T.C.
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL & APPLIED SCIENCES

FEEDFORWARD CONTROLLER SYNTHESIS FOR UNCERTAIN TIME DELAY SYSTEMS VIA DYNAMIC IQCs

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PREFACE

I would like to appreciate Assoc. Prof. İbrahim Beklan KÜÇÜKDEMİRAL who encourages and supports me during my Ph.D. study. I am also grateful to Prof. İbrahim Emre KÖSE for his valuable contribution to the thesis.

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January, 2012

Levent Ucun

CONTENTS

Page

LIST OF SYMBOLS

c_s	Suspension damping ratio
daug	Diagonal augmentation
diag	Diagonal
G	Nominal system
$\mathcal{H}$	Hilbert Space
$in(\Pi)$	Inertia of Π
k_s	Spring constant of the active suspension system
k_t	Wheel spring constant
m_s	Sprung mass
m_u	Unsprung mass
n_s	Order of the multiplier
n_+	Number of positive eigenvalues
n_-	Number of negative eigenvalues
$\Re$	Set of real numbers
sat	Saturation function
sup	Supremum
x	State vector
y	Measured output
z	Exogenous controlled output
θ	Constant time delay
$\bar{\theta}$	Upper bound of time delay
γ	L_2 gain
Δ	Bounded and causal operator
Π	Multiplier
$\varphi(t)$	Initial condition
$\lVert . \rVert$	Vector 2-norm
$*$	Off-diagonal block completion of a symmetric matrix
$\mathcal{L}_{2+}$	Spaces of vector-valued square integrable functions defined on $[0, \infty)$
$\langle \cdot, \cdot \rangle$	Inner product
$\mathbb{C}^0$	Extended imaginary axis
$\otimes$	Kronecker product

LIST OF ABBREVIATIONS

BRL	Bounded Real Lemma
FB	Feedback Controller
FF	Feedforward Controller
FWM	Free Weighting Matrix
IQC	Integral Quadratic Constraints
KYP	Kalman Yakubovich Popov
LMI	Linear Matrix Inequalities
NFDE	Neutral Functional Differential Equation
RFDE	Retarded Functional Differential Equation
L-K	Lyapunov-Krasovskii

LIST OF FIGURES

LIST OF TABLES

Page

ÖZET

BELİRSİZLİK İÇEREN ZAMAN GECİKMELİ SİSTEMLER İÇİN DİNAMİK ENTEGRAL KARESEL KISITLAR KULLANILARAK İLERİ-BESLEMELİ DENETLEYİCİ TASARIMI

Levent UCUN

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Tez Danışmanı: Doç. Dr. İbrahim Beklan KÜÇÜKDEMİRAL

Bu doktora tezi çalışmasında durumlarında ve kontrol sinyalinde zaman gecikmesi bulunan ve durumların türevlerinin (dinamiklerinin) zaman gecikmesi tarafından etkilendiği, literatürde "nötral" sistemler olarak da ele alınan sistemler için zaman gecikmesine bağlı gürbüz H_∞ en iyi ileri-beslemeli ve geri-beslemeli denetleyici tasarımı üzerine çalışılmıştır. Ele alınan sistem aynı zamanda $\mathcal{L}_2$ bozucular tarafından etkilenmektedir. Amaçlanan denetleyici tasarımı, durum geri beslemeli denetleyici ve dinamik ileri-beslemeli denetleyici olmak üzere iki temel kontrol döngüsü içermektedir. Geri beslemeli denetleyici ele alınan nominal sistemi kararlı kılma problemi için bir çözüm oluştururken ileri-beslemeli denetleyici bozucunun sistem çıkışına olan etkilerini minimize etmektedir. Frekansa bağlı çarpanlar (multiplier) içeren dinamik entegral karesel kısıtlar (EKK) sistemde bulunan zaman gecikmelerini ve sisteme etki eden parametrik belirsizlikleri ifade etmek için kullanılmışlardır. Kullanılan IQC'lerde bulunan çarpanların dereceleri elde edilen sonuçlardaki tutuculuğu azaltmak üzere mümkün olduğu mertebede yükseltilmişlerdir. Belirsiz zaman gecikmeli sistemin en küçük bozucu bastırma seviyesi ile evrensel ve asimtotik kararlı olmasını sağlayan gecikmeye bağlı yeter koşul, doğrusal matris eşitsizlikleri cinsinden verilmiştir. Doktora tezinin son bölümlerinde önerilen tasarımın başarımını göstermek için pek çok sayısal örnek verilmiştir.

Anahtar Kelimeler: Zaman Gecikmeli Sistemler, İleri-beslemeli Denetleyici, Integral Karesel Kısıtlar, Dayanıklı Kontrol, Optimal Kontrol

ABSTRACT

FEEDFORWARD CONTROLLER SYNTHESIS FOR UNCERTAIN TIME DELAY SYSTEMS VIA DYNAMIC IQCs

Levent UCUN

Department of Electrical Engineering

PhD. Thesis

Advisor: Assoc. Prof. Dr. İbrahim Beklan KÜÇÜKDEMİRAL

The thesis studies the design problem of a delay-dependent robust H_∞ optimal feedforward plus feedback controller for systems having state and control delays and neutral systems with delays affecting the derivative of states. The controlled system having parametric uncertainties is subject to $\mathcal{L}_2$ disturbances. The proposed controller involves two main control loops which are state-feedback and dynamic feedforward controller. The state feedback controller is used as a stabilizing controller whereas the feedforward controller performs the minimization of disturbance effects. Dynamic Integral Quadratic Constraints (IQCs) which consist of frequency dependent multipliers, have been introduced to represent the delays and parametric uncertainties in the system. The degree of the multipliers used in IQCs is increased in order to decrease the conservatism in the obtained results. Sufficient delay dependent criterion in terms of Linear Matrix Inequalities (LMIs) such that uncertain time-delay system is guaranteed to be globally, asymptotically stable with a minimum disturbance attenuation level, is presented in this study. Many numerical examples provided at the end, illustrating the usefulness of the proposed design.

Key words: Time-Delay Systems, Feedforward Controller, Integral Quadratic Constraints, Robust Control, Optimal Control

SECTION 1

INTRODUCTION

1.1 Literature Review

Many physical systems such as chemical engineering processes, electrical networks and systems with long transmission lines involve time-delay naturally. This fact is one of the main reasons for instability and poor control performance.

There are many studies in the literature which survey the work on time-delay systems such as [1], [2], [3] and [4]. For example, after presenting some motivations for the study of time-delay system, [2] recalls modifications (models, stability, structure) arising from the presence of the delay phenomenon. A brief overview of some control approaches is provided, together with control methods of time-delay systems. Lastly, some open problems such as the constructive use of the delayed inputs, the digital implementation of distributed delays, the control via the delay, and the handling of information related to the delay value are discussed.

Moreover, [4] considers methods such as the time-domain control of delayed systems and the robust filtering including Robust Kalman filtering and robust H_∞ filtering. The book begins with an introduction to time-delay systems and continues with the robust control of time-delay systems which involves robust stability, guaranteed cost control, passivity analysis and synthesis.

In the last decade, there has been a considerable amount of research effort in the literature for the analysis and design of robust H_∞ controllers both for continuous and discrete time-delay systems. These research activities can be classified basically into two main groups such as delay-dependent and delay-independent controller synthesis. Since it is a fact that delay-independent results tend to be more conservative, the researchers dealing with time-delay systems mostly deal with delay-dependent methods. In these studies, the time-delay systems are generally treated in time-domain by the use of different choice of Lyapunov-Krasovskii functionals [5], [6], [7], [8].

For example, [5] considers the problem of delay-dependent robust H_∞ control for uncertain systems with time-varying delays. An improved delay-dependent bounded real lemma (BRL) for time-delay systems is established in terms of a linear matrix inequality. Based on the obtained BRL, a delay-dependent condition for the existence of a state feedback controller, which ensures asymptotic stability and a prescribed H_∞ performance level of the closed-loop system for all admissible uncertainties, is proposed in terms of a matrix inequality.

Similarly, [6] discusses the problem of delay-dependent robust control for uncertain singular systems with time-delay. Firstly, based on Jensen inequality formula and Lyapunov stability theory, the sufficient condition is established for a delay-dependent singular system with time-delay, which guaranteed the nominal system to be regular, impulse free and stable. Secondly, based on the sufficient condition, the design method of robust state feedback controller is given for delay-dependent uncertain singular systems with time-delay, which guarantees that, for all admissible uncertainties, the resultant closed-loop system is regular, impulse free, and stable.

[7] studies the design problem of a robust delay-dependent H_∞ controller for a class of time-delay control systems with time-varying state and input delays, which are assumed to be noncoincident. Based on the selection of an augmented form of Lyapunov–Krasovskii (L-K) functional, first a Bounded Real Lemma (BRL) is obtained in terms of linear matrix inequalities (LMIs) such that the nominal, unforced time-delay system is guaranteed to be globally asymptotically stable with minimum allowable disturbance attenuation level. Extending BRL, sufficient delay-dependent criteria are developed for a stabilizing H_∞ controller synthesis involving a matrix inequality for

which a nonlinear optimization algorithm with LMIs is proposed to get feasible solution to the problem. Moreover, for the case of existence of norm-bounded uncertainties, both the BRL and H_∞ stabilization criteria are easily extended by employing a well-known bounding technique.

An alternative delay-dependent H_∞ controller design is proposed for linear, continuous, time-invariant systems with unknown state delay in [8]. The resulting delay-dependent H_∞ control criterion is obtained in terms of Park's inequality for bounding cross term. H_∞ controller determined by a convex optimization algorithm with linear matrix inequality (LMI) constraints, guarantees the asymptotic stability of the closed-loop systems and reduces the effect of the disturbance input on the controlled output to within a prescribed level.

The common and main purpose of these studies is to design a control scheme that provides minimum $\mathcal{L}_2$ gain and maximum allowable delay bound together with the minimum conservativeness.

Rather than using Lyapunov-Krasovskii functional approach, in the thesis, time-delay phenomenon is treated by means of IQCs. It is well-known that IQCs have played an efficient role especially in analysis of uncertain linear systems since the mid of 1990s.

Survey paper [9] is a well-known reference for the use of IQCs and their applications. The paper introduces a unified approach to robustness analysis with respect to nonlinearities, time variations, and uncertain parameters. It is also shown how a complex system can be described, using IQCs for its elementary components. A stability theorem for systems described by IQCs is presented that covers classical passivity/dissipativity arguments but simplifies the use of multipliers and the treatment of causality. The paper contains a summarizing list of IQCs for important types of system components.

Although IQCs have been widely used especially in the stability analysis of dynamical systems, the use of IQCs in the synthesis of robust controllers is still a challenging and mostly an open problem. To the best of authors knowledge, there are only a few results on this subject in the literature. Among these few studies, one can list the following results such as [10], [11] and [12].

Particularly, [10] deals with the design problem of controllers for linear systems having actuator saturation nonlinearity. They establish IQC-based conditions under which an ellipsoid is contractively invariant for a single input linear system under a saturated linear feedback law. While the advantages of the proposed IQC approach remain to be explored, it is shown in the paper that the largest contractively invariant ellipsoid determined by this approach is the same as the one determined by the existing approach based on expressing the saturated linear feedback as a linear differential inclusion (LDI), which is known to lead to less conservative result in determining the largest contractively invariant ellipsoid for single input systems. However, their method is based on the use of static IQCs rather than dynamic ones which mostly leads to conservative results.

Different from [10], there is a couple of studies in the literature dealing with time-delay systems via dynamic IQCs such as [11] and [12]. These papers describe a set of delay-dependent IQC's for time-delay uncertainty. The set is linearly parameterized in terms of the frequency-response of a complex valued multiplier. Using LMI optimization techniques, one may compute optimal multipliers and thereby obtain less conservative IQC stability robustness bounds for systems with uncertain time-delays. However, these researches focus on analysis rather than synthesis.

There are also some very few studies in the literature which deal with controller synthesis via dynamic IQCs. For instance, [13] studies robustness analysis with integral quadratic constraints, where they formulate a new positivity condition on the solution of the corresponding LMI which is necessary and sufficient for nominal stability of the underlying system. The application of the technical result is illustrated by a complete solution of the $\mathcal{L}_2$-gain and robust $\mathcal{H}_2$-estimator design problems if the uncertainties are characterized by dynamic integral quadratic constraints.

[14] and [15] deal with the feedforward dynamic controller synthesis for uncertain linear time invariant systems by the means of dynamic IQCs. They use IQCs for describing the uncertainty blocks in the system. A convex solution to the problem is obtained by using a state-space characterization of nominal stability that have been developed recently. Specifically, the solution consists of LMI conditions for the

existence of a feedforward controller that guarantees a given $\mathcal{L}_2$ gain for the closed-loop system.

1.2 Purpose of Thesis

Inspired by the work in the literature, in this thesis, a robust feedforward controller design problem for uncertain time-delay systems via dynamic IQCs is considered. Many different types of time-delay systems such as state-delay systems, control-delay systems and neutral systems have been analyzed and used in this study. The control scheme proposed in the thesis can be configured for many different types of time-delay systems mentioned above. Hence, even if there are some differences in the time-delay systems that the proposed controller is applied, the controller can be configured easily to accomodate for a different type of time-delay systems. Another important point that we consider in this thesis is the parametric uncertainty that affects the system. By utilization of dynamic IQCs, we can configure a dynamic feedforward controller that deals with both time-delay and parametric uncertainty. Also, another important fact is that, the proposed controller scheme can be easily adjusted to be used for a reference tracking problem althouh it is mainly aimed to deal with the disturbance attenuation problem. Hence, the controller construction proposed in the thesis, can be employed for both reference tracking and disturbance attenuation problems.

One of the main advantages to use dynamic IQCs for feedforward controller design is that different types of time-delay systems, nonlinearities and uncertainties can be handled by a single dynamic IQC. In order to perform this technique, we use combination properties of dynamic IQCs which will be explained in due course.

On the other hand, one of the most important problems that we should deal with, is the reduction of conservatism in design. As it is mentioned in the literature, the IQCs are very elegant mathematical tools which are used to describe uncertainties and nonlinearities in the system, in a way which is suitable to be used with the theory of linear system. Hence, in order to obtain better controllers, we should avoid unnecessary conservatism as much as possible while using the IQCs. For this purpose, during the design of the feedforward controller, we use dynamic multipliers instead of static ones. Another important point that might reduce conservatism is the selection of feedback controller in the system. Although the only assumption regarding the feedback controller is to make the closed-loop system stable, one might choose or design a high performance dynamic(static) feedback controller to reduce the conservatism. At this point, one can use less conservative feedback controllers such as suboptimal H_∞ controllers which have been designed by using recent L-K functionals from literature.

1.3 Hypothesis

The proposed control scheme is based on several ideas: First, the proposed controller is in the form of two degree of freedom control configuration which consists of feedback and feedforward loops. Feedback controller deals with the stabilization of the nominal system and feedforward controller which is supposed to be a robust optimal H_∞ disturbance attenuator, is used to improve the disturbance attenuation performance of the system. Second, the parametric uncertainty and time delay affecting states and control signal are described as dynamic IQCs in the controller synthesis. Finally, a convex optimization method is presented which accommodates the advantage of adjusting the degree of dynamic multipliers to reduce the conservatism as much as possible.

Hence, we are able to prove that we construct a feedforward controller that guarantees the uncertain time-delay system to be globally, asymptotically stable with a minimum disturbance attenuation level.

The contribution of the thesis to the control theory literature is the new general design method of a feedforward plus feedback controller loop for different kinds of uncertain time-delay systems by use of dynamic IQCs. Another important contribution of the thesis is the implementation and use of the combination of two or more dynamic multipliers for a controller design which is actually one more step forward than the usage of a couple of multipliers for stability analysis in the literature.

SECTION 2

TIME-DELAY SYSTEMS

One of the main reasons of instability and poor performance in many physical and dynamical systems, is the time-delay phenomenon concerning feedback control systems. Time-delay operator affecting the systems can be divided into two groups. Basically, the first group consists of the delays affecting system states and control signal applied to the system. The second group involves neutral systems which means that the delay operator is effective on state derivatives in the state dynamics.

In many physical and biological phenomena, the rate of variation in the system state depends on the past states. This characteristic is called a delay or a time delay, and a system with a time delay is called a time-delay system. Time-delay phenomena was first discovered in biological systems and was later found in many engineering systems, such as mechanical transmissions, fluid transmissions, metallurgical processes, and networked control systems. It is often a source of instability and poor control performance. Time-delay systems have attracted the attention of many researchers because of their importance and widespread occurrence. Basic theories describing such systems were established in the 1950s and 1960s. They covered topics such as the existence and uniqueness of solutions to dynamic equations, stability theory for trivial solutions. That work laid the foundation for the later analysis and design of time-delay systems.

Robust control of time-delay systems has been a very active field for the last 20 years and has spawned many branches, for example, stability analysis, stabilization design, H_∞ control, passive and dissipative control, reliable control, guaranteed-cost control,

H_∞ filtering, Kalman filtering and stochastic control. Regardless of the branch, stability remains the most significant objective. From this point of view, important developments in the field of time-delay systems that explore new directions have generally been launched from a consideration of stability as the starting point. This section of the thesis reviews methods of studying the stability of time-delay systems.

2.1 Brief History of Time-Delay Systems

Stability is a very basic issue in control theory and has been extensively discussed in many monographs [16], [17], [18]. Research on the stability of time-delay systems began in the 1950s, first by using frequency-domain methods and followed later also by using time-domain methods. Frequency-domain methods determine the stability of a system from the distribution of the roots of its characteristic equation or from the solutions of a complex Lyapunov matrix function equation [19]. This technique is mostly suitable for systems with constant delays. The main time-domain methods are generally based on Lyapunov-Krasovskii functional and Razumikhin function methods [20]. They are the most common approaches to the stability analysis of time-delay systems. Since it was very difficult to construct Lyapunov-Krasovskii functionals and Lyapunov functions until the 1990s, the stability criteria obtained were generally in the form of existence conditions; and it was impossible to derive a general solution. Then, Riccati equations, linear matrix inequalities (LMIs) [21] and Matlab toolboxes came into use; and the solutions they provided were used to construct Lyapunov-Krasovskii functionals and Lyapunov functions. These time-domain methods are now very important in the stability analysis of linear systems. This part reviews methods of examining stability and their limitations.

Consider the following linear system with a delay

$$\begin{aligned} \dot{x} &= Ax(t) + A_d x(t-\theta) \\ x(t) &= \varphi(t), \quad t \in [-\theta, 0], \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $\theta > 0$ is a delay in the state of the system, that is, it is a discrete delay; $\varphi(t)$ is the initial condition; and $A \in \mathbb{R}^{n\times n}$ and $A_d \in \mathbb{R}^{n\times n}$ are the system matrices. The future evolution of this system depends not only on its present state, but also on its history. The main methods of examining its stability can be classified into two types; frequency-domain and time-domain.

2.1.1 Frequency Domain Methods

Frequency-domain methods provide the most sophisticated approach to analyzing the stability of a system with no delay $(\theta = 0)$. The necessary and sufficient condition for the stability of such a system is $\lambda(A + A_d) < 0$. When $\theta > 0$, frequency-domain methods yield the result that system (2.1) is stable if and only if all the roots of its characteristic function,

$$f(\lambda) = det(\lambda I - A - A_d e^{-\theta\lambda}) = 0 \tag{2.2}$$

have negative real parts. However, this equation is transcendental, which makes it difficult to solve. Moreover, if the system has uncertainties and a time-varying delay, the solution is even more complicated. Thus, the use of a frequency-domain method to study time-delay systems has serious limitations.

2.1.2 Time Domain Methods

Time-domain methods are based primarily on two famous theorems, the Lyapunov-Krasovskii (L-K) stability theorem and the Razumikhin theorem. They were established in the 1950s by the Russian mathematicians Krasovskii and Razumikhin, respectively. The main idea is to obtain a sufficient condition for the stability of system (2.1) by constructing an appropriate L-K functional or an appropriate Lyapunov function. This idea is theoretically very important; but until the 1990s, there was no good way to implement it. Then the Matlab toolboxes emerged and made it easy to construct Lyapunov-Krasovskii functionals and Lyapunov functions, thus greatly promoting the development and application of these methods. Since then, significant results have continued to appear one after another [22]. Among them, two classes of sufficient conditions have received a great deal of attention. One class is independent of the length of the delay and its members are called delay-independent conditions. The other class makes use of information on the length of the delay, and its members are called delay-dependent conditions.

The Lyapunov-Krasovskii functional candidate is generally chosen to be

$$V_1(x_t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Qx(s)ds, \tag{2.3}$$

where $P = P^T > 0$ and $Q = Q^T > 0$ are to be determined and are called Lyapunov matrices and x_t denotes the translation operator acting on the trajectory, $x_t(\theta) = x_t(t+\theta)$ for some (non-zero) interval $[-h, 0](\theta \in [-h, 0])$. Calculating the derivative of $V_1(x_t)$ along the solutions of system (2.1) and restricting it to less than zero yield the delay-independent stability condition of the system.

Since

$$\begin{bmatrix} PA + A^T P + Q & PA_d \\ * & -Q \end{bmatrix} < 0 \tag{2.4}$$

is linear with respect to the matrix variables P and Q, it is called an LMI. If the LMI toolbox of Matlab yields solutions to (2.4) for these variables, then according to the Lyapunov-Krasovskii stability theorem, system (2.1) is asymptotically stable for all $\theta \geq 0$ and furthermore, an appropriate Lyapunov-Krasovskii functional is obtained.

Since delay-independent conditions contain no information on delay, they are overly conservative, especially when the delay is very small. This consideration has given rise to another important class of stability conditions, namely, delay-dependent conditions, which do contain information on the length of a delay. First of all, they assume that system (2.1) is stable when $\theta = 0$. Since the solutions of the system are continuous functions of θ, there must exist an upper bound, $\bar{\theta}$, on the delay such that system (2.1) is stable for all $\theta \in [0, \bar{\theta}]$. Thus, the maximum allowable upper bound on the delay is the main criterion for judging the conservativeness of a delay-dependent condition.

The hot topics in control theory are delay-dependent problems in stability analysis, robust control, H_∞ control, reliable control, guaranteed-cost control, saturation input control, and chaotic-system control.

Since the 1990s, the main approach to the study of delay-dependent stability has involved the addition of a quadratic double-integral term to the Lyapunov-Krasovskii functional (2.2)

$$V(x_t) = V_1(x_t) + V_2(x_t) \tag{2.5}$$

where

$$V_2(x_t) = \int_{-h}^{0} \int_{t+\theta}^{t} x^T(s) Z x(s) ds d\theta, \tag{2.6}$$

where derivative of $V_2(x_t)$ is

$$\dot{V}_2(x_t) = h x^T(t) Z x(t) - \int_{t-h}^{t} x^T(s) Z x(s) ds. \tag{2.7}$$

Delay-dependent conditions can be obtained from the Lyapunov-Krasovskii stability theorem. However, how to deal with the integral term on the right side of (2.7) is a problem. So far, three methods of studying delay-dependent problems have been developed, the discretized Lyapunov-Krasovskii functional method, fixed model transformations and parameterized model transformations.

The main use of the discretized Lyapunov-Krasovskii functional method is to study the stability of linear systems and neutral systems with a constant delay. It discretizes the Lyapunov-Krasovskii functional and the results can be written in the form of LMIs [23], [24] and [25]. The advantage of doing this is that the estimate of the maximum allowable delay that guarantees the stability of the system is very close to the actual delay bound. The drawbacks are that it is computationally expensive and that it cannot easily handle systems with a time-varying delay. Consequently, this method has not been widely studied or used since it was first proposed by Gu in 1997 [23].

2.2 Models of Time-Delay Systems

This section presents some basic definitions and theoretical results in the theory of time-delay systems.

In science and engineering, differential equations are often used as mathematical models of systems. A fundamental assumption about a system that is modeled in this way is that its future evolution depends solely on the current values of the state variables and is independent of their history. For example, consider the following first-order differential equation

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0. \tag{2.8}$$

The future evolution of the state variable x at time t depends only on t and $x(t)$, and does not depend on the values of x before time t.

If the future evolution of the state of a dynamic system depends not only on current values, but also on past ones, then the system is called a time delay system. Actual systems of this type cannot be satisfactorily modeled by an ordinary differential equation; that is, a differential equation is only an approximate model. One way to describe such systems precisely is to use functional differential equations.

In many systems, there may be a maximum delay, θ. In this case, we are often interested in the set of continuous functions that map $[-\theta, 0]$ to $\mathbb{R}^n$, which we denote simply by $\mathcal{C} = \mathcal{C}([\theta, 0], \mathbb{R}^n)$. For any $a > 0$, any continuous function of time $\psi \in \mathcal{C}([t_0 - \theta, t_0 + a], \mathbb{R}^n)$ and $t_0 \leq t \leq t_0 + a$, let $\psi_t \in \mathcal{C}$ be the segment of ψ given by $\psi_t(\theta) = \psi(t + \theta), \ -h \leq \theta \leq 0$. The general form of a retarded functional differential equation (RFDE) (or functional differential equation of retarded type) is

$$\dot{x} = f(t, x_t), \tag{2.9}$$

where $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$. This equation indicates that the derivative of the state variable x at time t depends on t and $x(\zeta)$ for $t - \theta \leq \zeta \leq t$. Thus, to determine the future evolution of the state, it is necessary to specify the initial value of the state variable, $x(t)$, in a time interval of length θ, say, from $t_0 - \theta$ to t_0; that is,

$$x_{t_0} = \phi, \tag{2.10}$$

where $\phi \in \mathcal{C}$ is given. In other words, $x(t_0 + \theta) = \phi(\theta), \ -h \leq \theta \leq 0$.

It is important to note that, in an RFDE, the derivative of the state contains no term with a delay. If such a term does appear, then we have a functional differential equation of neutral type. For example,

$$5\dot{x}(t) + 2\dot{x}(t - \theta) + x(t) - x(t - \theta) = 0 \tag{2.11}$$

is a neutral functional differential equation (NFDE). For an $a > 0$, a function x is said to be a solution of RFDE (2.9) in the interval $[t_0 - \theta, t - 0 + a)$ if x is continuous and satisfies that RFDE in that interval. Here, the time derivative should be interpreted as a one-sided derivative in the forward direction. Of course, a solution also implies that (t, x_t) is within the domain of the definition of f. If the solution also satisfies the initial condition (2.10), we say that it is a solution of the equation with the initial condition (2.10), or simply a solution through (t_0, ϕ). We write it as $x(t_0, \phi, f)$ when it is important to specify the particular RFDE and the given initial condition. The value of $x(t_0, \phi, f)$ at t is denoted by $x(t_0, \phi, f)$. We omit f and write $x(t_0, \phi)$ or $x(t; t_0, \phi)$ when f is clear from the context.

A fundamental issue in the study of both ordinary differential equations and functional differential equations is the existence and uniqueness of a solution. We state the following theorem without proof.

Theorem 2.1 (Uniqueness) [26] Suppose that $\Omega \subseteq \mathbb{R} \times \mathcal{C}$ is an open set, function $f : \Omega \to \mathbb{R}^n$ is continuous, and $f(t, \phi)$ is Lipschitzian in ϕ in each compact set in Ω. That is, for a given compact set, $\Omega_0 \subset \Omega$, there exists a constant L such that

$$\|f(t, \phi_1) - f(t, \phi_2)\| \leq L\|\phi_1 - \phi_2\| \tag{2.12}$$

for any $(t, \phi_1) \in \Omega_0$ and $(t, \phi_2) \in \Omega_0$. If $(t_0, \phi) \in \Omega$, then there exists a unique solution of RFDE (2.9) through (t_0, ϕ).

2.2.1 Concept of Stability

Let $y(t)$ be a solution of RFDE (2.9). The stability of the solution depends on the behavior of the system when the system trajectory, $x(t)$, deviates from $y(t)$. Without loss of generality, we assume that RFDE (2.9) admits the solution $x(t) = 0$, which will be referred to as the trivial solution. If the stability of a nontrivial solution, $y(t)$, needs to be studied, then we can use the variable transformation $z(t) = x(t) - y(t)$ to produce the new system

$$\dot{z}(t) = f(t, z_t + y_t) - f(t, y_t), \tag{2.13}$$

which has the trivial solution $z(t) = 0$.

For the function $\psi \in \mathcal{C}([a, b], \mathbb{R}^n)$, define the continuous norm $\|.\|_c$ to be

$$\|\phi\|_c = sup_{a \leq \theta \leq b}\|\phi(\theta)\|. \tag{2.14}$$

In this definition, the vector norm $\|.\|$ represents the 2-norm $\|.\|_2$.

We now define various types of stability for the trivial solution of time-delay system (2.9).

Definition 2.1 [27]

- If, for any $t_0 \in \mathbb{R}$ and $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|x_{t_0}\|_c < \delta$ implies $\|x(t)\| < \epsilon$ for $t \geq t_0$, then the trivial solution of (2.9) is stable.
- If the trivial solution of (2.9) is stable, and if, for any $t_0 \in \mathbb{R}$ and any $\epsilon > 0$, there exists a $\delta_a = \delta_a(t_0, \epsilon) > 0$ such that $\|x_{t_0}\|_c < \delta_a$ implies $\lim_{t \to \infty} x(t) = 0$, then the trivial solution of (2.9) is asymptotically stable.
- If the trivial solution of (2.9) is stable and if $\delta(t_0, \epsilon)$ can be chosen independently of t_0, then the trivial solution of (2.9) is uniformly stable.
- If the trivial solution of (2.9) is uniformly stable and if there exists a $\delta_a > 0$ such that, for any $\eta > 0$, there exists a $T = T(\delta_a, \eta)$ such that $\|x_{t_0}\|_c < \delta_a$ implies $\|x_t\| < \eta$ for $t \geq t_0 + T$, and $t_0 \in \mathbb{R}$, then the trivial solution of (2.9) is uniformly asymptotically stable.
- If the trivial solution of (2.9) is (uniformly) asymptotically stable and if δ_a can be an arbitrarily large, finite number, then the trivial solution of (2.9) is globally (uniformly) asymptotically stable.
- If there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\|x_t\| \leq \beta \sup_{-h \leq \theta \leq 0} \|x_\theta\| e^{-\alpha t}. \tag{2.15}$$

Then the trivial solution of (2.9) is globally exponentially stable; and α is called the exponential convergence rate.

2.2.2 Lyapunov-Krasovskii Stability Theorem

Just as for a system without a delay, the Lyapunov method is an effective way of determining the stability of a system with a delay. When there is no delay, this determination requires the construction of a Lyapunov function, $V(t, x(t))$, which can be viewed as a measure of how much the state, $x(t)$, deviates from the trivial solution. Now, in a delay-free system, we need $x(t)$ to specify the future evolution of the system beyond t. In a time-delay system, we need the "state" at time t for that purpose; it is the value of $x(t)$ in the interval $[t-\theta, t]$. So, it is natural to expect that, for a time-delay system, the Lyapunov function is a functional, $V(t, x_t)$, that depends on $x(t)$ and indicates how much $x(t)$ deviates from the trivial solution. This type of functional is called a Lyapunov-Krasovskii functional. More specifically, let $V(t,\phi) : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ be differentiable; and let $x_t(\tau,\phi)$ be the solution of RFDE (2.9) at time t for the initial condition $x_\tau = \phi$. Calculating the time derivative of $V(t, x_t)$ and evaluating it at $t = \tau$ yield

$$\dot{V}(\tau,\phi) = \frac{d}{dt}V(t,x_t)\mid_{t=\tau, x_t=\phi} = \limsup_{\Delta t \to 0} \frac{V(\tau+\Delta t, x_{\tau+\Delta t}(\tau,\phi)) - V(\tau,\phi)}{\Delta t} \quad (2.16)$$

If $\dot{V}(t, x_t)$ is non-positive, then x_t does not grow with t, which means that the system under consideration is stable in the sense of Definition 2.2.1. The following theorem states this more precisely.

Theorem 2.2 [16]

Suppose that $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ in (2.9) maps $\mathbb{R}\times$(bounded sets in $\mathcal{C}$) into bounded sets in $\mathbb{R}^n$, and that $u, v, \omega : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ are continuous non-decreasing functions, where $u(\tau)$ and $v(\tau)$ are positive for $\tau > 0$ and $u(0) = v(0) = 0$.

- If there exists a continuous differentiable functional $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ such that $u(\|\phi(0)\|) \le V(t,\phi) \le v(\|\phi\|_c)$ and $\dot{V}(t,\phi) \le -\omega(\|\phi(0)\|)$ then the trivial solution of (2.9) is uniformly stable.
- If the trivial solution of (2.9) is uniformly stable, and $\omega(\tau) > 0$ for $\tau > 0$, then the trivial solution of (2.9) is uniformly asymptotically stable.
- If the trivial solution of (2.9) is uniformly asymptotically stable and if $\lim_{\tau\to\infty} u(\tau) = \infty$ then the trivial solution of (2.9) is globally uniformly asymptotically stable.

2.2.3 Razumikhin Stability Theorem

That the Lyapunov-Krasovskii functional requires the state variable $x(t)$ in the interval $[t-h, t]$ requires the manipulation of functionals, which makes the Lyapunov-Krasovskii theorem difficult to apply. This difficulty can sometimes be circumvented by using the Razumikhin theorem, an alternative that involves only functions, but no functionals.

The key idea behind the Razumikhin theorem is the use of a function, $V(x)$, to represent the size of $x(t)$.

$$\bar{V}(x_t) = \max_{\theta \in [-h,0]} V(x(t+\theta)) \tag{2.17}$$

indicates the size of x_t. If $V(x(t)) < \bar{V}(x_t)$, then $\bar{V}(x_t)$ does not grow when $\dot{V}(x(t)) > 0$. In fact, for $\bar{V}(x_t)$ not to grow, it is only necessary that $\dot{V}(x(t))$ should not be positive whenever $V(x(t)) = \bar{V}(x_t)$. The precise statement is given in the next theorem.

Theorem 2.3 [16]

Suppose that $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ in (2.9) maps $\mathbb{R}\times$(bounded sets of $\mathcal{C}$) into bounded sets of $\mathbb{R}^n$ and also that $u, v, \omega : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ are continuous nondecreasing functions, $u(\tau)$ and $v(\tau)$ are positive for $\tau > 0$, $u(0) = v(0) = 0$ and v is always increasing.

- If there exists a continuously differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $u(\|x\|) \leq V(t,x) \leq v(\|x\|), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n$ and the derivative of V along the solution, $x(t)$, of system (2.9) satisfies $\dot{V}(t, x(t)) \leq -\omega(\|x(t)\|)$ whenever
$$V(t+\theta, x(t+\theta)) \leq V(t, x(t)), \tag{2.18}$$
for $\theta \in [-h, 0]$, then the trivial solution of (2.9) is uniformly stable.

- If there exists a continuously differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $u(\|x\|) \leq V(t,x) \leq v(\|x\|), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n$ if $\omega(\tau) > 0$ for $\tau > 0$, and if there exists a continuous nondecreasing function $p(\tau) > \tau$ for $\tau > 0$ such that condition (2.18) is strengthened to $\dot{V}(t, x(t)) \leq -\omega\|x(t)\|$ if $V(t+\theta, x(t+\theta)) \leq p(V(t, x(t)))$ for $\theta \in [-h, 0]$, then the trivial solution of (2.9) is uniformly asymptotically stable.

- If the trivial solution of (2.9) is uniformly asymptotically stable and if $\lim_{\tau \to \infty} u(\tau) = \infty$, then the trivial solution of (2.9) is globally uniformly asymptotically stable.

2.3 Systems with Multiple Delays

If a linear system with a single delay, θ, is not stable for a delay of some length, but is stable for $\theta = 0$, then there must exist a positive number $\bar{\theta}$ for which the system is stable for $0 \leq \theta \leq \bar{\theta}$. Many researchers have simply extended this idea to a system with multiple delays, but this simple extension may lead to conservativeness. For example, Fridman & Shaked [20], [27] investigated a linear system with two delays

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\theta_1) + A_2 x(t-\theta_2). \tag{2.19}$$

The upper bounds $\bar{\theta}_1$ and $\bar{\theta}_2$ on θ_1 and θ_2, respectively, are selected so that this system is stable for $0 \leq \theta_1 \leq \bar{\theta}_1$ and $0 \leq \theta_2 \leq \bar{\theta}_2$. However, the ranges of θ_1 and θ_2 that guarantee the stability of this system are conservative because they start from zero, even though that may not be necessary. One reason for this is that the relationship between θ_1 and θ_2 was not taken into account in the procedure for finding the upper bounds. Another point concerns a linear system with a single delay,

$$\dot{x}(t) = A_0 x(t) + (A_1 + A_2) x(t-\theta_1), \tag{2.20}$$

which is a special case of system (2.19), namely, the case $\theta_1 = \theta_2$. The stability criterion for system (2.20) should be equivalent to that for system (2.19) for $\theta_1 = \theta_2$; but this equivalence cannot be demonstrated by the methods in [20] and [27].

This section presents delay-dependent stability criteria for systems with multiple constant delays based on the Free Weighting Matrix (FWM) approach [16], [26]. Criteria are first established for a linear system with two delays. They take into account not only the relationships between $x(t-\theta_1)$ and $x(t) - \int_{t-h_1}^{t} \dot{x}(s)ds$, and $x(t-\theta_2)$ and $x(t) - \int_{t-h_2}^{t} \dot{x}(s)ds$ but also the one between $x(t-\theta_2)$ and $x(t-\theta_1) - \int_{t-\theta_2}^{t-\theta_1} \dot{x}(s)ds$. Note that the last relationship is between θ_1 and θ_2. All these relationships are expressed in terms of FWMs, and their parameters are determined based on the solutions of LMIs. In addition, the equivalence between system (2.20) and system (2.19) for $\theta_1 = \theta_2$ is demonstrated. Numerical examples show that the methods presented in this chapter are effective and are a significant improvement over others. Finally, these ideas are extended from systems with two delays to systems with multiple delays.

2.4 Neutral Systems

A neutral system is a system with a delay in both the state and the derivative of the state, with the one in the derivative being called a neutral delay. That makes it more complicated than a system with a delay in only the state. Neutral delays occur not only in physical systems, but also in control systems, where they are sometimes artificially added to boost the performance. For example, repetitive control systems constitute an important class of neutral systems. Stability criteria for neutral systems can be classified into two types: delay-independent and delay-dependent [28], [29], [30], [31] and [32]. Since the delay-independent type does not take the length of a delay into consideration, it is generally conservative. The basic methods for studying delay-dependent criteria for neutral systems are similar to those used to study linear systems, with the main ones being fixed model transformations. The four types of fixed model transformations impose limitations on possible solutions to delay-dependent stability problems.

The delay in the derivative of the state gives a neutral system special features not shared by linear systems. In a neutral system, a neutral delay can be the same as or different from a discrete delay. Neutral systems with identical constant discrete and neutral delays were studied in [18], [19], [28], [29], [30] and systems with different discrete and neutral delays were studied in [22], [23], [33]. The criteria in these reports usually require the neutral delay to be constant, but allow the discrete delay to be either constant [22], [23], [34] or time-varying [33], [35], [36], [37]. Almost all these criteria take only the length of a discrete delay into account and ignore the length of a neutral delay. They are thus called discrete-delay-dependent and neutral-delay-independent stability criteria. Discrete-delay and neutral-delay-dependent criteria are rarely investigated, with two exceptions being [38] and [39].

SECTION 3

IQC AND MULTIPLIER THEORY

This section presents the preliminaries and necessary mathematical background of IQC and multiplier theory.

3.1 Mathematical Background of Multiplier and IQC Theory

Integral Quadratic Constraints (IQCs) give useful characterizations of the structure of a given operator on an Hilbert space. The IQCs are defined in terms of quadratic forms which are defined in terms of self adjoint operators. The resulting stability theory unifies and extends the classical passivity based multiplier theory.

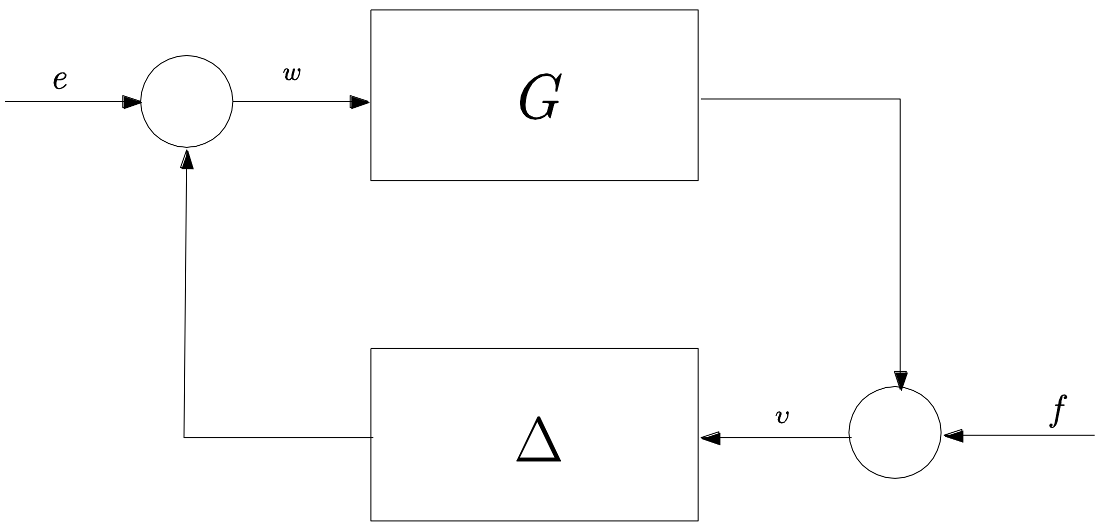

Figure 3. 1 Basic Feedback Configuration.

We consider systems of the form (3.8) where Δ is a bounded and causal operator on $\mathcal{H}$.

Let Π be a bounded and self adjoint operator. Then Δ satisfies the IQC defined by Π if

$$\sigma_{11}(v,\Delta(v)) = \langle \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}, \Pi \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} \rangle > 0, \quad \forall v \in \mathcal{H}. \tag{3.1}$$

We often call Π the multiplier that defines the IQC. We will sometimes use the shorthand notation $\Delta \in IQC(\Pi)$ to mean that Δ satisfies the IQC defined by Π.

If $\mathcal{H} = \mathbf{L}_2^m[0,\infty)$, then Π can be taken as a transfer function satisfying $\Pi(j\omega) = \Pi(j\omega)^*$. The condition in (3.1) reduces to

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{\Delta v}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{\Delta v}(j\omega) \end{bmatrix} d\omega \geq 0, \quad \forall v \in \mathbf{L}_2^m[0,\infty). \tag{3.2}$$

Theorem 3.1 (Multiplier Theorem)[9]

Assume that

- The feedback interconnection of G and Δ given in Figure 3.1 is well posed which means that $I - G\Delta$ is causally invertible since G is linear.
- Δ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}, \tag{3.3}$$

where M can be factorized into $M = M_- M_+$ where M_+, M_- and their inverses are all causal and bounded

- There exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall \omega \in \mathbb{R}. \tag{3.4}$$

Then the interconnection of G and Δ is stable.

If we compare the result in multiplier theorem with the corresponding result obtained in IQC stability theorem we see that the factorization condition is not needed in the IQC framework. The price paid for this is that well posedness is required for every feedback interconnection of G and $\tau\Delta$, when $\tau \in [0,1]$.

The concept integral quadratic constraint (IQC) is used for several purposes:

- to exploit structural information about perturbations,
- to characterize properties of external signals,

- to analyze combinations of several perturbations and external signals.

IQCs provide a way of representing relationships between processes envolving in a complex dynamical system, in a form that is convenient for analysis.

Depending on the particular application, various versions of IQC's are available. Two signals $\omega \in \mathbf{L_2^m}[0,\infty)$ and $v \in \mathbf{L_2^l}[0,\infty)$ are said to satisfy the IQC defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0, \tag{3.5}$$

where absolute integrability is assumed. Here, the Fourier transforms $\hat{w}(j\omega)$ and $\hat{v}(j\omega)$ represent the harmonic spectrum of the signals w and v at the frequency ω, and (3.1) describes the energy distribution in the spectrum of $\langle v, w \rangle$. In principle, $\Pi : \jmath\mathbf{R} \rightarrow \mathbf{C}^{(l+m)\times(l+m)}$ can be any measurable Hermitian valued function. In most situations, however, it is sufficient to use rational functions that are bounded on the imaginary axis.

A time-domain form of (3.1) is

$$\int_{0}^{\infty} \sigma(x_\pi(t), w(t), v(t))dt \geq 0 \tag{3.6}$$

where σ is a quadratic form, and x_π is defined by

$$\dot{x}_\pi(t) = A_\pi x_\pi(t) + B_w w(t) + B_v v(t), \quad x_\pi(0) = 0 \tag{3.7}$$

where A_π is a Hurwitz matrix. Intuitively, this state-space form IQC is a combination of a linear filter (3.7) and a "correlator" (3.6). For any bounded rational weighting function Π, (3.1) can be expressed in the form (3.6), (3.7) by first factorizing Π as $\Pi(\jmath\omega) = \psi(\jmath\omega)^* M \psi(\jmath\omega)$ with $\psi(\jmath\omega) = C_\psi(\jmath\omega I - A_\pi)^{-1}[B_w \quad B_v] + D_\psi$, then defining σ from C_ψ and D_ψ and M.

In system analysis, IQC's are useful to describe relations between signals in a system component. For example, to describe the saturation $w = sat(v)$, one can use the IQC defined by (3.1) with $\Pi = diag(1, -1)$, which holds for any square summable signals w, v related by $w = sat(v)$. In general, a bounded operator $\Delta : \mathbf{L_{2e}^l}[0,\infty) \rightarrow \mathbf{L_{2e}^m}[0,\infty)$ is said to satisfy the IQC defined by Π if (3.1) holds for all $w = \Delta(v)$, where $v \in \mathbf{L_2^l}[0,\infty)$.

There is, however, an evident problem in using IQC's in stability analysis. This is because both (3.1) and (3.6), (3.7) make sense only if the signals w, v are square summable. If it is not known apriori that the system is stable, then the signals might not be square summable. This will be resolved as follows. First, the system is considered as depending on a parameter $\tau \in [0,1]$, such that stability is obvious for $\tau = 0$, while $\tau = 1$ gives the system to be studied. Then, the IQC's are used to show that as τ increases from zero to one, there can be no transition from stability to instability.

The feedback configuration, illustrated in Figure 3.1, is the basic object of study,

$$\begin{aligned} v &= Gw + f \\ w &= \Delta(v) + e. \end{aligned} \tag{3.8}$$

Here $f \in \mathbf{L^l_{2e}}[0,\infty), e \in \mathbf{L^m_{2e}}[0,\infty)$ represent the "interconnection noise" and G and Δ are the two causal operators on $\mathbf{L^m_{2e}}[0,\infty)$ and $\mathbf{L^l_{2e}}[0,\infty)$, respectively. It is assumed that G is a linear time-invariant operator with the transfer function $G(s)$ in $\mathbf{RH^{l\times m}_\infty}$, and Δ has bounded gain.

In applications, Δ will be used to describe the "troublemaking" (nonlinear, time-varying, or uncertain) components of a system. The notation G will either denote a linear operator or a rational transfer matrix, depending on the context. The following definitions will be convenient.

We say that the feedback interconnection of G and Δ is well-posed if the map defined by (3.8) has a causal inverse on $\mathbf{L^{m+l}_{2e}}[0,\infty)$. The interconnection is stable if, in addition, the inverse is bounded, i.e., if there exists a constant $C > 0$ such that

$$\int_0^T (|v|^2 + |w|^2)dt \le C \int_0^T (|f|^2 + |e|^2)dt \tag{3.9}$$

for any $T \ge 0$ and for any solution of (3.8).

When G is linear, as it will be the case below, well-posedness means that $I - G\Delta$ is causally invertible. From boundedness of G and Δ, it also follows that the interconnection is stable if and only if $(I - G\Delta)^{-1}$ is a bounded causal operator on $\mathbf{L^l_2}[0,\infty)$.

In most applications, well-posedness is equivalent to the existence, uniqueness, and continuability of solutions of the underlying differential equations and is relatively easy to verify. Regarding stability, it is often desirable to verify some kind of exponential stability. However, for general classes of ordinary differential equations, exponential stability is equivalent to the input/output stability introduced above.

Consider ϕ with $sup_{x,t}\, \phi(x,t)/x(t) < \infty$. Assume that for any $g \in \mathbf{L_2^n}[0,\infty)$, $x_0 \in \mathbf{R^n}$, $t_0 \geq 0$ the system

$$\dot{x} = \phi(x(t),t) + g(t), \quad t \geq t_0, \tag{3.10}$$

has a solution $x(\cdot)$. Then the following two conditions are equivalent.

- There exists a constant $c > 0$ such that

$$\int_0^T (|x(t)|^2 dt \leq c \int_0^T (|g(t)|^2 dt, \quad \forall T > 0, \tag{3.11}$$

  for any solution of (3.10) with $x(0) = 0$.

- There exist $\epsilon, d > 0$ such that

$$|x(t_1)|^2 \leq d e^{\epsilon(t_0 - t_1)} x(t_0)^2 + d \int_{t_0}^{t_1} |g(t)|^2 dt, \tag{3.12}$$

  for any solution x of (3.10).

3.2 IQC Stability Analysis and Robust Performance Analysis

3.2.1 IQC Stability Analysis

We start with the well-known condition for robust stability by means of IQCs for an unforced feedback configuration shown in Figure 3.2

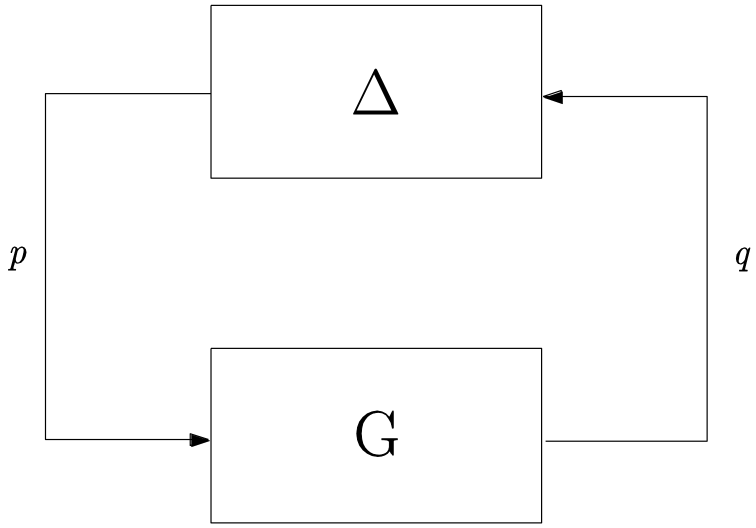

Figure 3. 2 Robust analysis problem.

Theorem 3.2 (IQC Stability Theorem)[14]

Suppose G is stable and

- The feedback interconnection of $\tau\Delta$ and G is well-posed for all $\tau \in [0,1]$,
- $\tau\Delta$ satisfies the IQC defined by Π for all $\tau \in [0,1]$. That is, for all $\tau \in [0,1]$,

$$\left\langle \begin{bmatrix} v \\ \tau\Delta v \end{bmatrix}, \Pi \begin{bmatrix} v \\ \tau\Delta v \end{bmatrix} \right\rangle \geq 0 \quad \forall v \in \mathcal{L}_{2+}. \tag{3.13}$$

- G satisfies

$$\begin{bmatrix} G \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I \end{bmatrix} \leq 0 \quad \text{on} \quad \mathbb{C}. \tag{3.14}$$

Then, the feedback interconnection of G and Δ is stable.

3.2.2 Robust Performance Analysis

Consider the system

$$\begin{aligned} \begin{bmatrix} z \\ v \end{bmatrix} &= G \begin{bmatrix} e \\ w \end{bmatrix} \\ w &= \Delta(v). \end{aligned} \tag{3.15}$$

Assume $G \in \mathbf{RH}_\infty^{(m+q)\times(m+q)}$. We want to investigate if the closed loop system satisfies various performance objectives. The most common performance measure is the L_2 gain of the system. This corresponds to the IQC

$$\sigma_P(z,e) = \int_0^\infty (|z(t)|^2 - \gamma^2|e(t)|^2)dt \le 0. \tag{3.16}$$

Assume $e \in \varepsilon \subset \mathbf{L}_2^q[0,\infty)$. Then the system in (3.15) has robust performance with respect to the performance IQC σ_P if

- the system is stable
- $\sigma_P(z,e) \le 0$ for all $z = \mathcal{F}_l(G,\Delta)e,\ e \in \varepsilon$.

To derive a condition for robust performance assume that we have the noise IQC

$$\sigma_\Psi(e) = \int_{-\infty}^\infty \hat{e}(\jmath\omega)^*\Psi(\jmath\omega)\hat{e}(\jmath\omega)d\omega \ge 0, \quad e \in \varepsilon \tag{3.17}$$

and the IQC

$$\sigma(v,\Delta(v)) = \int_{-\infty}^\infty \begin{bmatrix} \hat{v}(j\omega) \\ \hat{\Delta v}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{\Delta v}(j\omega) \end{bmatrix} d\omega \ge 0,\ \forall v \in \mathbf{L}_2^m[0,\infty) \tag{3.18}$$

for the uncertainty. We assume that Π has the block structure

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}. \tag{3.19}$$

We can now give the following robust L_2 performance result.

Assume that e satisfies (3.17) and Δ satisfies (3.18). Then the system (3.15) has robust L_2 gain γ if

- it is stable
- the frequency domain inequality

$$\left[\begin{array}{c} G(\jmath\omega) \\ I \end{array}\right]^* \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & \Pi_{11}(\jmath\omega) & 0 & \Pi_{12}(\jmath\omega) \\ \hline 0 & 0 & -\gamma^2 I + \Psi(\jmath\omega) & 0 \\ 0 & Pi_{12}^*(\jmath\omega) & 0 & \Pi_{22}(\jmath\omega) \end{array}\right] \left[\begin{array}{c} G(\jmath\omega) \\ I \end{array}\right] \le 0 \tag{3.20}$$

  holds for all $\omega \in [0,\infty)$.

3.2.3 The Kalman-Yakubovich-Popov Lemma

We show the frequency domain criterion

$$\left[\begin{array}{c} G(j\omega) \\ I \end{array}\right]^* \Pi(j\omega) \left[\begin{array}{c} G(j\omega) \\ I \end{array}\right] < 0, \quad \forall \omega \in [0, \infty) \tag{3.21}$$

is equivalent to a number of conditions on the system matrices in the realization of the transfer functions G and Π. The discrete time case can be treated similarly.

We will first derive an LQ optimal control formulation of (3.21). Let Π have the realization

$$\Pi = \left[\begin{array}{c} (j\omega I - A_\pi)^{-1} B_\pi \\ I \end{array}\right]^* M_\Pi \left[\begin{array}{c} (j\omega I - A_\pi)^{-1} B_\pi \\ I \end{array}\right] \tag{3.22}$$

where $B_\pi = [B_{\pi,v} \quad B_{\pi,\omega}]$ and A_π is Hurwitz. Using (3.22) and $G(s) = C_G(sI - A_G)^{-1}B_G + D_G$ shows that (3.21) can be formulated as

$$\left[\begin{array}{c} (j\omega I - A)^{-1} B \\ I \end{array}\right]^* \left[\begin{array}{cc} Q & S \\ S^T & R \end{array}\right] \left[\begin{array}{c} (j\omega I - A)^{-1} B \\ I \end{array}\right] > 0 \tag{3.23}$$

where

$$A = \left[\begin{array}{cc} A_\pi & B_{\pi,v} C_G \\ 0 & A_G \end{array}\right], \quad B = \left[\begin{array}{c} B_{\pi,v} D_G + B_{\pi,\omega} \\ B_G \end{array}\right] \tag{3.24}$$

and

$$\left[\begin{array}{cc} Q & S \\ S^T & R \end{array}\right] = \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_G & D_G \\ \hline 0 & 0 & I \end{array}\right]^T M_\Pi \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_G & D_G \\ \hline 0 & 0 & I \end{array}\right] \tag{3.25}$$

It follows that (3.23) is equivalent to existence of $\varepsilon > 0$ such that

$$\begin{aligned} \varepsilon \|w\|^2 &\leq \int_{-\infty}^{\infty} [\,*\,]^* \left[\begin{array}{cc} Q & S \\ S^T & R \end{array}\right] \left[\begin{array}{c} (j\omega I - A)^{-1} B \hat{w}(j\omega) \\ \hat{w}(j\omega) \end{array}\right] d\omega \\ &= \int_0^{\infty} (x^T Q X + 2x^T S w + w^T R w) dt \end{aligned} \tag{3.26}$$

for all pairs $(x, w) \in \mathbf{L}_2[0, \infty)$ such that $\dot{x} = Ax + Bw,\ x(0) = 0,\ w \in \mathbf{L}_2[0, \infty)$. This is an LQ optimal control problem. The Kalman-Yakubovich-Popov (KYP) Lemma shows that (3.23) and the LQ optimal control problem above are equivalent to an LMI condition, a Riccati equation condition and an eigenvalue condition on the Hamiltonian matrix corresponding to the LQ problem.

Theorem 3.3

Assume the pair of matrices (A,B) is stabilizable and A has no eigenvalues on the imaginary axis. Then the following statements are equivalent:

- There exists $\epsilon > 0$ such that for all pairs $(x, w) \in \mathbf{L}_2[0, \infty)$ such that

$$\dot{x} = Ax + Bw,\; x(0) = 0. \tag{3.27}$$

- We have

$$\left[\; * \;\right]^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1} B\hat{w}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} > 0,\; \forall \omega \in [0, \infty). \tag{3.28}$$

- There exists $P = P^T$ such that

$$\begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0. \tag{3.29}$$

- $R > 0$, and the Riccati equation

$$Q + PA + A^T P = (PB + S)R^{-1}(B^T P + S^T) \tag{3.30}$$

  has a stabilizing solution $P = P^T$, i.e., $\hat{A} = A - BR^{-1}(PB + S)^T$ is Hurwitz.

- $R > 0$ and the Hamiltonian matrix

$$H = \begin{bmatrix} A - BR^{-1}S^T & BR^{-1}B^T \\ Q - SR^{-1}S^T & -A^T + SR^{-1}B^T \end{bmatrix} \tag{3.31}$$

  has no eigenvalues on the imaginary axis.

3.3 Representation of Time-Delay and Parametric Uncertainties via IQCs

$(\Delta q)(t) = q(t - \theta) = p(t)$ where $\theta \in [0,\, \theta_0]$ which satisfies

$$|\hat{p}(j\omega)|^2 = |\hat{q}(j\omega)|^2 \tag{3.32}$$

$$\psi_1(\omega_\star)(|j\omega_\star \hat{p}(j\omega) + \hat{q}(j\omega)|^2 - (1 + \omega_\star^2)|\hat{q}(j\omega)|^2) \geq \psi_2(\omega_\star)|\hat{q}(j\omega) - \hat{p}(j\omega)|^2 \tag{3.33}$$

where $\omega_\star = \omega\theta_0/2$ and $\psi_{1,2}$ are the functions defined by

$$\psi_1(\omega) = \begin{matrix} \frac{\sin(\omega)}{\omega}, & |\omega| \leq \pi \\ 0, & |\omega| > \pi \end{matrix}$$

$$\psi_2(\omega) = \begin{matrix} \cos(\omega), & |\omega| \leq \pi \\ 0, & |\omega| > \pi \end{matrix}\;. \tag{3.34}$$

Note that (3.33) is just a sector inequality for the relation between $\hat{q}(j\omega) - \hat{p}(j\omega)$ and $j(\hat{q}(j\omega) + \hat{p}(j\omega))$

$$j(\hat{q}(j\omega) + \hat{p}(j\omega)) = \frac{\cos(\omega\theta/2)}{\sin(\omega\theta/2)}(\hat{q}(j\omega) - \hat{p}(j\omega)). \tag{3.35}$$

As explained in [9], multiplying (3.32) by any rational function and integrating over the imaginary axis yields a set of delay-independent IQC conditions. In order to decrease the conservatism of the description and obtain a delay-dependent condition, one can multiply (3.33) by any nonnegative weighting function and integrate over the imaginary axis. Unfortunately, the resulting IQCs have nonrational weighting matrices $\Pi(\cdot)$. However, one can use a rational upper bound ψ_{1+} of ψ_1 and rational lower bounds ψ_{1-} and ψ_{2-} of ψ_1 and ψ_2, respectively.

Then the point-wise inequality (3.33) holds with ψ_2 replaced by ψ_{2-} and with ψ_1 replaced by $\psi_{1\pm}$(the upper bound for the $|j\omega_\star \hat{p} + \hat{q}|^2$ multiplier, the lower bound for the $|\hat{q}|^2$ multiplier), respectively and can be integrated with a nonnegative rational weighting function to get rational IQCs utilizing delay bounds. With the inequality (3.33), the $\Pi_1(j\omega)$ for time-delay uncertainty can be expressed as

$$\Pi_1 = \begin{bmatrix} -cos\omega_* - \omega_* sin\omega_* & cos\omega_* + jsin\omega_* \\ cos\omega_* - jsin\omega_* & -cos\omega_* + \omega_* sin\omega_* \end{bmatrix}. \tag{3.36}$$

Instead of the multiplier given in (3.36), for a simpler identity which satisfies the previous sector inequality, a set of IQCs,

$$\Pi_1(j\omega, \theta_0) = \begin{bmatrix} \tau(j\omega)\psi_0(\omega\theta_0) & 0 \\ 0 & -\tau(j\omega) \end{bmatrix}, \tag{3.37}$$

is used to define the time-delay operator $(\Delta v)(t) = v(t-\theta) - v(t),\ \theta \leq \theta_0$ where $\tau(\cdot)$ is any nonnegative rational weighting function, and $\psi_0(\omega)$ is any rational upper bound of

$$\psi_*(\omega) = \max_{\theta\in[0,\theta_0]} |e^{-j\omega\frac{\theta}{\theta_0}} - 1|^2 = \begin{cases} 4\sin^2(\omega/2), & \omega < \Pi \\ 4 & \omega \geq \Pi \end{cases}. \tag{3.38}$$

For example, one can choose $\psi_0(\omega)$ as

$$\psi_0(\omega) = \frac{\omega^2 + 0.08\omega^4}{1 + 0.13\omega^2 + 0.02\omega^4}. \tag{3.39}$$

Due to the multiplier theory, to be able to prove that (3.37) is sufficient enough to define the time-delay operator $(\Delta v)(t) = v(t-\theta) - v(t),\ \theta \leq \theta_0$ one need to show that $\Pi_1(j\omega)$ satisfies

$$\begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix}^* \Pi_1(j\omega) \begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix} \geq 0 \tag{3.40}$$

When $\Delta(j\omega)$ is replaced with $e^{-j\omega\theta} - 1$ and (3.37) is used for $\Pi_1(j\omega)$, then one can easily show that (3.40) is satisfied by the help of (3.37) and (3.38).

On the other hand, since the parametric uncertainty affecting the system matrices, Δ is defined by multiplication with a real number of absolute value≤ 1. Then it satisfies all IQC's defined by matrix functions of the form

$$\Pi_2(j\omega) = \begin{bmatrix} P(j\omega) & Q(j\omega) \\ Q(j\omega)^* & -P(j\omega) \end{bmatrix}, \tag{3.41}$$

where $P(j\omega) = P(j\omega)^* \geq 0$ and $Q(j\omega) = -Q(j\omega)^*$are bounded and measurable matrix functions. This IQC is the basis for standard upper bounds for structured singular values. [9]

Δ in Figure 3.2 also involves the parametric uncertainties affecting the system matrices. From now on, Δ will be described as $\Delta = diag\{\Delta_1, \Delta_2\}$ where Δ_1 stands for the time delay on states, control inputs and derivatives of the states; Δ_2 describes the parametric uncertainty on the system matrices A and B_u.

Since Δ has the block-diagonal structure $\Delta = diag\{\Delta_1, \Delta_2\}$ and for $i = 1, 2$ Δ_i satisfies the IQC defined by

$$\Pi_i = \begin{bmatrix} \Pi_{i(11)} & \Pi_{i(12)} \\ \Pi_{i(12)}^* & \Pi_{i(22)} \end{bmatrix}, \tag{3.42}$$

where the block structures are consistent with the size of Δ_1 and Δ_2, respectively. Then, Δ satisfies the IQC defined by

$$daug(\Pi_1, \Pi_2) = \begin{bmatrix} \Pi_{1(11)} & 0 & \Pi_{1(12)} & 0 \\ 0 & \Pi_{2(11)} & 0 & \Pi_{2(12)} \\ \Pi_{1(12)}^* & 0 & \Pi_{1(22)} & 0 \\ 0 & \Pi_{2(12)}^* & 0 & \Pi_{2(22)} \end{bmatrix}. \tag{3.43}$$

If, for $i = 1, 2$ Δ_i is described by the cone Π_{Δ_i}, $\Delta = diag\{\Delta_1, \Delta_2\}$ is described by $daug\{\Pi_{\Delta_1}, \Pi_{\Delta_2}\} = \{daug(\Pi_1, \Pi_2) : \Pi_1 \in \Pi_{\Delta_1}, \Pi_2 \in \Pi_{\Delta_2}\}$. Addition and diagonal augmentation of any finite number of cones can be done in same way [40].

SECTION 4

FEEDFORWARD CONTROLLER SYNTHESIS VIA IQCs

In this section, the feedforward controller synthesis via dynamic IQCs is provided. The general problem is defined in the first subsection and the duality in multiplier theory is explained. The section is concluded with the theory of feedforward controller synthesis for uncertain time-delay systems.

4.1 General Problem Definition

Consider a class of time-delay system with stationary time-delay given as

$$\begin{aligned} \dot{x} &= [A+\Delta A]x(t) + A_h x(t-\theta_1) \\ &\qquad + A_d \dot{x}(t-\theta_2) + [B_u + \Delta B_u]u(t-\theta_3) + B_w w(t) \\ z &= Cx(t) + D_w w(t), \qquad x(0) = x_0 \end{aligned} \tag{4.1}$$

where $x \in \Re^n$ real-time measurable states, $x(0) = x_0$ initial conditions, θ_1 and θ_2 are constant time delays which satisfies $0 \leq \theta_1 \leq \theta_{u1}$, $0 \leq \theta_2 \leq \theta_{u2}$ and $0 \leq \theta_3 \leq \theta_{u3}$ where θ_{u1}, θ_{u2} and θ_{u3} are the known upper limits of delay, $u \in \Re^m$ denotes the control inputs, $w \in \Re^q$ is the disturbance vector acting on the system which is assumed to be restricted in a set of the form

$$\mathcal{W} := \left\{ w : \mathbb{R}_+ \to \mathbb{R}^q \ : \ \int_0^\infty w^{\mathrm{T}}(t) w(t) dt < \infty \right\}. \tag{4.2}$$

$z \in \Re^p$ is an exogenous controlled output. Then A, A_d, A_h, B_u, B_w, C and D_w are known real constant state-space system matrices of appropriate dimensions. The uncertainties ΔA and ΔB_u are matrix valued functions with appropriate dimensions.

4.2 Duality in Multiplier Theory

One needs to use the dual forms of the conditions involving the multiplier Π and G to carry out the feedforward controller synthesis. It can be shown that with $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$, it can be assumed that $\Pi_{11} > 0$. Due to (3.4), Π has the same number of negative eigenvalues as the number of inputs of G [41]. Since $in(\Pi) = in(\Pi_{11}) + in(\Pi_{22} - \Pi_{12}^* \Pi_{11}^{-1} \Pi_{12})$, it can be shown that $\Pi_{22} - \Pi_{12}^* \Pi_{11}^{-1} \Pi_{12} < 0$ on $\mathbb{C}^0$.

Lemma 4.1 [14]

Let $S \in \mathbb{C}^{(m+n)\times m}$ have full column-rank and $M = M^* \in \mathbb{C}^{(m+n)\times(m+n)}$ be such that $in(M) = (m, n, 0)$. Then, $S^* M S > 0$ iff $S_\perp^* M^{-1} S_\perp < 0$ where $S_\perp$ forms a basis for the orthogonal complement of the image of S.

Lemma 4.2 [14]

Let $\Delta : \mathcal{L}_2^\rho \rightarrow: \mathcal{L}_2^\mu$ be linear and suppose that $\Pi = \Pi^* \in R\mathcal{L}_\infty^{(\rho+\mu)\times(\rho+\mu)}$ is such that $in(\Pi) = (\rho, \mu, 0)$ on $\mathbb{C}^0$. Then, the following statements are equivalent,

i. $\langle \begin{pmatrix} v \\ \Delta v \end{pmatrix}, \Pi \begin{pmatrix} v \\ \Delta v \end{pmatrix} \rangle \geq 0 \quad \forall v \in \mathcal{L}_2^{\bar{\rho}}$.

ii. $\langle \begin{pmatrix} \Delta^* w \\ w \end{pmatrix}, \Pi^{-1} \begin{pmatrix} \Delta^* w \\ w \end{pmatrix} \rangle \leq 0 \quad \forall w \in \mathcal{L}_2^\mu$.

Now by the help of these lemmas, one can define the equivalent condition for the dual form

$$\begin{bmatrix} I \\ -G^* \end{bmatrix}^* \Pi^{-1} \begin{bmatrix} I \\ -G^* \end{bmatrix} > 0 \quad on \quad \mathbb{C}^0, \tag{4.3}$$

where

$$\Pi^{-1} =: \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^* & \Theta_{22} \end{bmatrix} \tag{4.4}$$

and deduce $\Theta_{22} < 0$ and $\Theta_{11} - \Theta_{12}^* \Theta_{22}^{-1} \Theta_{12} > 0$ on $\mathbb{C}^0$ [15].

It is desirable to restrict Θ to a subspace of $R\mathcal{L}_\infty$ and optimize over it. We wish to do this by specifying $\phi \in R\mathcal{H}_\infty$ and optimizing over N, where $\Theta = \phi N \phi^*$ and Θ^{-1}

satisfies the IQC associated with Δ. For several perturbation blocks, parameterization of a suitable multiplier (i.e. Θ^{-1}) is available in the literature. However, since it is Θ, and not Θ^{-1}, that appears linearly in our formulation, the structure of the inverse of the multiplier is paramount. The structures of some multipliers are inversion-invariant. An immediate example is multipliers of the form

$$\Pi_1(j\omega) = \begin{bmatrix} \Pi_{11}(j\omega) & 0 \\ 0 & \Pi_{22}(j\omega) \end{bmatrix}, \quad \Pi_{11} = -\Pi_{22} > 0, \tag{4.5}$$

where Π_{22} is unstructured. Yet some structures are not inversion-invariant. In general, Θ must be parameterized in a case-by-case manner. However, for linear Δ blocks defined not only over $\mathcal{L}_{2+}$ but over $\mathcal{L}_2$, we can use the lemma and parameterize Θ via the dual IQC,

$$\langle \begin{pmatrix} \Delta^* w \\ w \end{pmatrix}, \Theta \begin{pmatrix} \Delta^* w \\ w \end{pmatrix} \rangle \leq 0 \quad \forall w \in \mathcal{L}_2^{\mu}. \tag{4.6}$$

Hence, if Θ_is are known to satisfy the dual IQC, Θ can be parameterized as $\Theta = \sum_{i=1}^{k} \alpha_i \Theta_i$, where $\alpha_i \geq 0$. Alternatively, if Π_is satisfy the primal IQC, one can take $\Theta = \sum_{i=1}^{k} \alpha_i \Pi_i^{-1}$ with $\alpha_i \geq 0$. Based on a finite set of primal multipliers, this provides a generic recipe for constructing an affinely parameterized family of dual multipliers that is suited for our synthesis procedure.

4.3 Theory of Feedforward Controller Synthesis via Dynamic IQCs

Θ can be decomposed as

$$\Theta = \phi N \phi^* = \begin{bmatrix} -\phi_1 \\ -\phi_2 \end{bmatrix} N \begin{bmatrix} -\phi_1^* & -\phi_2^* \end{bmatrix}, \tag{4.7}$$

by using the minimal state-space realization

$$\begin{bmatrix} -\phi_1^* & -\phi_2^* \end{bmatrix} = \left[\begin{array}{cc|cc} -A_{11}^{\mathrm{T}} & 0 & -C_{11}^{\mathrm{T}} & 0 \\ -A_{12}^{\mathrm{T}} & -A_{22}^{\mathrm{T}} & -C_{12}^{\mathrm{T}} & -C_{22}^{\mathrm{T}} \\ \hline -B_1^{\mathrm{T}} & -B_2^{\mathrm{T}} & -D_1^{\mathrm{T}} & -D_2^{\mathrm{T}} \end{array}\right]. \tag{4.8}$$

For the decomposition of Θ one can assume that (C_{22}, A_{22}) is observable and A_{11}, A_{22} are Hurwitz. With the definitions of

$$\mathcal{G} \triangleq \begin{bmatrix} -\phi_1^* & -\phi_2^* \end{bmatrix} \begin{bmatrix} I \\ -G^* \end{bmatrix}, \tag{4.9}$$

one can reobtain (4.3) [14] as $\mathcal{G}(j\omega)^* N \mathcal{G}(j\omega) > 0$ whereas $\mathcal{G}(j\omega)$ has the realization of

$$\mathcal{G}(j\omega) = \left[\begin{array}{ccc|c} -A_{11}^{\mathrm{T}} & 0 & 0 & -C_{11}^{\mathrm{T}} \\ -A_{12}^{\mathrm{T}} & -A_{22}^{\mathrm{T}} & C_{22}^{\mathrm{T}} B^{\mathrm{T}} & -C_{12}^{\mathrm{T}} + C_{22}^{\mathrm{T}} D^{\mathrm{T}} \\ 0 & 0 & -A^{\mathrm{T}} & -B^{\mathrm{T}} \\ \hline -B_1^{\mathrm{T}} & -B_2^{\mathrm{T}} & D_2^{\mathrm{T}} B^{\mathrm{T}} & -D_1^{\mathrm{T}} + D_2^{\mathrm{T}} D^{\mathrm{T}} \end{array}\right]. \tag{4.10}$$

On the other hand, Θ_{22} can be defined as

$$\Theta_{22} = \left[\begin{array}{c|c} -A_{22}^{\mathrm{T}} & -C_{22}^{\mathrm{T}} \\ \hline -B_2^{\mathrm{T}} & -D_2^{\mathrm{T}} \end{array}\right]^{\mathrm{T}} N \left[\begin{array}{c|c} -A_{22}^{\mathrm{T}} & -C_{22}^{\mathrm{T}} \\ \hline -B_2^{\mathrm{T}} & -D_2^{\mathrm{T}} \end{array}\right] \tag{4.11}$$

and for $\Theta_{22} < 0$, one can apply the KYP Lemma to show that there exists an $X = X^{\mathrm{T}}$ such that

$$\begin{bmatrix} -A_{22}^{\mathrm{T}} & -C_{22}^{\mathrm{T}} \\ I & 0 \\ -B_2^{\mathrm{T}} & -D_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & N \end{bmatrix} \begin{bmatrix} -A_{22}^{\mathrm{T}} & -C_{22}^{\mathrm{T}} \\ I & 0 \\ -B_2^{\mathrm{T}} & -D_2^{\mathrm{T}} \end{bmatrix} < 0. \tag{4.12}$$

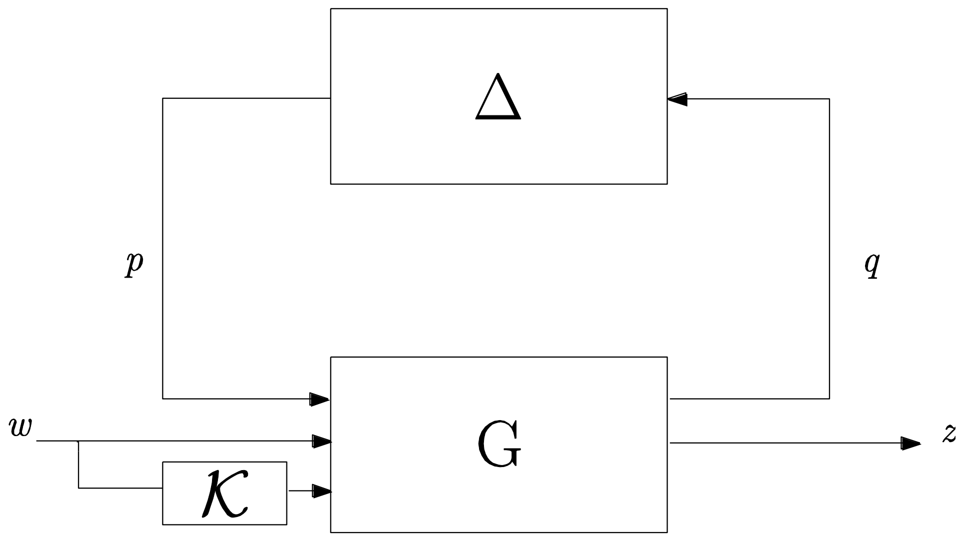

Figure 4. 3 Robust feedforward problem

Let us deal with the robust feedforward problem shown in Figure 4.3. The objective is to design a controller $\mathcal{K}(s)$ such that the closed-loop system has a minimum $\mathcal{H}_\infty$-norm from input w to the output z. We consider a plant with the state-space representation such as

$$\begin{bmatrix} q \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_p & B_w & B_u \\ \hline C_q & D_{qp} & D_{qw} & D_{qu} \\ C_z & D_{zp} & D_{zw} & D_{zw} \\ 0 & 0 & I & 0 \end{array}\right] \begin{bmatrix} p \\ w \\ u \end{bmatrix}, \tag{4.13}$$

where A is Hurwitz. $\mathcal{K}$ is realized as $u = \left[\begin{array}{c|c} A_C & B_C \\ \hline C_C & D_C \end{array}\right] w$. Then, the nominal (delay-free) closed-loop system becomes

$$G_{cl} := \left[\begin{array}{cc|cc} A & B_u C_C & B_p & B_w + B_u D_C \\ 0 & A_C & 0 & B_C \\ \hline C_q & D_{qu} C_C & D_{qp} & D_{qw} + D_{qu} C_C \\ C_z & D_{zu} C_C & D_{zp} & D_{zw} + D_{zu} D_C \end{array}\right]. \tag{4.14}$$

On the other hand, involving the multiplier realization, one can define

$$\left[\begin{array}{ccc|cc} -A_{11}^{\mathrm{T}} & 0 & 0 & -C_{11}^{\mathrm{T}} & 0 \\ -A_{12}^{\mathrm{T}} & -A_{22}^{\mathrm{T}} & C_{22}^{\mathrm{T}} B_p^{\mathrm{T}} & -C_{12}^{\mathrm{T}} + C_{22}^{\mathrm{T}} D_{qp}^{\mathrm{T}} & C_{22}^{\mathrm{T}} D_{zp}^{\mathrm{T}} \\ 0 & 0 & -A^T & -C_q^T & -C_z^T \\ \hline -B_1^{\mathrm{T}} & -B_2^{\mathrm{T}} & D_2^{\mathrm{T}} B_p^{\mathrm{T}} & -D_1^{\mathrm{T}} + D_2^{\mathrm{T}} D_{qp}^{\mathrm{T}} & D_2^{\mathrm{T}} D_{zp}^{\mathrm{T}} \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & -B_w^{\mathrm{T}} & -D_{qw}^{\mathrm{T}} & -D_{zw}^{\mathrm{T}} \\ 0 & 0 & -B_u^{\mathrm{T}} & -D_{qu}^{\mathrm{T}} & -D_{zu}^{\mathrm{T}} \end{array}\right]$$

$$=: \left[\begin{array}{c|cc} -\mathcal{A}^{\mathrm{T}} & -\mathcal{C}_q^{\mathrm{T}} & -\mathcal{C}_z^{\mathrm{T}} \\ \hline -\mathcal{B}_\phi^{\mathrm{T}} & -\mathcal{D}_{\phi_q}^{\mathrm{T}} & -\mathcal{D}_{\phi_z}^{\mathrm{T}} \\ 0 & 0 & -\mathcal{D}_{zp}^{\mathrm{T}} \\ -\mathcal{B}_w^{\mathrm{T}} & -\mathcal{D}_{qw}^{\mathrm{T}} & -\mathcal{D}_{zw}^{\mathrm{T}} \\ -\mathcal{B}_u^{\mathrm{T}} & -\mathcal{D}_{qu}^{\mathrm{T}} & -\mathcal{D}_{zu}^{\mathrm{T}} \end{array}\right] =: \left[\begin{array}{c|c} -\mathcal{A}^{\mathrm{T}} & -\mathcal{C}^{\mathrm{T}} \\ \hline -\mathcal{B}_\phi^{\mathrm{T}} & -\mathcal{D}_\phi^{\mathrm{T}} \\ 0 & -\mathcal{D}_p^{\mathrm{T}} \\ -\mathcal{B}_w^{\mathrm{T}} & -\mathcal{D}_w^{\mathrm{T}} \\ -\mathcal{B}_u^{\mathrm{T}} & -\mathcal{D}_u^{\mathrm{T}} \end{array}\right], \tag{4.15}$$

and $T := \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}$ with a suitable row partition with $\mathcal{A}$.

4.4 Feedforward Controller Synthesis for Time-delay Systems Having Parametric Uncertainties

A robust optimal stabilizing $\mathcal{H}_\infty$ controller synthesis problem for nominal time-delay system (4.1) is considered so that the closed-loop system has minimum $\mathcal{L}_2$-gain, γ, which is defined as

$$\sup_{w \in \mathcal{L}_{2+}, w \neq 0} \frac{\|z\|_2}{\|w\|_2}. \tag{4.16}$$

Theorem 4.1

Given positive constants $\theta_0 > 0$, $\gamma > 0$ the feedforward control law $u = \left[\begin{array}{c|c} A_C & B_C \\ \hline C_C & D_C \end{array}\right] w$ globally asymptotically stabilizes the system (4.1) with an $\mathcal{H}_\infty$

disturbance attenuation level of γ where the state, control and neutral delays are defined by the multiplier

$$\Pi_1(j\omega, \theta_0) \triangleq \begin{bmatrix} \tau(j\omega)\psi_0(\omega\theta_0) & 0 \\ 0 & -\tau(j\omega) \end{bmatrix}, \tag{4.17}$$

and the parametric uncertainty affecting the system matrices defined by the multiplier

$$\Pi_2(j\omega) \triangleq \begin{bmatrix} P(j\omega) & Q(j\omega) \\ Q(j\omega)^* & -P(j\omega) \end{bmatrix}, \tag{4.18}$$

if there exists matrices $X = X^T$, $Y = Y^T$, $\hat{X} = \hat{X}^T$, A_C, B_C, C_C and D_C with appropriate dimensions such that

$$\begin{bmatrix} * \end{bmatrix}^T \begin{bmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & N \end{bmatrix} \begin{bmatrix} -A_{22}^{\mathrm{T}} & -C_{22}^{\mathrm{T}} \\ I & 0 \\ -B_2^{\mathrm{T}} & -D_2^{\mathrm{T}} \end{bmatrix} < 0, \tag{4.19}$$

$$R^{\mathrm{T}} \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & N & 0 & 0 \\ 0 & 0 & 0 & \gamma I & 0 \\ 0 & 0 & 0 & 0 & -\gamma^{-1} \end{bmatrix} R > 0 \tag{4.20}$$

where

$$R = \begin{bmatrix} -\hat{X}\mathcal{A}^{\mathrm{T}} & -\hat{X}\mathcal{A}^{\mathrm{T}} & -\hat{X}\mathcal{C}^{\mathrm{T}} \\ \mathbf{A}_{\mathbf{C}}^{\mathrm{T}} & -Y\mathcal{A}^{\mathrm{T}} - \mathbf{C}_{\mathbf{C}}^{\mathrm{T}}\mathcal{B}_u^{\mathrm{T}} & -Y\mathcal{C}^{\mathrm{T}} - \mathbf{C}_{\mathbf{C}}^{\mathrm{T}}\mathcal{D}_u^{\mathrm{T}} \\ I & 0 & 0 \\ 0 & I & 0 \\ -B_\phi^{\mathrm{T}} & -B_\phi^{\mathrm{T}} & -D_\phi^{\mathrm{T}} \\ 0 & 0 & -D_p^{\mathrm{T}} \\ -\mathcal{B}_\omega^{\mathrm{T}} - \mathbf{D}_{\mathbf{C}}^{\mathrm{T}} - \mathcal{B}_u^{\mathrm{T}} - \mathbf{B}_{\mathbf{C}}^{\mathrm{T}} & -\mathcal{B}_\omega^{\mathrm{T}} - \mathbf{D}_{\mathbf{C}}^{\mathrm{T}} - \mathcal{B}_u^{\mathrm{T}} & -\mathcal{D}_\omega^{\mathrm{T}} - \mathbf{D}_{\mathbf{C}}^{\mathrm{T}} - \mathcal{D}_u^{\mathrm{T}} \end{bmatrix},$$

$$T^{\mathrm{T}}\hat{X}T - \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} > 0 \quad and \quad Y - \hat{X} > 0. \tag{4.21}$$

One can obtain the system matrices of dynamic controller such as

$$\begin{aligned} A_C^{\mathrm{T}} &= \left(-\mathbf{A}_{\mathbf{C}}^{\mathrm{T}} - YA^{\mathrm{T}} + \mathbf{C}_{\mathbf{C}}^{\mathrm{T}}B_u^{\mathrm{T}}\right)\left(\hat{X} - Y\right)^{-1} \\ B_C^{\mathrm{T}} &= \mathbf{B}_{\mathbf{C}}^{\mathrm{T}}\left(\hat{X} - Y\right)^{-1} \\ C_C^{\mathrm{T}} &= \mathbf{C}_{\mathbf{C}}^{\mathrm{T}} \\ D_C^{\mathrm{T}} &= \mathbf{D}_{\mathbf{C}}^{\mathrm{T}}. \end{aligned} \tag{4.22}$$

Proof: For the system with the time-delay and parametric uncertainty (4.1), dynamic IQCs are used to represent the delay operator and parametric uncertainty. Then for the time-delay operator, using the multiplier $\Pi_1(j\omega,\, \theta_0)$ allows to write

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \tau(j\omega)\psi_0(\omega\theta_0) & 0 \\ 0 & -\tau(j\omega) \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0. \tag{4.23}$$

Factorizing $\psi_0(\omega\theta_0)$ function in the multiplier $\Pi_1(j\omega,\, \theta_0)$ leads to

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \tau(j\omega)\Gamma^*(\omega\theta_0)\Gamma(\omega\theta_0) & 0 \\ 0 & -\tau(j\omega) \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0. \tag{4.24}$$

Then, by the help of the factorization we obtain

$$\begin{bmatrix} \Gamma(\omega\theta_0)G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \tau(j\omega) & 0 \\ 0 & -\tau(j\omega) \end{bmatrix} \begin{bmatrix} \Gamma(\omega\theta_0)G(j\omega) \\ I \end{bmatrix} < 0. \tag{4.25}$$

As explained in the previous sections, we have two different multipliers for two different types of uncertainties which are delay operator and the parametric uncertainty. Since there are two multipliers, one needs to combine them and define a compact multiplier which consists of the two multipliers,

$$\begin{bmatrix} \tau(j\omega) & 0 \\ 0 & -\tau(j\omega) \end{bmatrix} = \begin{bmatrix} \varphi_1^*\bar{M}\varphi_1 & 0 \\ 0 & -\varphi_1^*\bar{M}\varphi_1 \end{bmatrix}, \tag{4.26}$$

and

$$\begin{bmatrix} P(j\omega) & Q(j\omega) \\ Q(j\omega)^* & -P(j\omega) \end{bmatrix} = \begin{bmatrix} \varphi_2^*\bar{P}\varphi_2 & \varphi_2^*\bar{R}\varphi_2 \\ \varphi_2^*\bar{R}^T\varphi_2 & -\varphi_2^*\bar{P}\varphi_2 \end{bmatrix}. \tag{4.27}$$

Here, (4.26) stands for the time-delay operator whereas the multiplier given in (4.27) stands for parametric uncertainty affecting the system matrices. As explained in Section 3.3, the combination of the two multipliers can be carried out and the final multiplier is given in the form of $\Pi_{total}(j\omega) = \varphi^*\Pi\varphi$ as follows

$$\Pi_{total}(j\omega) = \begin{bmatrix} \varphi_1^* \bar{M}\varphi_1 & 0 & 0 & 0 \\ 0 & \varphi_2^* \bar{P}\varphi_2 & 0 & \varphi_2^* \bar{R}\varphi_2 \\ 0 & 0 & -\varphi_1^* \bar{M}\varphi_1 & 0 \\ 0 & \varphi_2^* \bar{R}^T\varphi_2 & 0 & -\varphi_2^* \bar{P}\varphi_2 \end{bmatrix} =$$

$$\begin{bmatrix} \varphi_1 & 0 & 0 & 0 \\ 0 & \varphi_2 & 0 & 0 \\ 0 & 0 & \varphi_1 & 0 \\ 0 & 0 & 0 & \varphi_2 \end{bmatrix}^* \begin{bmatrix} \bar{M} & 0 & 0 & 0 \\ 0 & \bar{P} & 0 & \bar{R} \\ 0 & 0 & -\bar{M} & 0 \\ 0 & \bar{R}^T & 0 & -\bar{P} \end{bmatrix} \begin{bmatrix} \varphi_1 & 0 & 0 & 0 \\ 0 & \varphi_2 & 0 & 0 \\ 0 & 0 & \varphi_1 & 0 \\ 0 & 0 & 0 & \varphi_2 \end{bmatrix}. \quad (4.28)$$

As explained in Section 4.2, by using the duality properties we have $\Theta(j\omega) = \Pi_{total}^{-1}(j\omega)$ similar to the one in (4.4) which have the factorization in (4.7). So that we have the dual form of the total multiplier involving the time delay and parametric uncertainty in the form of $\Theta = \phi N \phi^*$ where

$$N = \begin{bmatrix} M & 0 & 0 & 0 \\ 0 & P & 0 & R \\ 0 & 0 & -M & 0 \\ 0 & R^T & 0 & -P \end{bmatrix} \quad (4.29)$$

and $\phi = diag(S, S)$ with [40]

$$S = \left[\begin{array}{c|c} A_S & B_S \\ \hline C_S & D_S \end{array}\right]. \quad (4.30)$$

Suppose $\Pi_{total} = (\phi N \phi^*)^{-1}$ satisfies the IQC associated with Δ which has the block-diagonal structure $\Delta = diag\{\Delta_1, \Delta_2\}$ where Δ_1 and Δ_2 stand for the delay operator and parametric uncertainty, respectively. Then, the system has $\mathcal{L}_2$ gain less than γ if G_{cl} is stable and

$$[\,*\,]^* \left[\begin{array}{cc|cc} \phi_1 N \phi_1^* & 0 & \phi_1 N \phi_2^* & 0 \\ 0 & \gamma I & 0 & 0 \\ \hline \phi_2 N \phi_1^* & 0 & \phi_2 N \phi_2^* & 0 \\ 0 & 0 & 0 & -\gamma^{-1} I \end{array}\right] \begin{bmatrix} I \\ -G_{cl}^* \end{bmatrix} > 0. \quad (4.31)$$

With the help of (4.15), one can obtain

$$\begin{bmatrix} -\phi_1^* & 0 & -\phi_2^* & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I \\ -G_{cl}^* \end{bmatrix} = \left[\begin{array}{cc|c} -\mathcal{A}^{\mathrm{T}} & 0 & -\mathcal{C}^{\mathrm{T}} \\ C_C^{\mathrm{T}} \mathcal{B}_u^{\mathrm{T}} & -A_C^{\mathrm{T}} & C_C^{\mathrm{T}} \mathcal{D}_u^{\mathrm{T}} \\ \hline -B_\phi^{\mathrm{T}} & 0 & -D_\phi^{\mathrm{T}} \\ 0 & 0 & -\mathcal{D}_p^{\mathrm{T}} \\ -\mathcal{B}_w^{\mathrm{T}} + D_C^{\mathrm{T}} \mathcal{B}_u^{\mathrm{T}} & -B_C^{\mathrm{T}} & -\mathcal{D}_w^{\mathrm{T}} + D_C^{\mathrm{T}} \mathcal{D}_u^{\mathrm{T}} \end{array}\right]. \quad (4.32)$$

So the frequency domain inequality is equivalent to the existence of $\mathcal{Y} = \mathcal{Y}^{\mathrm{T}}$ such as

$$\left[\,\star\,\right]^{\mathrm{T}} \left[\begin{array}{c|c|ccc} 0 & \mathcal{Y} & 0 & 0 & 0 \\ \hline \mathcal{Y} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & N & 0 & 0 \\ 0 & 0 & 0 & \gamma I & 0 \\ 0 & 0 & 0 & 0 & -\gamma^{-1} \end{array}\right] \left[\begin{array}{ccc} -\mathcal{A}^{\mathrm{T}} & 0 & -\mathcal{C}^{\mathrm{T}} \\ C_C^{\mathrm{T}}\mathcal{B}_u^{\mathrm{T}} & -A_C^{\mathrm{T}} & C_C^{\mathrm{T}}\mathcal{D}_u^{\mathrm{T}} \\ \hline I & 0 & 0 \\ 0 & I & 0 \\ \hline -B_\phi^{\mathrm{T}} & 0 & -D_\phi^{\mathrm{T}} \\ 0 & 0 & -\mathcal{D}_p^{\mathrm{T}} \\ -\mathcal{B}_w^{\mathrm{T}} + D_C^{\mathrm{T}}\mathcal{B}_u^{\mathrm{T}} & -B_C^{\mathrm{T}} & -\mathcal{D}_w^{\mathrm{T}} + D_C^{\mathrm{T}}\mathcal{D}_u^{\mathrm{T}} \end{array}\right] > 0. \quad (4.33)$$

Let us partition $\mathcal{Y}$ as

$$\left[\begin{array}{ccc|c} \mathcal{Y}_{11} & \mathcal{Y}_{12} & \mathcal{Y}_{13} & \mathcal{Y}_{14} \\ * & \mathcal{Y}_{22} & \mathcal{Y}_{23} & \mathcal{Y}_{24} \\ * & * & \mathcal{Y}_{33} & \mathcal{Y}_{34} \\ \hline * & * & * & \mathcal{Y}_{44} \end{array}\right] = \left[\begin{array}{c|c} Y & I \\ \hline * & (Y - \hat{X})^{-1} \end{array}\right] \quad (4.34)$$

with nonsingular $\hat{X}$ and $Y - \hat{X}$. The congruence transformation is applied to (4.33)

$$\left[\begin{array}{cc|c} I & I & 0 \\ \hat{X} - Y & 0 & 0 \\ \hline 0 & 0 & I \end{array}\right]^{\mathrm{T}} \left[\,*\,\right] \left[\begin{array}{cc|c} I & I & 0 \\ \hat{X} - Y & 0 & 0 \\ \hline 0 & 0 & I \end{array}\right] > 0 \quad (4.35)$$

and with the definitions of $\mathbf{A}_{\mathbf{C}}^{\mathrm{T}} := -Y\mathcal{A}^{\mathrm{T}} + C_C^{\mathrm{T}}\mathcal{B}_u^{\mathrm{T}} - A_C^{\mathrm{T}}\left(\hat{X} - Y\right)$, $\mathbf{C}_{\mathbf{C}}^{\mathrm{T}} := C_C^{\mathrm{T}}$, $\mathbf{B}_{\mathbf{C}}^{\mathrm{T}} := B_C^{\mathrm{T}}\left(\hat{X} - Y\right)$ and $\mathbf{D}_{\mathbf{C}}^{\mathrm{T}} := D_C^{\mathrm{T}}$, it leads to (4.20). On the other hand, it is needed to solve the LMI (4.19) because of the $\Theta_{22} = \phi_2 N \phi_2^* < 0$ condition. Then, the closed-loop system is uniformly exponentially γ-stable if (4.21) holds which concludes the proof.

SECTION 5

NUMERICAL EXAMPLES AND SIMULATION STUDIES

The results obtained by using the proposed controller synthesis, are illustrated by using different examples that can be classified into three groups: State and control delayed systems including an active suspension system, delayed systems with parametric uncertainties and neutral time-delay systems. All of the examples are the benchmark problems used in the literature for many times [42], [43], [44], [45], [46].

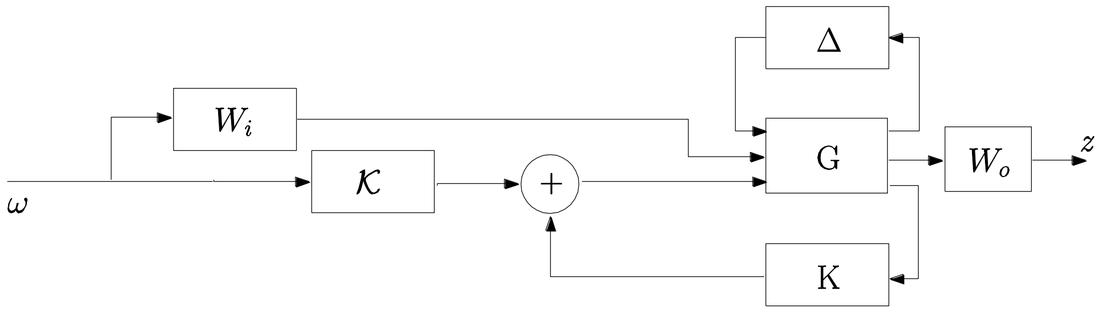

Figure 5. 4 Combined feedback and feedforward control scheme.

The examples are grouped into three case studies which involve state and control delay, time-delay and parametric uncertainty, neutral system (delay on state derivatives).

5.1 Case 1: State and Control Delay

Example 5.1

Consider the following linear time-delay system.

$$\begin{aligned} \dot{x} &= Ax(t) + A_d x(t-\theta_1) + B_u u(t-\theta_2) + B_w w(t) \\ z &= \begin{bmatrix} Cx(t) + D_w w(t) \\ C_d x(t-\theta_1) \\ Du(t) \end{bmatrix} \end{aligned} \tag{5.1}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_d = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix},\ C_d = \begin{bmatrix} 0 & 0 \end{bmatrix},\ D_w = \begin{bmatrix} 0 \end{bmatrix},\ D = \begin{bmatrix} 0.1 \end{bmatrix}. \quad (5.2)$$

$\theta_{u1} = 0.4$ and $\theta_{u2} = 0.15$ are the upper bounds for the time delay that affect the states and control input of the system, respectively. An $\mathcal{H}_\infty$ state-feedback controller whose gains are chosen as $K = \begin{bmatrix} 1.120 & -0.439 \end{bmatrix}$ is employed to stabilize and control this system. Note that this feedback compensator is obtained from the well-known $\mathcal{H}_\infty$ synthesis given in [47].

The aim of this study is to design a feedforward controller, $\mathcal{K}$, such that the closed-loop system is stable and has minimum $\mathcal{H}_\infty$ gain, γ as an addition to the existing stabilizing feedback controller in a standard two-degrees of freedom control structure as shown in Figure 5.4.

Note that the disturbance is assumed to be measurable in this system. The multiplier is assigned in the form of $\Theta(j\omega) = \phi(j\omega)N\phi^*(j\omega)$ where $N = diag(V, -V)$ and $\phi = diag(S, S)$ which satisfies the dual of the IQC dealing with Δ which here stands for the state and control input delay. Here we choose

$$S(s) = \begin{pmatrix} 1 & \frac{1}{s+p} & \cdots & \left(\frac{1}{s+p}\right)^{n_s} \end{pmatrix} \otimes I_2 \quad \text{where} \quad p = 1. \quad (5.3)$$

W_i is also chosen as a second order Butterworth low-pass filter.

The cut-off frequency of this filter is chosen as 1rad/sec. where this filter is selected as

$$W_i(s) = \frac{0.001(s^2 + s) + 1}{s^2 + \sqrt{2}s + 1}. \quad (5.4)$$

On the other hand, it is needed to choose W_o as an approximate integrator which is $W_o(s) = 1/(s + 0.01)$ to eliminate steady-state errors.

We obtain the $\mathcal{L}_2$ gains with changing order, n_s which symbolizes the effect of the applied disturbances to the outputs of the system. Table 5.1 shows the obtained values of γ. Here, "FB" symbolizes the case of pure feedback control, whereas "FB+FF" stands for the combined feedback feedforward control scheme. Notice that the feedforward controller significantly reduces the $\mathcal{H}_\infty$ gain of the closed-loop system even for the static case, $n_s = 0$. Besides, as n_s increases, the obtained $\mathcal{H}_\infty$ gain tends to decrease in some extend where it converges to a limit beyond $n_s \geq 2$. The Matlab code and Simulink file of the example is also provided in Appendix A.

Table 5. 1 Minimum allowable γ values for FB+FF and only FB in Example 1.

n_s	0	1	2
FB	27.5	14.6	14.55
FB+FF	1.82	0.76	0.71

As a detailed analysis of time-delay effect, Table 5.2 illustrates the change of θ_{u2} which is the upper bound of the time-delay on the control signal with respect to the value of n_s. Here, the value of γ is fixed to 1.82 which is obtained for $n_s = 0$ in Table 5.1. Moreover, the time-delay affecting the state is also fixed to 0.4 and the upper bound of θ_{u2} is increased as much as possible by the help of the increase in the value of n_s. Table 5.2 shows that increase of n_s also results with a rise in the upper bound of delays affecting the system. It is also obvious that similar to γ, the value of θ_{u2} converges to a certain value for the higher values of n_s.

Table 5. 2 The obtained values of θ_{u2} with respect to n_s.

n_s	0	1	2
θ_{u2}	0.15	0.18	0.19

Example 5.2

To demonstrate effectiveness of the proposed method, the proposed controller is applied to the active suspension system model which has a $0.01\mathrm{sec.}$ delay on the control input. Here, we focus on minimizing the $\mathcal{L}_2$ gain from disturbances w to performance outputs, z [48].

Figure 5. 5 Active suspension system experimental model.

Detailed information about the active suspension system is given in the beginning of Appendix B. Let the state space representation of active suspension system be defined as

$$\begin{aligned} \dot{x} &= Ax(t) + B_u u(t-\theta_1) + B_w w(t) \\ z &= Cx(t) + D_u u(t-\theta_1) \end{aligned} \quad (5.5)$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_s/m_s & k_s/m_s & -c_s/m_s & c_s/m_s \\ k_s/m_u & -(k_s+k_t)/m_u & c_s/m_u & -c_s/m_u \end{pmatrix}$$
$$B_u^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 1/m_s & -1/m_u \end{pmatrix} * 74.4$$
$$B_w^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & k_t/m_u \end{pmatrix}. \quad (5.6)$$

Here, c_s stands for the suspension damping ratio, k_t is the wheel spring constant, k_s is the spring constant of the active suspension system, m_s is the sprung mass and m_u is the unsprung mass. The parameters used in the system are given as follows; $k_t = 100000[\mathrm{N/m}]$, $k_s = 8800[\mathrm{N/m}]$, $c_s = 10[\mathrm{Ns/m}]$, $m_s = 20.4[\mathrm{kg}]$ and

$m_u = 15.9[\mathrm{kg}]$. With the given values of system parameters, the numerical values of active suspension system matrices can be obtained as

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -431.32 & 431.32 & -0.49 & 0.49 \\ 553.4 & -6842.7 & 0.63 & -0.63 \end{pmatrix}$$
$$B_u^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 3.647 & -4.68 \end{pmatrix} \qquad B_w^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 6289.3 \end{pmatrix}$$
$$C = \begin{pmatrix} -431.32 & 431.32 & -0.49 & 0.49 \end{pmatrix} \qquad D_u^{\mathrm{T}} = \begin{pmatrix} 3.647 & 0 & 0 & 0 \end{pmatrix}. \quad (5.7)$$

Here $\theta_1 = 0.01$ is the upper bound for the time delay that affects the control signal of the system. The active suspension system is controlled with a stabilizing $\mathcal{H}_\infty$ state feedback controller whose gains are given as $K = \begin{bmatrix} -43.04 & 18.45 & -4.56 & 0.95 \end{bmatrix}$. The calculation of the $\mathcal{H}_\infty$ state feedback controller gains is given in detail in Appendix B.

Since there is only delay operator affecting the control signal in the active suspension system, similar to Example 5.1, $N = diag(V, -V)$ is used for the factorization of the multiplier in controller synthesis. Table 5.3 shows that there is a significant decrease in $\mathcal{L}_2$ gain of the system if FB+FF controller scheme is used instead of only FB controller. It is a fact that the increase of n_s has also an effect in decreasing the $\mathcal{L}_2$ gain of the active suspension system.

Table 5. 3 Minimum allowable γ values for FB+FF and only FB in Example 2.

n_s	0	1	2	3
FB	0.61	0.60	0.60	0.60
FB+FF	0.29	0.28	0.26	0.25

The random road profile data applied to the active suspension system is shown in Figure 5.6. The applied road profile data is the disturbance signal affecting the system.

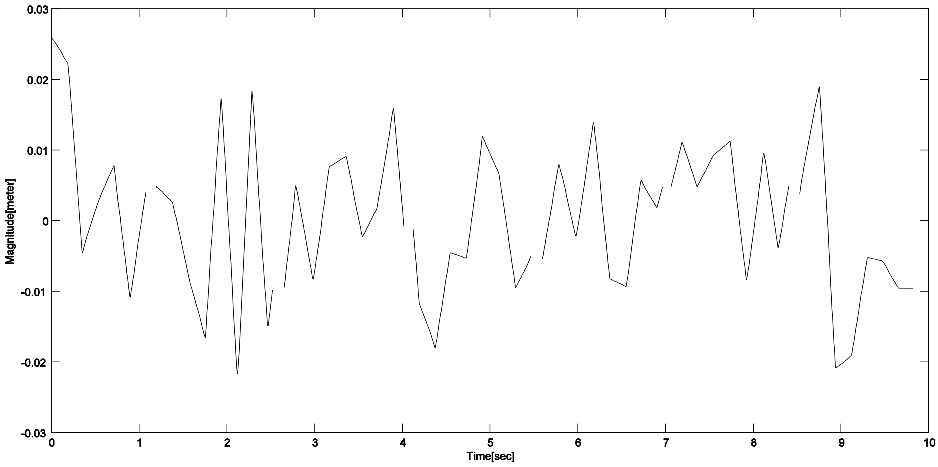

Figure 5. 6 Random road profile.

The Bode magnitude plots of uncontrolled (dotted line), only FB controlled (dashed line) and FF+FB controlled (solid line) systems are illustrated in Figure 5.7. The Bode magnitude plots show that FF+FB controller scheme comes out with improved results especially at low frequencies compared to only FB controller and uncontrolled system. Observe that the proposed controller significantly reduces the disturbance effects at the first mode whose effect is generally most important in vibration control systems.

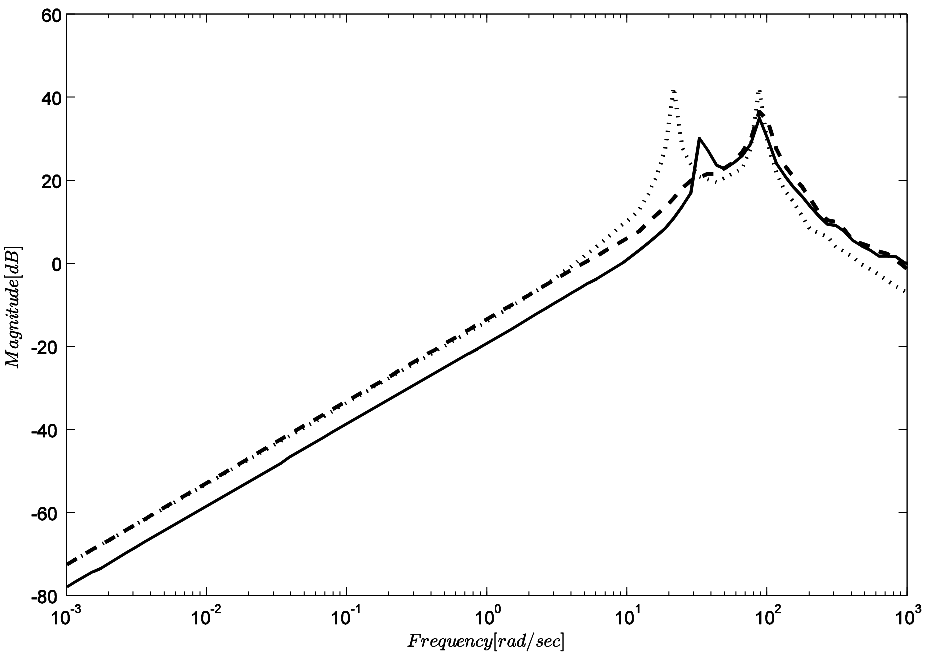

Figure 5. 7 Bode magnitude plots of uncontrolled, only FB controlled and FF+FB controlled systems.

The comparison of FF+FB controller (red line), only FB controller (green line) and open loop system (black line) in terms of variation of $\ddot{x}_s$ which symbolizes the acceleration of the main body of the car has been given in Figure 5.8. It is obvious that the proposed FF+FB controller successfully attenuates the peak values in $\ddot{x}_s$ which is directly related to the passenger ride comfort in the vehicle.

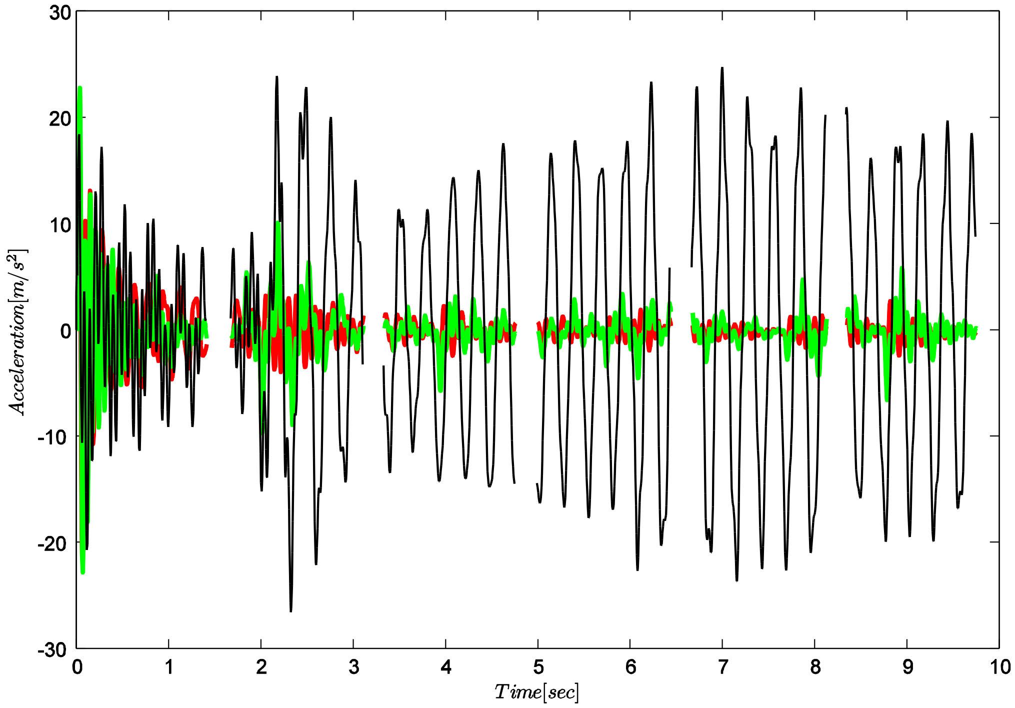

Figure 5. 8 Acceleration of the main body.

Example 5.3

The ship-steering problem which is given in [14] and [15] is also analyzed in this study.

The equation of motion is given as $\ddot{\psi}(t) = av(t)\dot{\psi}(t) + bv(t)^2\delta(t-\theta)$ where $\delta(t)$ is the rudder angle which is also the control signal for the system. There is also an uncertainty in the system such as $v(t) = v_0 + \vartheta\cos(\omega_0 t)$. The nominal system is controlled with a classical PD controller. Different from the example in [14] and [15] where the feed-forward controller synthesis is given for the ship-steering problem, there is time-delay θ affecting the control signal in the system. θ_u stands for the upper bound of the time-delay.

The γ values obtained for the system without time-delay in [14] is given in Table 5.4.

Table 5. 4 The obtained values of γ for ship steering problem without time-delay.

n_s	0	1	2
FB	42.0	30.2	9.1
FB+FF	0.87	0.83	0.74

As we deal with the effect of time-delay over the system, the aim is to make the upper bound of the time-delay affecting the control signal as much as possible by the help of the proposed feedforward controller involving the time-delay multiplier. When the proposed controller is applied to the system, we obtain the results given in Table 5.5.

Table 5. 5 The obtained values of θ_u for ship steering problem with time-delay.

$\gamma\backslash n_s$	0	1	2
0.87	0	0.02	0.08
0.83	Infeasible	0	0.06
0.74	Infeasible	Infeasible	0

The θ_u values given in Table 5.5 show us that by the help of proposed controller there is a significant increase in the value of time-delay upper bound with respect to n_s. The values of γ obtained in [14] and [15] are used with respect to n_s in Table 5.5. It is obvious that we have the value of 0 as the upper bound of the delay for $\gamma\backslash n_s = (0.87, 0), (0.83, 1)\ and\ (0.74, 2)$. This result actually makes sense since this is the same structure that is obtained in [14] and [15] where the time-delay is not taken in account. On the other hand, for the other values of $\gamma\backslash n_s$, as the time-delay multiplier is used for the design of the proposed feedforward controller, different from [14] and [15], the upper bound of the time-delay increases as higher values are used for the degree of the multiplier n_s.

It is also important that different from the previous examples, the ship-steering problem is a reference tracking problem which shows that the proposed controller can be used for reference tracking as well as disturbance attenuation.

5.2 Case 2: Time Delay and Parametric Uncertainty

Example 5.4

Consider the following linear time-delay system:

$$\begin{aligned} \dot{x} &= (A+\Delta A)x(t) + (A_d + \Delta A_d)x(t-\theta_1) + Bu(t) + B_w w(t) \\ z &= Cx(t) + D_w w(t) \end{aligned} \tag{5.8}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_d = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \end{bmatrix},\ D_w = \begin{bmatrix} 0 \end{bmatrix} \tag{5.9}$$

Here $\theta_1 = 0.9710$ is the upper bound for the time delay that affects the states of the system. On the other hand, assume $\|\Delta A_d\| = \|\Delta A\| \leq 0.2$. The system is stabilized and controlled with a $\mathcal{H}_\infty$ state feedback controller whose gains are chosen as $K = \begin{bmatrix} -0.1202 & -134.5322 \end{bmatrix}$.

Since there is also parametric uncertainty in addition to the time delay operator, the multiplier used in feedforward controller synthesis is different from the one used in Example 5.1. In this example, multiplier is in the form of $\Theta(j\omega) = \phi(j\omega)N\phi^*(j\omega)$ where

$$N = \begin{bmatrix} M & 0 & 0 & 0 \\ 0 & P & 0 & R \\ 0 & 0 & -M & 0 \\ 0 & R^T & 0 & -P \end{bmatrix}. \tag{5.10}$$

The rest of the parameters are chosen same as in Example 5.1.

Table 5. 6 Minimum allowable γ values for FB+FF and only FB in Example 4.

n_s	0	1	2
FB	0.0071	0.0069	0.0069
FB+FF	0.0020	0.0005	0.0005

The values of $\mathcal{L}_2$ gains are given in the Table 5.6. It is obvious that there is significant reduction of $\mathcal{L}_2$ gain by using the proposed feedforward scheme. The values given in Table 5.6 show that there is an obvious improvement in $\mathcal{L}_2$ gain of the system by means of using dynamic multipliers instead of static ones. When the obtained γ values are compared to the results in [7], it is a fact that by increasing the degree of the multiplier, better results than the ones in [7] have been obtained. The value of γ for $\theta_1 = 0.9710$ is obtained as 0.0022 in [7] where as 0.0005 is obtained as the minimum value for γ with the proposed theory in the thesis.

Table 5. 7 The obtained values of θ_1 with respect to n_s.

n_s	0	1	2
θ_1	0.97	23.3	23.4

Similar to the previous example, Table 5.7 shows that as we increase the degree of the multiplier n_s the upper bound of the delay affecting the system also reaches to higher values. The value of γ is fixed to 0.0020 which is obtained for $\theta_1 = 0.97$ and $n_s = 0$. Then, the given values in Table 5.7 are obtained for the higher values of n_s. It is obvious that n_s has a significant effect on the upper bound of the delay and it also converges to a certain value.

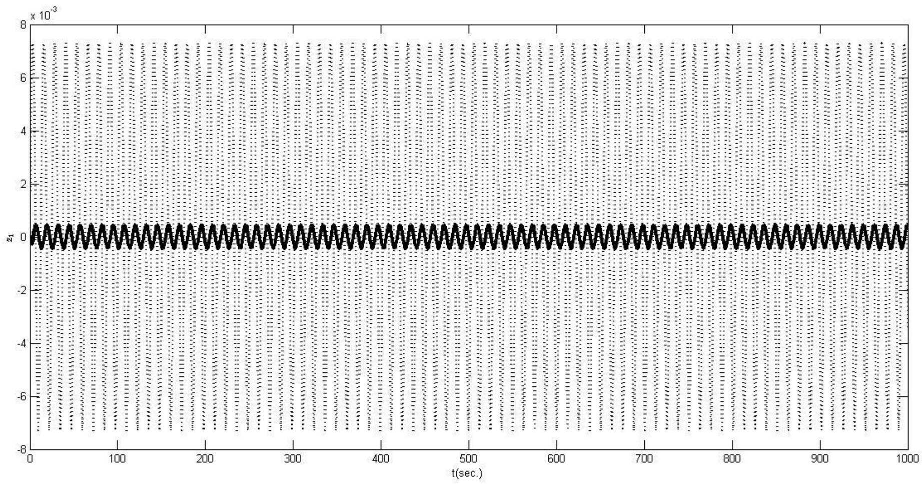

Figure 5. 9 Time-domain response.

The time domain responses for FB and FB+FF configurations are also illustrated in Figure 5.9. During the time-domain simulations, a sinusoidal disturbance signal with a unit amplitude and frequency of 0.5 rad/sec has been applied to the system.

Notice that the proposed scheme provides remarkable disturbance attenuation performance improvements when compared with the results of the single state-feedback $\mathcal{H}_\infty$ controller. The time-domain results show that the proposed scheme has a significant improvement on the reduction of the effect of the disturbance to z.

5.3 Case 3: Neutral Systems

Example 5.5

Consider the following linear time-delay system:

$$\begin{aligned} \dot{x} &= Ax(t) + A_h x(t-\theta_1) + A_d \dot{x}(t-\theta_2) + Bu(t) + B_w w(t) \\ z &= Cx(t) + D_w w(t) \end{aligned} \tag{5.11}$$

where

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} A_h = \begin{bmatrix} 0.15 & 0.05 \\ 0 & 0.1 \end{bmatrix} A_d = \begin{bmatrix} -0.05 & 0.02 \\ 0.01 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} B_w = \begin{bmatrix} -0.01 \\ 0.03 \end{bmatrix} C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} D = \begin{bmatrix} 0.5 & 0.01 \end{bmatrix}. \tag{5.12}$$

Here $\theta_1 = \theta_2 = 196.45$ is the upper bound for the time delay that affects the states and the derivatives of the states of the system. The system has been stabilized and controlled with an $\mathcal{H}_\infty$ state feedback controller whose gains are chosen as $K = \begin{bmatrix} -6.6439 & -5.1930 \end{bmatrix}$.

The values of $\mathcal{L}_2$ gains with respect to n_s are listed in the Table 5.8. Again considerable amount of $\mathcal{H}_\infty$ gain reduction is obtained by use of the proposed scheme. However, for this example, nearly the same amount of disturbance attenuation performances are obtained for static and dynamic cases. This is due to the fact that the values used for θ_1 and θ_2 in numerical example are very tight and very close to the exact maximum allowable delay bound which can be tolerated by the system. As a comparison with [45], the minimum value of γ for the same system with the same value of θ_1 and θ_2, has been calculated as 0.0095 in [45] which is a better result than the results in Table

5.8. This result can be interpreted as the method involving the multiplier for the neutral delay that is used in the thesis is more conservative than the method in [45]. In addition to time-delay analysis, time-delay θ_3 which affects the control signal in the system has been added to the given system and the upper bound of θ_3 is pulled up as much as possible by the help of increasing multiplier degree n_s with fixed γ value which is 0.017. The calculated results for the upper bound of θ_3 are given in Table 5.9.

Table 5. 8 Minimum allowable γ values for FB+FF and only FB in Example 5.

n_s	0	1	2
FB	0.50	0.49	0.49
FB+FF	0.017	0.013	0.012

Table 5. 9 The obtained values of θ_3 with respect to n_s.

n_s	0	1	2
θ_1	0.03	0.27	0.29

The time domain responses for FB and FB+FF configurations are illustrated in Figure 5.10. For time-domain simulations, a sinusoidal disturbance signal with a unit amplitude and frequency of 0.05 rad/sec has been applied to the system. Notice that the proposed scheme provides remarkable disturbance attenuation performance improvements when compared with the results of the single state-feedback $\mathcal{H}_\infty$ controller.

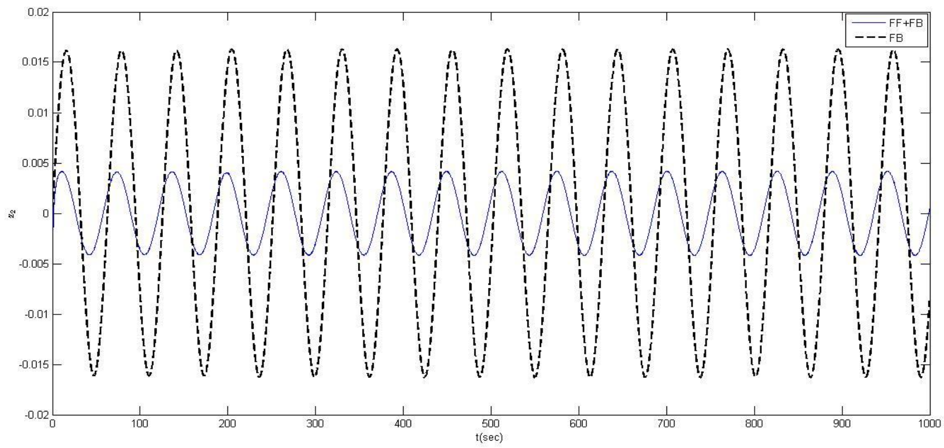

Figure 5. 10 Comparison of FF+FB and only FB in terms of variation of z_2(t).

Example 5.6

Consider the following linear time-delay system

$$\begin{aligned} \dot{x} &= Ax(t) + (A_h + \Delta A_h)x(t-\theta_1) \\ &\quad +(A_d + \Delta A_d)\dot{x}(t-\theta_2) + Bu(t) + B_w w(t) \\ z &= Cx(t) + D_w w(t) \end{aligned} \tag{5.13}$$

where

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} A_h = \begin{bmatrix} 0.15 & 0.05 \\ 0 & 0.1 \end{bmatrix} A_d = \begin{bmatrix} -0.05 & 0.02 \\ 0.01 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} B_w = \begin{bmatrix} -0.01 \\ 0.03 \end{bmatrix} C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} D = \begin{bmatrix} 0.5 & 0.01 \end{bmatrix}. \tag{5.14}$$

Here $\theta_1 = \theta_2 = 196.45$ is the upper bound for the time delay that affects the states and the derivatives of the states of the system. On the other hand, assume $\|\Delta A_h\| = \|\Delta A_d\| \leq 0.2$. The system has been stabilized and controlled with an $\mathcal{H}_\infty$ state feedback controller whose gains are chosen as $K = \begin{bmatrix} -6.6439 & -5.1930 \end{bmatrix}$.

In the third example, we deal with a system that involves both delay operator and parametric uncertainty so the multiplier that we used in the first two examples has changed.

In this example, we have used the multiplier in the form of $\Theta(j\omega) = \phi(j\omega)N\phi^*(j\omega)$ where

$$N = \begin{bmatrix} M & 0 & 0 & 0 \\ 0 & P & 0 & R \\ 0 & 0 & -M & 0 \\ 0 & R^T & 0 & -P \end{bmatrix}$$

and $\phi = diag(S, S)$ satisfy the dual of the IQC dealing with Δ which here stands for delay operator and parametric uncertainty. Here we choose

$$S(s) = \begin{pmatrix} 1 & \frac{1}{s+p} & \cdots & \left(\frac{1}{s+p}\right)^{n_s} \end{pmatrix} \otimes I_2 \quad \text{where} \quad p = 1, \tag{5.15}$$

as it was used in the first two examples.

W_i second order Butterworth low-pass filter and W_o approximate integrator are chosen similar to the ones that are used in the first two examples.

Table 5. 10 Minimum allowable γ values for FB+FF and only FB in Example 6.

n_s	0	1
FB	0.51	0.51
FB+FF	0.05	0.03

The values of $\mathcal{L}_2$ gains with respect to n_s are listed in the Table 5.10. One more time, considerable amount of $\mathcal{H}_\infty$ gain reduction is obtained by use of the proposed scheme. Here we can only obtain the result for multiplier degrees $n_s = 0$ and $n_s = 1$ due to the high computational loads increasing proportionally with n_s. However, it is obvious that the value of $\mathcal{L}_2$ gains converges to a limit.

Another important point in this example is that we have consistent results when the numerical results are compared with the results obtained in the second example. There is an additional parametric uncertainty in the third example compared to the second one. All the system matrices and gains of the state feedback controller are the same so it is expected that the $\mathcal{L}_2$ gain in the third example should be higher than the second example because of the additional parametric uncertainty which is supported by Example 5.3 and Example 5.4.

The detailed Matlab code and necessary Simulink files of the example are also given in Appendix A.

SECTION 6

CONCLUSION

Convex solutions to the robust H_∞ feedforward control problems for time-delayed uncertain systems have been given in IQC framework by the help of dynamic multipliers in this thesis. Employing two independent control loops; state feedback controller for stabilization of nominal system and dynamic feedforward controller for disturbance attenuation, we have given a control scheme for state, control time-delayed and neutral uncertain systems. The main advantage of using dynamic IQCs for controller design is to reduce the conservatism of the results as much as possible by the help of adjusting the multiplier degree.

In Section 1.2, it has been mentioned that one of the most important issues that should be taken care of, is to reduce the conservatism. To be able to do that, two mathematical tools have been used. One of them is dynamic multipliers and the other one is the degree of the multiplier used in the design of feedforward controller. As it has been given in the tables and the frequency and time-domain plots of the simulations, it is obvious that increasing the degree of the multiplier has a significant effect on the reduction of conservatism. It is clear that there is always a limit for the degree of the multiplier because it is directly related with the dimensions of the matrices in the LMIs. Hence, as long as we increase the degree of the multiplier, we also increase the computational work that is needed to be done to design the feedforward controller which is also the main drawback of the method.

We also have the purpose of dealing with more than one uncertainty/nonlinearity by the design of a feedforward controller via dynamic IQCs. The simulations and the tables have demonstrated that the proposed controller could deal with both time-delay and parametric uncertainty affecting the system and it reduces the effect of time-delay and parametric uncertainty over γ. The ability to combine more than one IQC and to represent all of the IQCs in one block of IQC helps to achieve the purpose that we mentioned above.

Examples involving neutral systems are also given in the simulation part of the thesis. Neutral systems as mentioned in previous sections are one of the most diffucult types of time-delay systems. The delay operator in the state dynamics(derivative of the states) decreases the performance of the controllers. The proposed feedforward controller deals with time-delay in the operator point of view so the difficulty that can happen in the neutral systems doesn't make much difference for the dynamic IQCs because it can be represented in a similar way to other types of time-delay systems. These are the reasons of a good controller performance also in neutral systems.

In conclusion, numerical benchmark problems selected from the literature, are used to illustrate the efficiency of the controller design approach studied. The simulation results both in time-domain and frequency-domain proved that the simulations of the proposed control methodology involving feedforward controller resulted with better performance compared to the results with the controllers having only state feedback.

As the drawbacks of the thesis, IQCs that are used for the controller synthesis are not the least conservative ones so as far as they are suitable for the duality conditions of the controller design some less conservative multipliers such as the ones in Safonov's paper can be used. It is also a fact that to be able to apply H_∞ static state feedback gains to the system, the states of the system needs to be measurable which is not always possible for the application so observer design can be needed in such conditions in practice. Since we deal with disturbance attenuation problem and design of a feedforward controller we assume that the disturbance is known or can be observed but there are some applications where this assumption can not be accepted. As it is assumed that the disturbance is known or can be observed it should be considered that there is the possibility of delay existence in the disturbance. Thus, the delay affecting the disturbance signal also needs to be included in Δ which is trivial and similar to the methods that are used in the thesis. The lower bounds of the time-delays are assumed as zero during the studies in the thesis. Some future study can be carried out where the lower bound of the delay is different from zero. As an additional future work, controller synthesis for the systems involving time-varying delays can be studied.

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APPENDIX-A

MATLAB SIMULATION M-FILES

Example 5.1 State and Control Delay

```
clear
clc
s=tf('s');
%Input values;ns,ps,qs,ws,zs,us,ys

%System matrices of the nominal system
Amodel=[0 0;
        0 1];
Admodel=[-1 -1;
         0 -0.9];
Bumodel=[0;
         1];
Bwmodel=[1;
         1];
Cmodel=[0 1];
Cdmodel=[0 0];
Dumodel=0.1;
Dwmodel=0;

[aa,bb]=size(Amodel);
[cc,dd]=size(Bumodel);
[ee,ff]=size(Bwmodel);
[gg,hh]=size(Cmodel);
[kk,ll]=size(Cdmodel);
[mm,nn]=size(Dumodel);

Asys=Amodel+Admodel;
Bsys=[Admodel Bumodel Bwmodel];
```

```
Csys=[eye(aa);
        Cmodel;
        Cdmodel;
        zeros(1,aa)];
Dsys=[zeros(aa,aa) zeros(aa,dd) zeros(aa,ff);
        zeros(gg,aa) zeros(gg,dd) Dwmodel;
        Cdmodel     zeros(kk,dd) zeros(kk,ff);
        zeros(mm,aa) Dumodel     zeros(mm,ff)];

%Size of input-output vectors
ps=3;
qs=3;
ws=1;
zs=3;
us=1;
ys=1;
gamma=0.77 %sdpvar(1);
dgamma=.01;

theta=0.4; %upper bound of state delay
AAA=[sqrt(0.08)*theta^2 theta 0];
BBB=[sqrt(0.02)*theta^2 0.642*theta 1];

theta2=0.15; %upper bound of control delay

CCC=[sqrt(0.08)*theta2^2 theta2 0];
DDD=[sqrt(0.02)*theta2^2 0.642*theta2 1];

%By the help of the simulink file we have the state-space matrices of generalized plant
[Agen,Bgen,Cgen,Dgen]=linmod('iqc_control_state_time_delay');

sys=ss(Agen,Bgen,Cgen,Dgen);
sys1=balreal(sys);
sys2=minreal(sys1);
Agen=sys2.a;
Bgen=sys2.b;
Cgen=sys2.c;
Dgen=sys2.d;

[t,u]=size(Agen);

%Constants for the basis of the multiplier
U=tf('s');
ns=1;
for n=0:1:ns
  U(1,n+1)=((s-1)/(s+1))^n;
```

```
end
[A,B,C,D]=ssdata(U);
As=kron(A,eye(ps));
Bs=kron(B,eye(ps));
Cs=kron(C,eye(ps));
Ds=kron(D,eye(ps));

[a,b]=size(As);
[c,d]=size(Bs);
[e,f]=size(Cs);
[g,h]=size(Ds);

%System variables
while true
Bp=Bgen(:,1:ps);

Dqp=Dgen(1:qs,1:ps);

Dzp=Dgen((qs+1):(qs+zs),1:ps);

B2=[zeros(size(Bs)) Bs];
[l,m]=size(B2);

D1=[Ds zeros(size(Ds))];

D2=[zeros(size(Ds)) Ds];
[v,y]=size(D2);

A22=As;

Cq=Cgen(1:qs,:);
Cz=Cgen((qs+1):(qs+zs),:);
C22=Cs;

AA=[As zeros(a,b) zeros(a,t);
   zeros(a,b) A22 zeros(a,t);
   zeros(t,b) -(Bp*C22) Agen];

Bphi=[Bs             zeros(size(Bs));
     zeros(size(Bs)) Bs;
     -(Bp*D2)];

[i,k]=size(Bphi);

CC=[Cs         -(Dqp*C22) Cq;
       zeros(zs,f) -(Dzp*C22) Cz];
```

```
Dphi=[D1-(Dqp*D2);
        -(Dzp*D2)];

Bw=[zeros(c,ws);zeros(l,ws);Bgen(:,(ps+1):(ps+ws))];
Bu=[zeros(c,us);zeros(l,us);Bgen(:,(ps+ws+1):(ps+ws+us))];

Dqu=Dgen((1:qs),(ps+ws+1):(ps+ws+us));
Dzu=Dgen((qs+1):(qs+zs),(ps+ws+1):(ps+ws+us));
Du=[Dqu;
   Dzu];

Dp=[zeros(qs,zs);-eye(zs)];

Dqw=Dgen((1:qs),(ps+1):(ps+ws));
Dzw=Dgen((qs+1):(qs+zs),(ps+1):(ps+ws));
Dw=[Dqw;
   Dzw];

T=[zeros(a,b) zeros(a,t);
  eye(b) zeros(a,t);
  zeros(t,b) eye(t)];

%Unknown Variables which will be solved by the help of LMIs
Xh=sdpvar(2*b+t);
Y=sdpvar(2*b+t);
X=sdpvar(l);

%Controller Variables

%For Feedback+feedforward;
Ac=sdpvar(i,i,'full');
Bc=sdpvar(i,1, 'full');
Cc=sdpvar(1,i,'full');
Dc=sdpvar(1,1,'full');

%Only for feedback;
%Cc=zeros(1,i);
%Dc=zeros(1,1);

V=sdpvar(k/2);
N=blkdiag(V,-V);
[q,r]=size(N);

% Obtaining LMIs
```

```
Phi11=-Xh*AA'-AA*Xh+Bphi*N*Bphi';
Phi12=-Xh*AA'+Ac+Bphi*N*Bphi';
Phi13=-Xh*CC'+Bphi*N*Dphi';
Phi14=-Bw-Bu*Dc-Bc;

Phi22=-Y*AA'-Cc'*Bu'-AA*Y-Bu*Cc+Bphi*N*Bphi';
Phi23=-Y*CC'-Cc'*Du'+Bphi*N*Dphi';
Phi24=-Bw-Bu*Dc;

Phi33=Dphi*N*Dphi'+Dp*gamma*Dp';
Phi34=-Dw-Du*Dc;

Phi44=gamma;

Phi=[Phi11 Phi12 Phi13 Phi14;
   Phi12' Phi22 Phi23 Phi24;
   Phi13' Phi23' Phi33 Phi34;
   Phi14' Phi24' Phi34' Phi44];

XXN=blkdiag([zeros(l) X;X zeros(l)],N);

%LMI Solution

F=set(Phi>0);
F=F+set(((T'*Xh*T)-[X zeros(l,t);zeros(t,l) zeros(t)])>0);
F=F+set((Y-Xh)>0);
F=F+set([-(A22') -(C22');eye(l) zeros(l,v);
        -(B2') -(D2')]'*XXN*[-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')]<0);
sol=solvesdp(F,'', sdpsettings('solver','sedumi','verbose',0))
eig(real(double(Phi)));
eig(real(double((T'*Xh*T)-[X zeros(l,t);zeros(t,l) zeros(t)])));
eig(real(double(Y-Xh)));
eig(real(double([-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')]'*XXN*...
                   [-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')])));

if min(eig(real(double(Phi))))<0
break
else gamma=gamma-dgamma
end
end
gamma=gamma+dgamma

% Calculation of the Controller State-Space Matrices with the help of Optimal Gamma
Bp=Bgen(:,1:ps);

Dqp=Dgen(1:qs,1:ps);
```

```
Dzp=Dgen((qs+1):(qs+zs),1:ps);

B2=[zeros(size(Bs)) Bs];
[l,m]=size(B2);

D1=[Ds zeros(size(Ds))];

D2=[zeros(size(Ds)) Ds];
[v,y]=size(D2);

A22=As;

Cq=Cgen(1:qs,:);
Cz=Cgen((qs+1):(qs+zs),:);
C22=Cs;

AA=[As zeros(a,b) zeros(a,t);
      zeros(a,b) A22 zeros(a,t);
      zeros(t,b) -(Bp*C22) Agen];

Bphi=[Bs          zeros(size(Bs));
        zeros(size(Bs)) Bs;
       -(Bp*D2)];
[i,k]=size(Bphi);

CC=[Cs      -(Dqp*C22) Cq;
   zeros(zs,f) -(Dzp*C22) Cz];

Dphi=[D1-(Dqp*D2);
     -(Dzp*D2)];

Bw=[zeros(c,ws);zeros(l,ws);Bgen(:,(ps+1):(ps+ws))];
Bu=[zeros(c,us);zeros(l,us);Bgen(:,(ps+ws+1):(ps+ws+us))];

Dqu=Dgen((1:qs),(ps+ws+1):(ps+ws+us));
Dzu=Dgen((qs+1):(qs+zs),(ps+ws+1):(ps+ws+us));
Du=[Dqu;
   Dzu];

Dp=[zeros(qs,zs);-eye(zs)];

Dqw=Dgen((1:qs),(ps+1):(ps+ws));
Dzw=Dgen((qs+1):(qs+zs),(ps+1):(ps+ws));
Dw=[Dqw;
   Dzw];
```

```
T=[zeros(a,b) zeros(a,t);
   eye(b) zeros(a,t);
   zeros(t,b) eye(t)];

%Unknown Variables
Xh=sdpvar(2*b+t);
Y=sdpvar(2*b+t);
X=sdpvar(l);

%Controller Variables
Ac=sdpvar(i,i,'full');
Bc=sdpvar(i,1, 'full');
Cc=sdpvar(1,i, 'full');
Dc=sdpvar(1,1, 'full');


V=sdpvar(k/2);
N=blkdiag(V,-V);
[q,r]=size(N);

%Obtaining LMIs

Phi11=-Xh*AA'-AA*Xh+Bphi*N*Bphi';
Phi12=-Xh*AA'+Ac+Bphi*N*Bphi';
Phi13=-Xh*CC'+Bphi*N*Dphi';
Phi14=-Bw-Bu*Dc-Bc;

Phi22=-Y*AA'-Cc'*Bu'-AA*Y-Bu*Cc+Bphi*N*Bphi';
Phi23=-Y*CC'-Cc'*Du'+Bphi*N*Dphi';
Phi24=-Bw-Bu*Dc;

Phi33=Dphi*N*Dphi'+Dp*gamma*Dp';
Phi34=-Dw-Du*Dc;

Phi44=gamma;

Phi=[Phi11  Phi12  Phi13  Phi14;
    Phi12' Phi22  Phi23  Phi24;
    Phi13' Phi23' Phi33  Phi34;
    Phi14' Phi24' Phi34' Phi44];

XXN=blkdiag([zeros(l) X;X zeros(l)],N);

%LMI Solution
```

```
F=set(Phi>0);
F=F+set(((T'*Xh*T)-[X zeros(l,t);zeros(t,l) zeros(t)])>0);
F=F+set((Y-Xh)>0);
F=F+set([-(A22') -(C22');eye(l) zeros(l,v);
       -(B2') -(D2')]'*XXN*[-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')]<0);
sol=solvesdp(F,'', sdpsettings('solver','sedumi','verbose',0))
eig(real(double(Phi)));
eig(real(double((T'*Xh*T)-[X zeros(l,t);zeros(t,l) zeros(t)])));
eig(real(double(Y-Xh)));
eig(real(double([-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')]'*XXN*...
                 [-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')])));

%Controller Synthesis
Dc=Dc;
Cc=Cc;

Ac=double(Ac);
Bc=double(Bc);
Cc=double(Cc);
Dc=double(Dc);
Xh=double(Xh);
Y=double(Y);
Bc=inv(Xh-Y)*Bc;
Ac=inv(Xh-Y)*(-Ac-AA*Y-Bu*Cc);
```

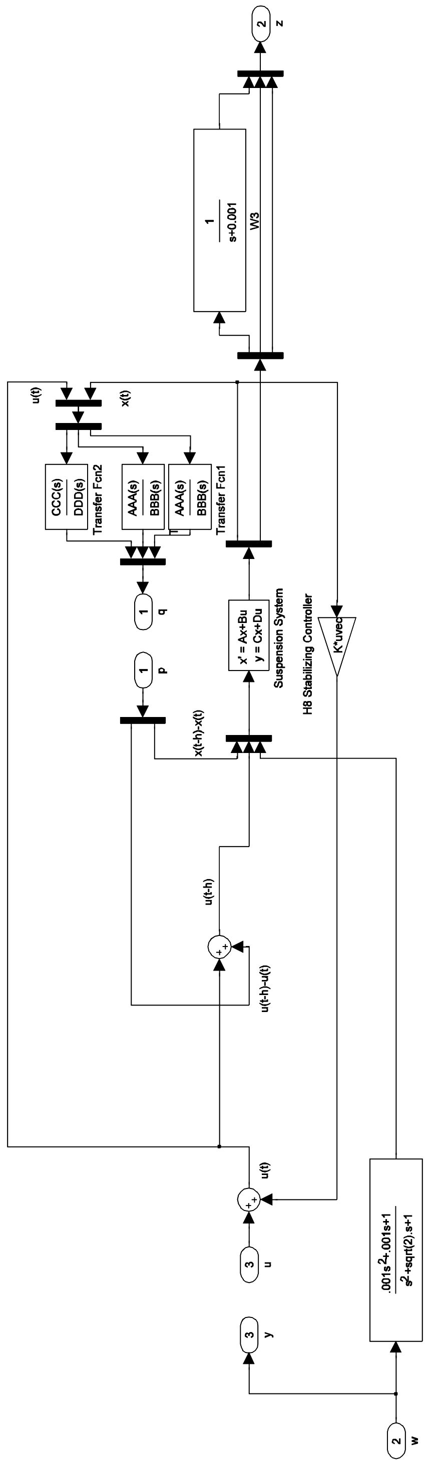

Figure 7. 11 Simulink .mdl file for state and control delay example.

Example 5.6: Neutral System with Parametric Uncertainty

```
clear
clc
s=tf('s');

%Necessary input values;ns,ps,qs,ws,zs,us,ys
%Size of input-output vectors
ps=9;
qs=9;
ws=1;
zs=1;
us=1;
ys=1;
gamma=.018 %sdpvar(1);
dgamma=.001;
par=0; %parametric uncertainty
theta=196.45; %upper bound of state delay
AAA=[sqrt(0.08)*theta^2 theta 0];
BBB=[sqrt(0.02)*theta^2 0.642*theta 1];

theta2=0; %upper bound of control delay

CCC=[sqrt(0.08)*theta2^2 theta2 0];
DDD=[sqrt(0.02)*theta2^2 0.642*theta2 1];

theta3=196.45; %upper bound of state dot delay(neutral delay)

EEE=[sqrt(0.08)*theta3^2 theta3 0];
FFF=[sqrt(0.02)*theta3^2 0.642*theta3 1];


[Agen,Bgen,Cgen,Dgen]=linmod('iqc_neutral_state_time_delay_par');

sys=ss(Agen,Bgen,Cgen,Dgen);
sys1=balreal(sys);
sys2=minreal(sys1);
Agen=sys2.a;
Bgen=sys2.b;
Cgen=sys2.c;
Dgen=sys2.d;

[t,u]=size(Agen);

%Constants for the basis of the multiplier
```

```
U=tf('s');
ns=0;
for n=0:1:ns
   U(1,n+1)=((s-1)/(s+1))^n;
end
[A,B,C,D]=ssdata(U);
As=kron(A,eye(ps));
Bs=kron(B,eye(ps));
Cs=kron(C,eye(ps));
Ds=kron(D,eye(ps));

[a,b]=size(As);
[c,d]=size(Bs);
[e,f]=size(Cs);
[g,h]=size(Ds);

%System matrices
while true
Bp=Bgen(:,1:ps);

Dqp=Dgen(1:qs,1:ps);

Dzp=Dgen((qs+1):(qs+zs),1:ps);

B2=[zeros(size(Bs)) Bs];
[l,m]=size(B2);

D1=[Ds zeros(size(Ds))];

D2=[zeros(size(Ds)) Ds];
[v,y]=size(D2);

A22=As;

Cq=Cgen(1:qs,:);
Cz=Cgen((qs+1):(qs+zs),:);
C22=Cs;

AA=[As zeros(a,b) zeros(a,t);
   zeros(a,b) A22 zeros(a,t);
   zeros(t,b) -(Bp*C22) Agen];

Bphi=[Bs           zeros(size(Bs));
         zeros(size(Bs)) Bs;
         -(Bp*D2)];
```

```
[i,k]=size(Bphi);


CC=[Cs         -(Dqp*C22) Cq;
        zeros(zs,f) -(Dzp*C22) Cz];

Dphi=[D1-(Dqp*D2);
          -(Dzp*D2)];

Bw=[zeros(c,ws);zeros(l,ws);Bgen(:,(ps+1):(ps+ws))];
Bu=[zeros(c,us);zeros(l,us);Bgen(:,(ps+ws+1):(ps+ws+us))];

Dqu=Dgen((1:qs),(ps+ws+1):(ps+ws+us));
Dzu=Dgen((qs+1):(qs+zs),(ps+ws+1):(ps+ws+us));
Du=[Dqu;
        Dzu];

Dp=[zeros(qs,zs);-eye(zs)];

Dqw=Dgen((1:qs),(ps+1):(ps+ws));
Dzw=Dgen((qs+1):(qs+zs),(ps+1):(ps+ws));
Dw=[Dqw;
         Dzw];

T=[zeros(a,b) zeros(a,t);
   eye(b) zeros(a,t);
   zeros(t,b) eye(t)];

%Unknown Variables which will be solved with thehelp of LMIs
Xh=sdpvar(2*b+t);
Y=sdpvar(2*b+t);
X=sdpvar(l);

%Controller state space matrices

%For Feedback+feedforward;
Ac=sdpvar(i,i,'full');
Bc=sdpvar(i,1, 'full');
Cc=sdpvar(1,i,'full');
Dc=sdpvar(1,1,'full');

%For only feedback;
%Cc=zeros(1,i);
%Dc=zeros(1,1);

V=sdpvar(5*k/18);
```

```
P=sdpvar(4*k/18);
R=sdpvar(4*k/18);


N=[V                zeros(5*k/18,4*k/18)     zeros(5*k/18,5*k/18)    zeros(5*k/18,4*k/18);
   zeros(4*k/18,5*k/18) P                   zeros(4*k/18,5*k/18)    R;
   zeros(5*k/18,5*k/18) zeros(5*k/18,4*k/18)   -V                  zeros(5*k/18,4*k/18);
   zeros(4*k/18,5*k/18) R'                  zeros(4*k/18,5*k/18)    -P];
[q,r]=size(N);

%Obtaining LMIs

Phi11=-Xh*AA'-AA*Xh+Bphi*N*Bphi';
Phi12=-Xh*AA'+Ac+Bphi*N*Bphi';
Phi13=-Xh*CC'+Bphi*N*Dphi';
Phi14=-Bw-Bu*Dc-Bc;

Phi22=-Y*AA'-Cc'*Bu'-AA*Y-Bu*Cc+Bphi*N*Bphi';
Phi23=-Y*CC'-Cc'*Du'+Bphi*N*Dphi';
Phi24=-Bw-Bu*Dc;

Phi33=Dphi*N*Dphi'+Dp*gamma*Dp';
Phi34=-Dw-Du*Dc;

Phi44=gamma;

Phi=[Phi11 Phi12 Phi13 Phi14;
     Phi12' Phi22 Phi23 Phi24;
     Phi13' Phi23' Phi33 Phi34;
     Phi14' Phi24' Phi34' Phi44];

XXN=blkdiag([zeros(l) X;X zeros(l)],N);

%LMI Solution
F=set(P>0);
F=F+set(Phi>0);
F=F+set(((T'*Xh*T)-[X zeros(l,t);zeros(t,l) zeros(t)])>0);
F=F+set((Y-Xh)>0);
F=F+set([-(A22') -(C22');eye(l) zeros(l,v);
     -(B2') -(D2')]'*XXN*[-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')]<0);
sol=solvesdp(F,'', sdpsettings('solver','sedumi','verbose',0))
eig(real(double(Phi)));
eig(real(double((T'*Xh*T)-[X zeros(l,t);zeros(t,l) zeros(t)])));
eig(real(double(Y-Xh)));
eig(real(double([-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')]'*XXN*...
            [-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')])));
```

```
if min(eig(real(double(Phi))))<0
break
else gamma=gamma-dgamma
end
end
gamma=gamma+dgamma

%Calculation of the Controller State-Space Matrices with the help of Optimal Gamma

Bp=Bgen(:,1:ps);

Dqp=Dgen(1:qs,1:ps);

Dzp=Dgen((qs+1):(qs+zs),1:ps);

B2=[zeros(size(Bs)) Bs];
[l,m]=size(B2);

D1=[Ds zeros(size(Ds))];

D2=[zeros(size(Ds)) Ds];
[v,y]=size(D2);

A22=As;

Cq=Cgen(1:qs,:);
Cz=Cgen((qs+1):(qs+zs),:);
C22=Cs;

AA=[As zeros(a,b) zeros(a,t);
   zeros(a,b) A22 zeros(a,t);
   zeros(t,b) -(Bp*C22) Agen];

Bphi=[Bs          zeros(size(Bs));
     zeros(size(Bs)) Bs;
     -(Bp*D2)];
[i,k]=size(Bphi);

CC=[Cs       -(Dqp*C22) Cq;
   zeros(zs,f) -(Dzp*C22) Cz];

Dphi=[D1-(Dqp*D2);
     -(Dzp*D2)];

Bw=[zeros(c,ws);zeros(l,ws);Bgen(:,(ps+1):(ps+ws))];
```

```
Bu=[zeros(c,us);zeros(l,us);Bgen(:,(ps+ws+1):(ps+ws+us))];

Dqu=Dgen((1:qs),(ps+ws+1):(ps+ws+us));
Dzu=Dgen((qs+1):(qs+zs),(ps+ws+1):(ps+ws+us));
Du=[Dqu;
   Dzu];

Dp=[zeros(qs,zs);-eye(zs)];

Dqw=Dgen((1:qs),(ps+1):(ps+ws));
Dzw=Dgen((qs+1):(qs+zs),(ps+1):(ps+ws));
Dw=[Dqw;
   Dzw];

T=[zeros(a,b) zeros(a,t);
  eye(b) zeros(a,t);
  zeros(t,b) eye(t)];

%Defining Unknown Variables
Xh=sdpvar(2*b+t);
Y=sdpvar(2*b+t);
X=sdpvar(l);

%Defining Controller State Space Matrices
Ac=sdpvar(i,i, 'full');
Bc=sdpvar(i,1, 'full');
Cc=sdpvar(1,i, 'full');
Dc=sdpvar(1,1, 'full');

V=sdpvar(5*k/18);
P=sdpvar(4*k/18);
R=sdpvar(4*k/18);

N=[V                 zeros(5*k/18,4*k/18)   zeros(5*k/18,5*k/18)    zeros(5*k/18,4*k/18);
  zeros(4*k/18,5*k/18) P                    zeros(4*k/18,5*k/18)    R;
  zeros(5*k/18,5*k/18) zeros(5*k/18,4*k/18)   -V                    zeros(5*k/18,4*k/18);
  zeros(4*k/18,5*k/18) R'                   zeros(4*k/18,5*k/18)    -P];
[q,r]=size(N);

%Obtaining LMIs

Phi11=-Xh*AA'-AA*Xh+Bphi*N*Bphi';
Phi12=-Xh*AA'+Ac+Bphi*N*Bphi';
Phi13=-Xh*CC'+Bphi*N*Dphi';
Phi14=-Bw-Bu*Dc-Bc;
```

```
Phi22=-Y*AA'-Cc'*Bu'-AA*Y-Bu*Cc+Bphi*N*Bphi';
Phi23=-Y*CC'-Cc'*Du'+Bphi*N*Dphi';
Phi24=-Bw-Bu*Dc;

Phi33=Dphi*N*Dphi'+Dp*gamma*Dp';
Phi34=-Dw-Du*Dc;

Phi44=gamma;

Phi=[Phi11  Phi12  Phi13  Phi14;
    Phi12' Phi22  Phi23  Phi24;
    Phi13' Phi23' Phi33  Phi34;
    Phi14' Phi24' Phi34' Phi44];

XXN=blkdiag([zeros(l) X;X zeros(l)],N);

%LMI Solution
F=set(P>0);
F=F+set(Phi>0);
F=F+set(((T'*Xh*T)-[X zeros(l,t);zeros(t,l) zeros(t)])>0);
F=F+set((Y-Xh)>0);
F=F+set([-(A22') -(C22');eye(l) zeros(l,v);
       -(B2') -(D2')]'*XXN*[-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')]<0);
sol=solvesdp(F,'', sdpsettings('solver','sedumi','verbose',0))
eig(real(double(Phi)));
eig(real(double((T'*Xh*T)-[X zeros(l,t);zeros(t,l) zeros(t)])));
eig(real(double(Y-Xh)));
eig(real(double([-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')]'*XXN*...
                  [-(A22') -(C22');eye(l) zeros(l,v);-(B2') -(D2')])));

%Controller Synthesis
Dc=Dc;
Cc=Cc;

Ac=double(Ac);
Bc=double(Bc);
Cc=double(Cc);
Dc=double(Dc);
Xh=double(Xh);
Y=double(Y);
Bc=inv(Xh-Y)*Bc;
Ac=inv(Xh-Y)*(-Ac-AA*Y-Bu*Cc);
```

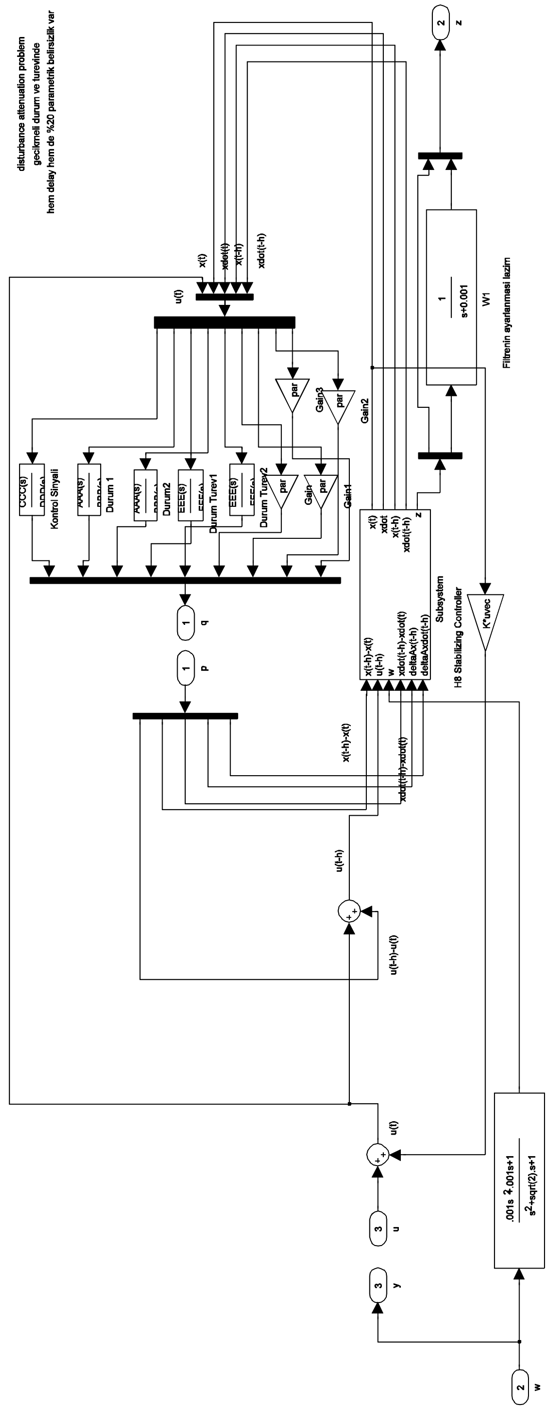

Figure 7. 12 Simulink .mdl file for neutral system example.

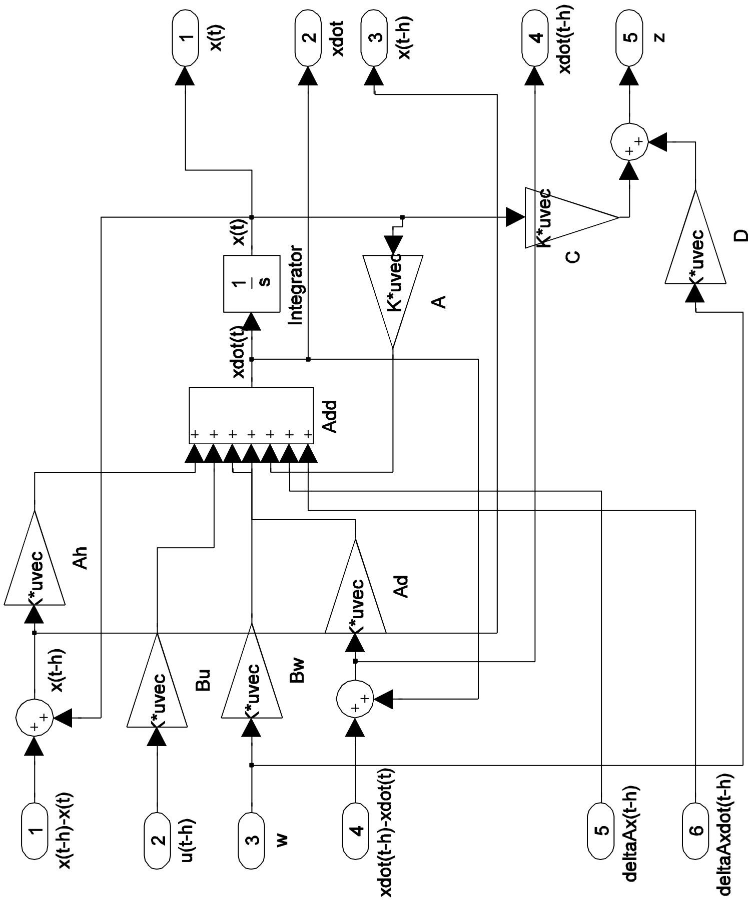

Figure 7. 13 Simulink nominal system .mdl file for neutral system example.

APPENDIX-B

H_∞ STATE FEEDBACK CONTROLLER FOR ACTIVE SUSPENSION SYSTEM

Active suspension system is introduced to illustrate the effectiveness of the proposed feedforward controller in "Simulation Results" part of the thesis. As it is explained in the previous sections, there are two different controllers; feedback and feedforward controller in the proposed controller scheme. Active suspension system problem and theoretical background of the state-feedback controller used in the active suspension system is introduced in Appendix B.

Performance requirements for advanced vehicle suspensions include isolating passengers from vibration and shock arising from road roughness (ride comfort), suppressing the hop of the wheels so as to maintain firm, uninterrupted contact of wheels to road (good handling or good road holding) and keeping suspension strokes within an allowable maximum. In fact, the active suspension control problem can be formulated as a constrained disturbance attenuation problem. To quantify ride comfort, the body acceleration is chosen as controlled performance output. Here, there is an additional control delay to illustrate the efficiency of the time-delay multiplier used in the thesis.

The theoretical background used to design the state-feedback controller is given in the following part.

Let us consider the system

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_w w \qquad u = Kx \\ z &= Cx + D_w w. \end{aligned} \tag{8.1}$$

Closed loop system is given as

$$\begin{aligned} \dot{x} &= A_{cl}x + B_w w \\ z &= Cx + D_w w, \end{aligned} \tag{8.2}$$

where $A_{cl} = A + BK$. Let us choose a Lyapunov function of the form

$$V(x(t)) = x^T(t)Px(t), \tag{8.3}$$

and assume that

$$\dot{V}(x(t)) + z^T z - \gamma^2 w^T w \leq 0 \quad where \quad \gamma > 0. \tag{8.4}$$

Taking the time-derivative of (8.3) along the system trajectory (8.1) lets us write

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A_{cl}^T P + PA_{cl} + C^T C & C^T D_w + PB_w \\ * & {D_w}^T D_w - \gamma^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \leq 0. \tag{8.5}$$

Pre and post multiply

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} + C^T C & C^T D_w + PB_w \\ * & {D_w}^T D_w - \gamma^2 I \end{bmatrix} \tag{8.6}$$

by $diag\{P^{-1}, I\}$ gives

$$\begin{bmatrix} P^{-1}A_{cl}^T + A_{cl}P^{-1} + P^{-1}C^T C P^{-1} & P^{-1}C^T D_w + B_w \\ * & {D_w}^T D_w - \gamma^2 I \end{bmatrix} \leq 0. \tag{8.7}$$

We know that $A_{cl} = A + BK$ so we have

$$\begin{bmatrix} P^{-1}(A+BK)^T + (A+BK)P^{-1} & P^{-1}C^T D_w + B_w & P^{-1}C^T \\ * & {D_w}^T D_w - \gamma^2 I & 0 \\ * & * & -I \end{bmatrix} \leq 0. \tag{8.8}$$

By the help of the definitions $P^{-1} \triangleq Y$, $KY \triangleq L$ and given $\gamma > 0$ we have the following optimization problem

$$\min_{Y,L} \gamma$$

$$\begin{bmatrix} AY + BL + YA^T + L^T B^T & YC^T D_w + B_w & YC^T \\ * & {D_w}^T D_w - \gamma^2 I & 0 \\ * & * & -I \end{bmatrix} \leq 0. \tag{8.9}$$

Hence, as long as there is a feasible solution for (8.9), one can easily calculate the state feedback controller K with the help of the definitions.

In addition, when we integrate both sides of (8.4), we have

$$x^T P x - x^T(0) P x(0) + \int_0^\infty z^T z - \gamma^2 \int_0^\infty w^T w \leq 0, \tag{8.10}$$

$$\int_0^\infty z^T z \leq \gamma^2 \int_0^\infty w^T w, \tag{8.11}$$

$$\frac{\int_0^\infty z^T z}{\int_0^\infty w^T w} \leq \gamma^2, \tag{8.12}$$

which is nothing but

$$\|G\|_\infty = \frac{\|z\|_2}{\|w\|_2} \leq \gamma^2. \tag{8.13}$$

In conclusion, (8.9) and (8.13) are equivalent conditions.

CV

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EDUCATION

Degree	Subject	School/University	Graduation
M.Sc.	Control and Automation	YTÜ	2007
Bachelor	Electrical Engineering	YTÜ	2005
High School		Kadıköy Anatolium High Sch.	2001

EMPLOYMENT

Year	Corporation/Institution	Duty
2005	YTÜ	Research Assistant

PUBLICATIONS

Article

1. I.B. Kucukdemiral, **L. Ucun**, Y. Eren, H. Gorgun, G. Cansever	Ellipsoid Based L_2 Controller Design for LPV Systems with Saturating Actuators, Turkish Journal of Electrical Engineering and Computer Sciences, (2011)

Proceedings (International + National)

1. **L. Ucun**, İ.B. Kucukdemiral	Zaman Gecikmeli Sistemlerin Dinamik IQC Tabanlı İleribeslemeli Kontrolü, TOK'11 Otomatik Kontrol Ulusal Toplantısı, (2011)
2. **L. Ucun**, İ.B. Kucukdemiral, A. Delibasi, G. Cansever	LPV Control of Active Suspension System, IEEE International Conference on Mechatronics (ICM 2011), (2011)
3. İ.B. Kucukdemiral, **L. Ucun**, Y. Eren, H. Gorgun, A. Delibasi, G. Cansever,	L_2 Controller Design for LPV Systems with Saturating Actuators, Proc. of IFAC Symposium on Mechatronic Systems 2010, (2010)
4. İ. B. Küçükdemiral, **L. Ucun**, Y. Eren, H. Görgün, G. Cansever	Eyleyici Doyumlu Sistemler için Elipsoit Tabanlı L_2 Denetleyici Tasarımı, TOK'09, (2009)
5. Y. Eren, **L. Ucun**, H. Görgün, İ. B. Küçükdemiral, G. Cansever	PEM Tipi Yakıt Hücresi Sisteminde Kullanılan Kompresör Modelinin Adaptif Denetleyici ile Kontrolü, Elektrik Elektronik ve Bilgisayar Mühendisliği Sempozyumu (ELECO’08), (2008)
6. **L. Ucun**, İ. B. Küçükdemiral	Genelleştirilmiş Öngörülü Kontrol Algoritması ile Van De Vusse Reaktör denetimi, 49.Otomatik Kontrol Ulusal Toplantısı, (2007)

Research Projects

1.TÜBİTAK(2008-2011) Researcher	Doğrusal Motor Eyleyicili, Çeyrek Taşıt Aktif Süspansiyon Sisteminin Geliştirilmesi ve Doğrusal Parametreleri Değişen Optimal Kontrol Tekniği ile Denetimi
2.AB LLP Project (2009-2012) Araştırmacı	Co-Operative Network Training (CONET)

Awards

1.Phoenix Contact Xplore 2008 Third place in TCP/IP	Quadrotor Air Vehicle Assisted By Ground Vehicle
2.TOK'09 Best Student Paper Award	Eyleyici Doyumlu Sistemler için Elipsoit Tabanlı L_2 Denetleyici Tasarımı