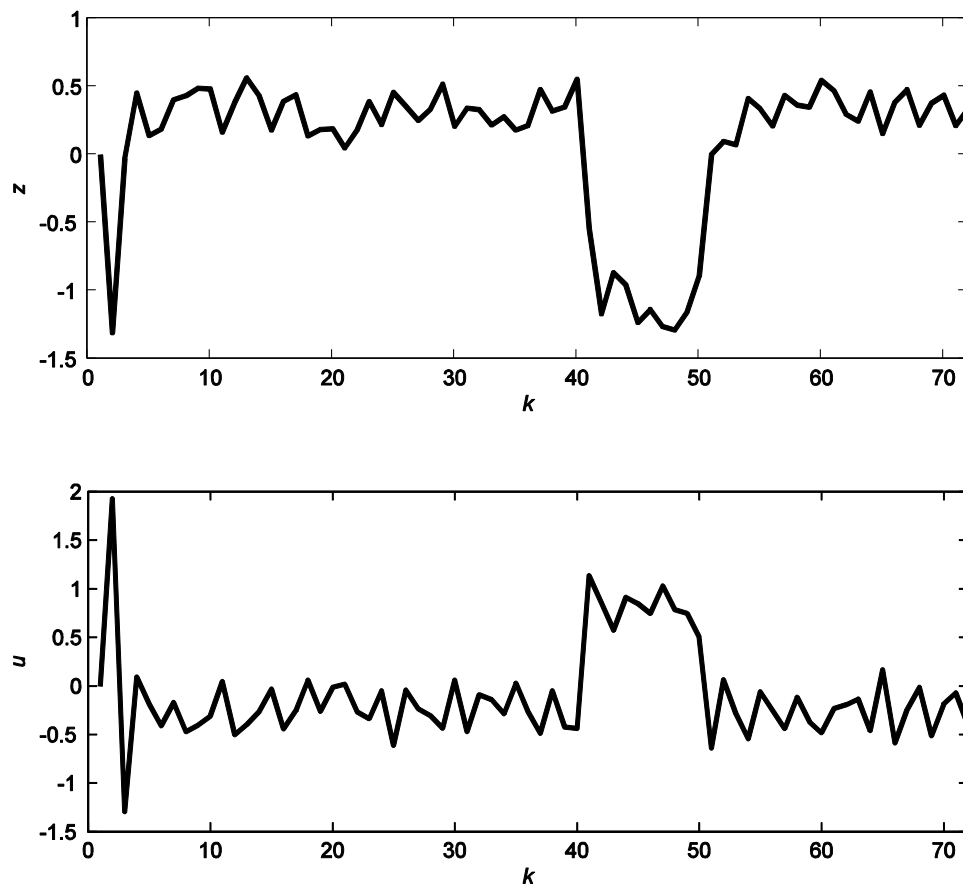


Figure 6.8 The time-history of the controlled output,  $z$ , and control input,  $u$ , for  $d_{max} = 3, d_{min} = 2$ .

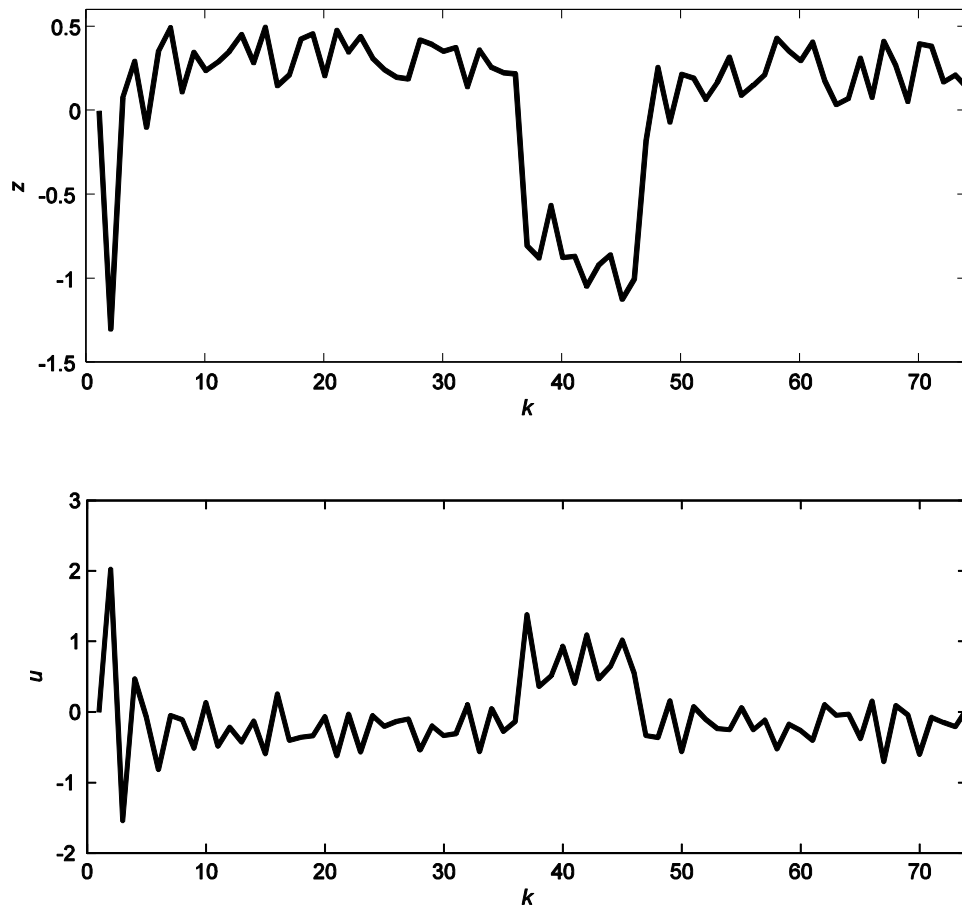


Figure 6.9 The time-history of the controlled output,  $z$ , and control input,  $u$ , for  $d_{max} = 6, d_{min} = 1$ .

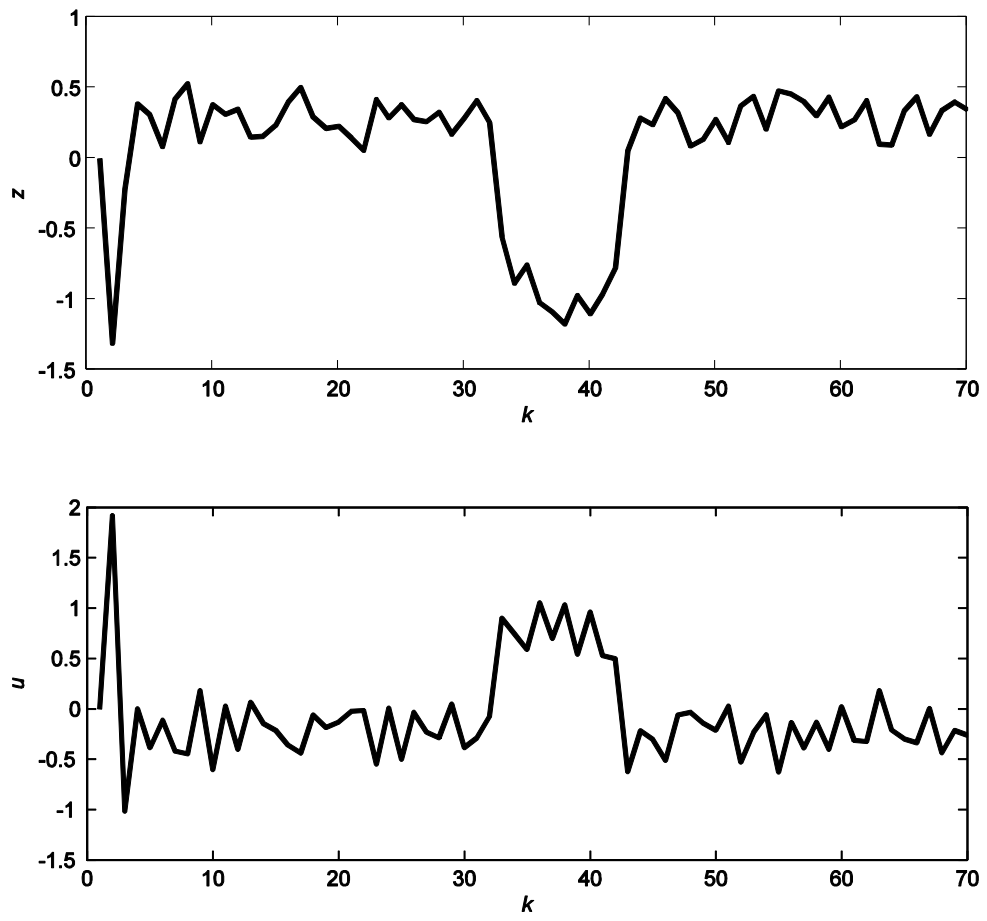


Figure 6.10 The time-history of the controlled output,  $z$  and control input,  $u$  for  $d_{max} = 10, d_{min} = 1$ .



**Example 3:** We utilize the example given in Ref. [17] in revised form as follows:

$$A = \begin{bmatrix} 0 & 0.1 \\ -0.14 & 0.9 \end{bmatrix} \quad A_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix} \quad B_w = B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{0.1} \end{bmatrix} \quad C_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad D_w = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad D_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x(i) = [0 \ 0]^T \quad i = 0, -1, \dots, -d_{max}. \quad \|u(k)\| \leq 1, \forall k \geq 0.$$

where  $\|\Delta A\| = \|\Delta A_d\| = \|\Delta B_u\| \leq 0.1$ .

The time-history of the disturbance applied to the system is as shown in Figure 6.5. During the simulations, the best variation of  $r$  is obtained as given in Table 6.4. Figure 6.11 shows the variations in  $\gamma$  with respect to different delay-bounds. For various delay bounds,  $d_{max}$  and  $d_{min}$ , we obtain the time-history of performance outputs and the control efforts as shown in Figure 6.12, Figure 6.13 and Figure 6.14.

The disturbance signal applied to the system is as shown in Figure 6.5. During the simulations, the best variation of  $r$  is obtained as given in Table 6.4. Figure 6.11 shows the variations in  $\gamma$  with respect to different delay-bounds. For various delay bounds,  $d_{max}$  and  $d_{min}$ , we obtain the time-history of performance outputs and the control efforts as shown in Figures 6.12, 6.13 and 6.14.

Table 6.4  $r$  values at each step in Example 3.

$k$	<b>1</b>	<b>2-40</b>	<b>41</b>	<b>42-48</b>	<b>49-50</b>	<b>51-58</b>
$r$	0.3	0.4	0.6	0.7	0.75	0.45

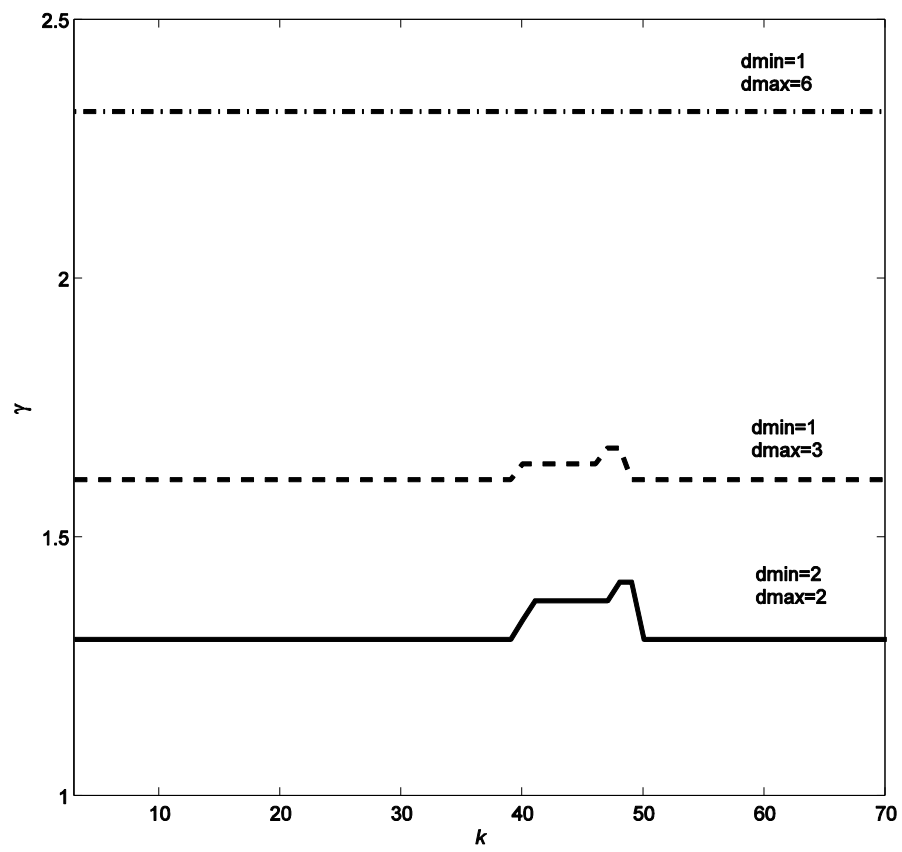


Figure 6.11 The variations of disturbance attenuation level,  $\gamma$ , for different delay intervals.

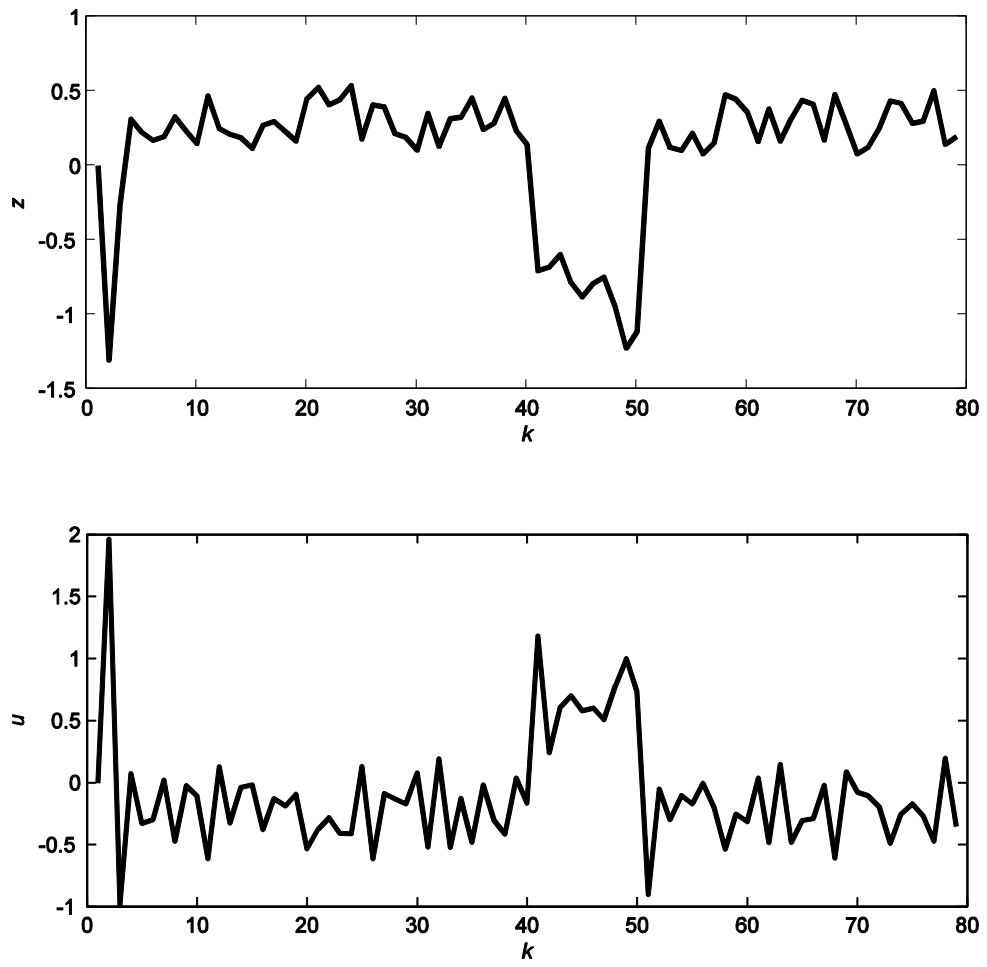


Figure 6.12 The time-history of the controlled output,  $z$ , and control input,  $u$ , for  $d_{max} = d_{min} = 2$ .

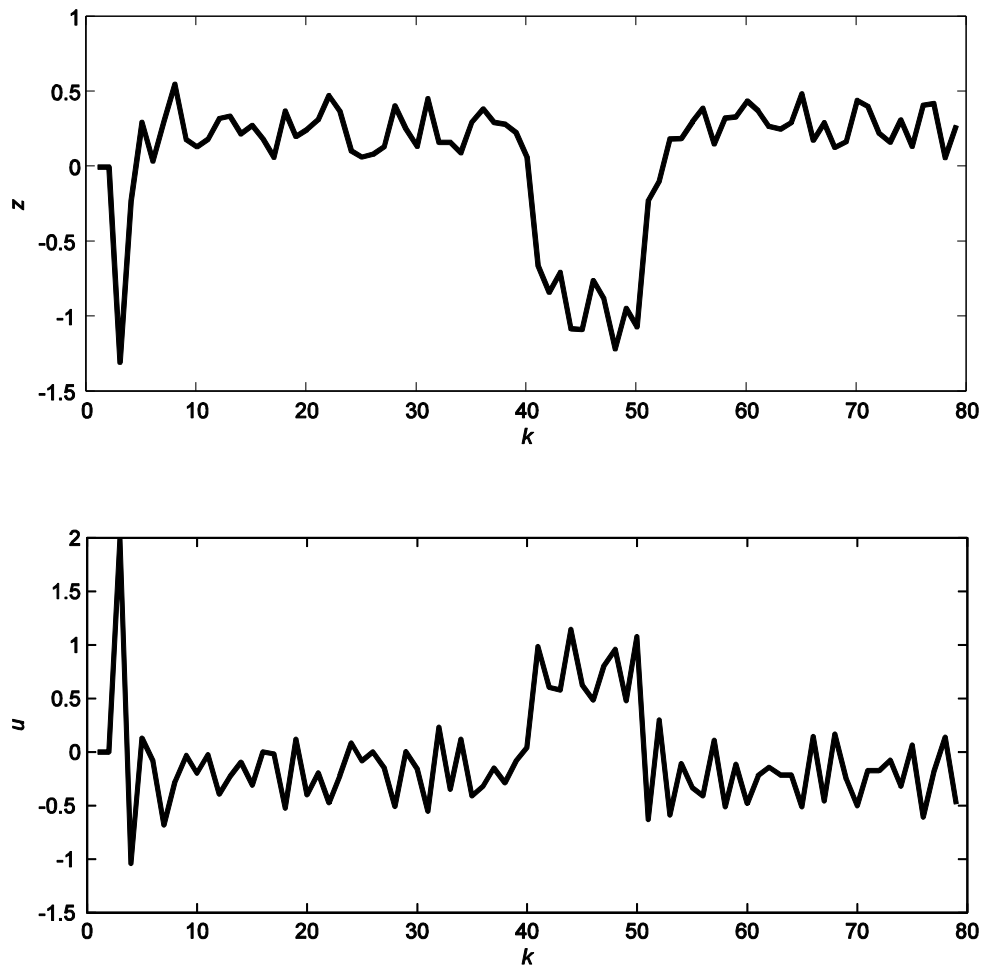


Figure 6.13 The time-history of the controlled output,  $z$ , and control input,  $u$ , for  $d_{max} = 3, d_{min} = 1$ .

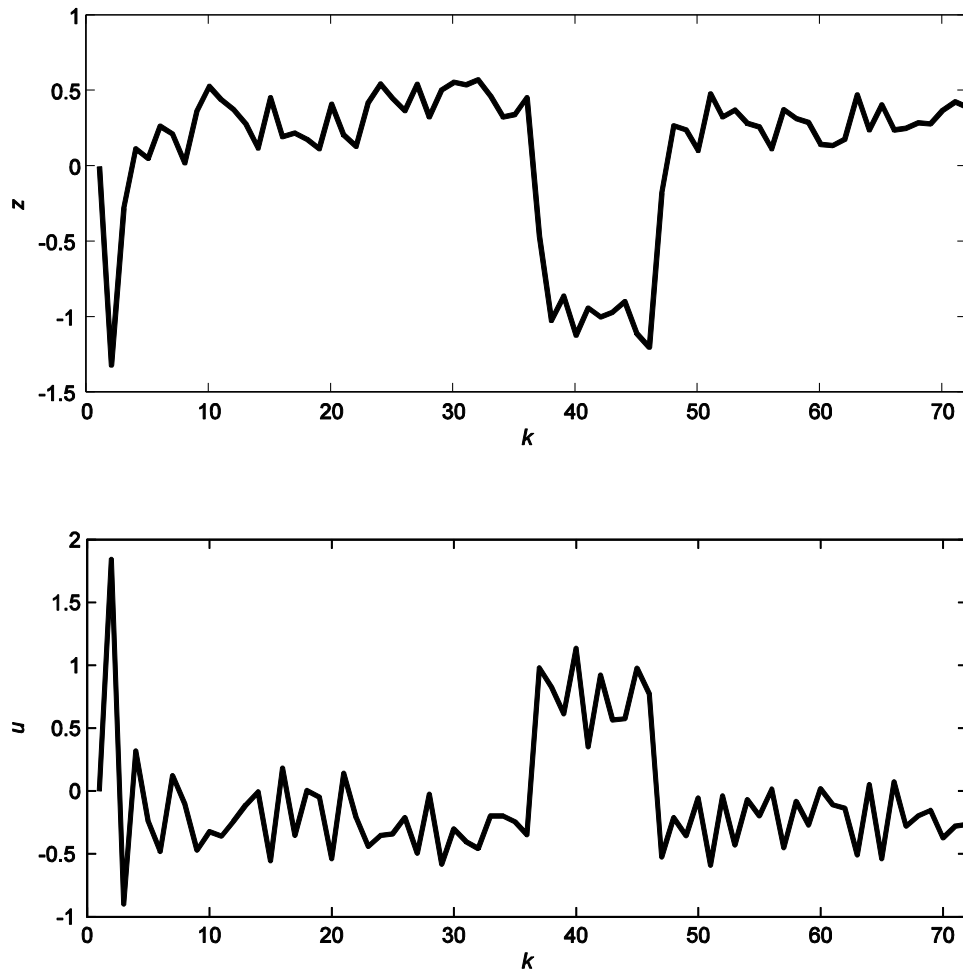


Figure 6.14 The time-history of the controlled output,  $z$ , and control input,  $u$ , for  $d_{max} = 6, d_{min} = 1$ .

## CONCLUSIONS

The MHHC of a class of linear discrete-time, uncertain time-delayed systems having time-varying interval delays was considered. Utilizing a standard discrete-time Lyapunov-Krasovskii functional, some delay dependent, Linear Matrix Inequality (LMI) based conditions which need to be solved iteratively in each step of run-time were provided. The provided LMI based conditions guarantee the closed-loop asymptotic stability, maximum disturbance attenuation performance and closed-loop dissipativity in consideration of the physical limitations of the actuator. The dissipation constraint that is utilized to guarantee closed-loop system dissipativity and  $\mathcal{H}_\infty$  performance is less conservative than requiring monotonicity of the value function.

The proposed MHHC has a capability of taking the performance criterion into account, while some of other stabilizing state-feedback controls for time-delay systems can only achieve stability. On the other hand, the proposed algorithm comes with a computational burden that complicates the real-time applications. To overcome this problem, certain calculations can be carried out offline, which can be stated as a future study on MHHC for time-delay systems.

Three numerical examples which consist of nominal and uncertain system models were considered to demonstrate the applicability of the proposed approach. Both the numerical results and simulation results validated that the proposed approach of this note can be efficiently used for the control of discrete-time, time-delayed systems having uncertainties and physical control limitations.

For a further study, the conservatism of the approach can be alleviated by using complete L-K functional and/or some delay-decomposition methods. However, it is probable that both strategies would impose noticeable amount of additional computational loads to the controller which will prohibit the use of the proposed method

in real-time. Also these types of methods cannot be solved by using convex optimization methods and would require the use of some linearization techniques such as cone-complementary methods. Extension of the proposed results to time varying delayed systems with input constraint is also considered to be a future work.

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10.3906/elk-1301-203

2. M. Ince, F. Yildiz, G. Onkal Engin, S.N. Engin, B. Keskinler **Respirometric kinetic parameter calculations of a batch jet loop bioreactor treating leachate and oxygen uptake rate estimation by DTM , Journal of Hazardous Materials, Vol. 153, Issue 3, pp. 991-998, (2008)**

#### **Conference Papers**

1. M. S. Arslan, F. Yıldız, N. Fukushima, I. Hagiwara **Bir Otonom Su Altı Aracının Pozisyon ve Yönelim Kontrollerinde Enerji Optimal Kontrol Yönteminin Model Öngörülü Kontrol ile Karşılaştırılması, TOK, (2010)**
2. F. Yıldız, İ. Alışkan, Ş.N. Engin **Jet Çevrimli Biyoreaktörde O<sub>2</sub> Tüketiminin Lyapunov Tabanlı Doğrusal Olmayan Denetleyici ile Kontrolü, ELECO, (2008)**
3. Ş. N. Engin, F. Yıldız, A.B. Ülkü ve G. Engin **Pilot Ölçekli Bir Havasız Atıksu Arıtma Sürecinin İzlenmesi, Modellenmesi, Kontrolü ve Otomasyonu Üzerine, Türk Otomatik Kontrol Ulusal Konferansı, (2006)**
4. S.N. Engin, F. Yıldız, M. Ince, G. Onkal Engin **Modeling and Parameter Identification of a Jet-Loop Bioreactor, American Control Conference, (2007)**

#### **Projects**

1. BAP (2011-2013) Researcher **Bir rüzgar türbini laboratuvar deney düzeneğinin kurulması ve gerçek zamanlı model öngörülü kontrolü (MPC)**
2. KISC NRF (2007-2009) Researcher **Coordinated and joint research activities in wastewater treatment monitoring, modelling, estimation and control in real time**
3. Brunel University (2008-2008) Researcher **PLC based Automobile Active Suspension System**
4. Phoenix Contact **A Four Rotor Air Vehicle with an Assisting Ground Vehicle**

Xplore 2008 (2007-  
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2007) Researcher

**Anaerobik Atıksu Arıtma Proses Sisteminin Modellenmesi  
ve Etkin Kontrol Algoritmalarının Geliştirilmesi**

## **AWARDS**

1. Phoenix Contact  
Xplore 2008 Third  
place in TCP/IP

**Quadrotor Air Vehicle Assisted By Ground Vehicle**