REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

DETECTION METHODS OF ASYMPTOTIC CRITICAL VALUES OF POLYNOMIAL MAPPINGS

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Abuzer GÜNDÜZ

Signature

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Dedicated to my wife and my children

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Abuzer GÜNDÜZ

TABLE OF CONTENTS

LIST OF SYMBOLS

$\Delta(f)$	A convex polyhedron of f
Γ	A facet Γ of the polyhedron Δ^* verifying $dim\,(\Gamma \cap \mathbb{R}^{n-k}) = n-k-1$
X_σ	Affine toric variety
$I(X)$	All polynomials vanishing on V
$\Phi(\sigma)$	An algebraic torus of dimension $n-k$ related to the cone σ
S_σ	Finitely generated monoid
$(\mathrm{C}^*)^n$	n- dimensional algebraic torus
$g_\rho^j(\mathbf{c})$	The coefficient of $\left\langle \mu_j, \vartheta_u f^W \right\rangle (Q(t))$ for each $j \in \mathbb{J}$
$V(T)$	The common zeroes of all the elements of T
$\widetilde{\Gamma}_-(f)$	The convex hull of $supp(f) \cup \{0\}$ in $\mathbb{R}^n$
$(\widetilde{\Gamma}_-(f))^*$	The dual space of $\widetilde{\Gamma}_-(f)$
$F_b = f^{-1}(b)$	The fiber over b
Q(t)	The form of parametric curve
$\vartheta_u f^W(u)$	The logarithmic gradient
L_0	The maximum value of $\left\langle (q', 0), w_i - w_j \right\rangle$
$\mu(f)$	The Milnor number of f
ρ	The minimum value of $\langle \tilde{\alpha}, q \rangle$
$\mathscr{K}_\infty(f)$	The set of asymptotic critical value at infinity
$\mathscr{B}(f)$	The set of bifurcation value of f
$\mathscr{B}_\infty(f)$	The set of bifurcation value of f at infinity
$f(\mathrm{Sing} f)$	The set of critical value of f
$\mathbb{J}$	The set of indices that subset of $[1; n]$

LIST OF ABBREVIATIONS

Sing$f$	Critical values of f
h.o.t.	Higher order terms
L.H.S.	Left hand side
R.H.S.	Right hand side

LIST OF FIGURES

ABSTRACT

Detection Methods of Asymptotic Critical Values of Polynomial Mappings

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Doctor of Philosophy Thesis

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In algebraic geometry, the problem of detecting the bifurcation values of a polynomial is very important. The bifurcation values of a polynomial mapping consist of the bifurcation value at infinity and the set of critical values of its. This problem is generally encountered as detecting bifurcation value at infinity, which is a subset of the bifurcation values of the polynomial. This corresponds to determining some supersets containing bifurcation values at infinity. In addition, it is another important problem to determine the cases where the bifurcation values consist only of the values of the polynomial at the critical points. This is equivalent to bifurcation values at infinity is empty. In this thesis, we firstly construct a curve that approaching an asymptotic critical value which is a superset of the bifurcation value at infinity with very few coefficients. We used toric geometry as the main tool. By aids of, we get the corollary that says every critical value of polynomial mappings over the bad face of Newton polyhedron is an element of asymptotic critical value. Finally, we give a method to construct a curve approaching an asymptotic critical value of a real polynomial map, corresponding to detect real coefficients of the parametric representation of the curve. Asymptotic critical values sometimes correspond to the infimum or supremum of the polynomial. We hope that the study can be applied to optimization problems.

Keywords: Newton polyhedron, regularity at infinity, critical values, toric geometry, singularity theory

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ÖZET

Polinom Fonksiyonların Asimptotik Kritik Değerlerinin Tespit Yöntemleri

Abuzer GÜNDÜZ

Matematik Anabilim Dalı
Doktora Tezi

Danışman: Prof. Dr. Bayram Ali ERSOY
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Cebirsel geometride, bir polinomun çatallanma değerlerini (Bifurcation values) belirleme problemi oldukça önemlidir. Bir polinomun çatallanma değerleri kümesi polinomun sonsuzda çatallanma değerleri kümesi ve polinomun kritik noktalarda aldığı değerler kümesinin birleşiminden oluşur. Bu problem genellikle polinomun çatallanma değerlerinin bir alt kümesi olan sonsuzda çatallanma değerlerinin (bifurcation value at infinity) tespit edilmesi olarak karşımıza çıkar. Bu ise sonsuzda çatallanma değerlerini kapsayan bir kısım özel kümeler (superset) belirlemeye karşılık gelir. Bunun yanı sıra çatallanma değerlerinin sadece polinomun kritik noktalarda aldığı değerlerden oluştuğu durumları tespit etmek bir diğer önemli problemdir. Bu ise sonsuzda çatallanma değerleri kümesinin boş küme olmasına denktir. Biz bu çalışmada ilk olarak, söz konusu polinomun sonsuzda çatallanma değerlerini kapsayan bir özel küme (superset) olan asimptotik kritik değerlere yaklaşan bir parametrik eğrinin inşası için bir yöntem verdik ve dahası bu eğriyi oldukça az katsayı ile inşaa ettik. Bu süreç için temel araçlarımızı Torsal geometriden aldık. Buradan söz konusu polinomun Newton çok yüzlüsünün, kötü yüzleri (bad face) üzerine kısıtladığımızda oluşan yeni polinomun tüm kritik değerlerinin tam olarak en baştaki polinomun asimptotik kritik değerleri olduğunu söyleyen bir sonuç verdik. Son olarak, tüm katsayıların reel sayılardan oluşmasına denk olan, reel eğri inşası için bir yöntem verdik. Asimptotik kritik değerler bazen polinomun infimum veya spremum değerlerine karşılık gelir. Bilhassa çalışmanın bu kısmının optimizasyon çalışmalarına katkı verebileceğini umuyoruz.

Anahtar Kelimeler: Newton çokyüzlüsü, sonsuzda düzenlilik, kritik değerler, torsal geometri, tekillik teorisi

1
INTRODUCTION

1.1 Literature Review

The bifurcation value of polynomial mapping was introduced by [1] and [2] that say if f is a "tame polynomial" that the global Milnor number is finite, then the bifurcation value at infinity is the empty set. Namely, bifurcation value of f comprises only the critical values ($f(\mathrm{Sing} f)$). So, in this thesis, we focus on f that is not a "tame polynomial".

On the other hand, in [3], authors showed that if f is Newton non-degenerate, convenient and $f(0) = 0$ the following conclusion;

$$\mathscr{B}(f) \subseteq \Sigma_f \cup \{0\} \cup \bigcup_{\gamma \in \mathbb{B}} \Sigma_\gamma \tag{1.1}$$

holds, where $\Sigma_\gamma = \{f_\gamma(z^0) : \ z^0 \in (\mathbb{C}^*)^n \ and \ grad \ (f_\gamma(z^0)) = 0\} \ and \ \Sigma_f = f(\mathrm{Sing} f)$ and $\mathbb{B}$ is the set of bad faces γ and $f_\gamma(z) = \sum_{\alpha \in \gamma} a_\alpha z^\alpha$ for $f(x) = \sum_{\alpha \in \Delta(f)} a_\alpha z^\alpha$.

This inclusion is investigated in section 2.5.

In [4], the author showed for a bad face σ a special so that there is a a critical point on the algebraic torus (toric part) and so $t_0 \in \sum_\gamma$. Toric geometry was first used by Zaharia in [4] for this problem.

Jelonek and Kurdyka [5, 6] found an algorithm for detecting the set of asymptotic critical values $\mathscr{K}_\infty(f)$. They showes that $\mathscr{K}_\infty(f)$ is finite. Also, it is called a superset of $\mathscr{B}_\infty(f)$, it means that it includes $\mathscr{B}_\infty(f)$.

In [7], authors consider a real rational curve $\{X(t)\} \subset \mathbb{R}^n$ satisfies $lim_{t \to 0} \|X(t)\| \to \infty$, for a real polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree $\leq d$, with parametric representation with length $(d+1)d^{n-1}+1$ to attain the asymptotic critical value $lim_{t \to 0} f(X(t)) \in \mathscr{K}_\infty(f)$.

In [8], the author presented the relation between the $\mathscr{K}_\infty(f)$ and optimization problems. He said that supremum of $f(x)$ is finite if and only if the supremum of its an element of $\mathscr{K}_\infty(f)$. Moreover, [8], he found that if $sup_{x \in \mathbb{R}} f(x)$ is finite, then the elements of $\mathscr{K}_\infty(f)$ are $sup_{x \in \mathbb{R}} f(x)$.

Also, the problem was studied in [9], [10], [11],[12], [13]. Moreover, detecting of a superset of $\mathscr{B}_\infty(f)$ was studied in [14], [15], [16], [17], [18], [19] etc.

1.2 Objective of the Thesis

In this thesis, we aimed to construct a curve approaching an asymptotic critical value and getting a relation between Newton polyhedron of polynomial and an asymptotic critical value of a polynomial (moreover a real polynomial) map. As a result, we aimed to contribute to the optimization problem of the polynomial map.

1.3 Hypothesis

In this thesis, we first construct a curve approaching an asymptotic critical value which is a superset of the bifurcation value at infinity with very few coefficients. By the aid of this, we get a corollary that says every critical value of polynomial mappings over the bad face of Newton polyhedron is an element of asymptotic critical value.
Secondly, we give a method to construct a curve approaching an asymptotic critical value of a real polynomial map, corresponding to detect real coefficients of the parametric representation of the curve. We hope that the study can be applied to optimization problems.

2
PREMINARIES

In this section, we will basically utilize the book [20] for definitions and theorems.

2.1 Affine Variety

Let k be a fixed field (for example $k = \mathbb{R}$ or $k = \mathbb{C}$), the set of $n-$ tuples

$$\mathbb{A}_k^n = \{(x_1, \ldots, x_n) : \ x_i \in k, \ \forall i \in [1; n]\} \tag{2.1}$$

is called $n-$ dimensional affine $k-$space.
According to the above definition if $n = 2, \ k = \mathbb{R}$, then $\mathbb{A}_{\mathbb{R}}^2 = \mathbb{R}^2$ and for $k = \mathbb{C}$, then $\mathbb{A}_{\mathbb{C}}^2 = \mathbb{C}^2 \cong \mathbb{R}^4$.
Let $k[x_1, \ldots, x_n]$ be the polynomial ring in variables over k. Then each polynomial $f \in k[x_1, \ldots, x_n]$ is a function as $f : \mathbb{A}_k^n \to \mathbb{A}_k^1 = k$. Therefore if $f \in k[x_1, \ldots, x_n]$ is a polynomial then the set of zeroes of f as

$$V(f) = \{(x_1, \ldots, x_n) \in \mathbb{A}_k^n : \ f(x_1, \ldots, x_n) = 0\} \subseteq \mathbb{A}_k^n = f^{-1}(0). \tag{2.2}$$

Example 1. *Let $f(x_1, x_2) = x_1^2 + x_2^2 - 4 \in k[x_1, x_2]$ and $k = \mathbb{R}$ as specially, Then* $V(f) = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 - 4 = 0\} \subseteq \mathbb{R}^2$.

More generally, if T is any subset of $k[x_1, \ldots, x_n]$, we define the zero set of T to be common zeroes of all the elements of T;

$$i.e. V(T) = \{(x_1, \ldots, x_n) \in \mathbb{A}_k^n : \ f(x_1, \ldots, x_n) = 0, \ \forall f \in T\}. \tag{2.3}$$

Example 2. *Consider $T = \{x_1^2 + x_2^2 - 1 = 0, \ x_1 + x_2 = 0\} \subseteq \mathbb{R}^2$, to find common solution $x_1^2 + x_2^2 - 1 = 0$ and $x_1 + x_2 = 0$ that implies $x_2 = -x_1$ and so $x_1^2 + (-x_1)^2 - 1 = 0 \Rightarrow 2x_1^2 = 1$ and so $x_1 = \mp \frac{1}{\sqrt{2}}$ and $x_2 = \pm \frac{1}{\sqrt{2}}$.*
As a result, $V(T) = \{(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$.

Definition 2.1. [20] Let $k = \mathbb{R}$ or $k = \mathbb{C}$. A subset $X \subset \mathbb{A}_k^n$ is an algebraic set if there exists a subset $T \subseteq k[x_1, \ldots, x_n]$ verifying $X = V(T)$.

On the other hand, we can ask this question: Does it correspond to an algebraic set if we have a geometric object?

Definition 2.2. [20] A set X is a irreducible set if $X = X_1 \cup X_2$ with $X_1, X_2 \subseteq X$ then $X = X_1$ or $X = X_2$.

Proposition 1. *[20] An algebraic set is irreducible if and only if "its defining ideal" is prime.*

Definition 2.3. [20] For any subset $X \subseteq \mathbb{A}_k^n$ we define the ideal of $X \in k[x_1, \ldots, x_n]$ by

$$I(X) = \{f \in k[x_1, \ldots, x_n] : f(p) = 0, \forall p \in X\}. \tag{2.4}$$

Theorem 2.1. *[20] Let $f_1, f_2 \in k[x_1, \ldots, x_n]$*
1) $V(0) = \mathbb{A}^n$
2)$V(f_1) \cup V(f_2) = V(f_1 f_2)$,
3) $V(\sum_{i \in \Lambda} I_i) = \bigcap_{i \in \Lambda} V(I_i)$.

Theorem 2.2 (Hilbert Nullstellensatz)**.** *[20] Let k' be an algebraic closed field and J is an ideal of $k'[x_1, \ldots, x_n]$ polynomial ring. If $f \in k'[x_1, \ldots, x_n]$ that vanishes at all points of $V(J) \Rightarrow f^r \in J$ for some $r > 0$.*
Namely, $\sqrt{J} = I(V(J))$, if we define $\sqrt{J} = \{f : f^r \in J \; for \; some \; r > 0\}$.

We can construct Zariski topology on $\mathbb{A}_k^n$ as open sets to be the complement of an algebraic variety.

2.2 Singularity Theory

In this section, we will basically utilize the book [20] for definitions and theorems.

Definition 2.4. [20] Let $V = V(f_1, \ldots, f_k) \subset \mathbb{A}^n$ be an algebraic variety. If the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix} (p_1) \tag{2.5}$$

has rank strictly less than $min(n, k)$, then p_1 is said to be the singular point of V.

Example 3. *Let $I = (y^2 - x^3, z - x^2)$ and $V(I) \subseteq \mathbb{A}^3$. Then the Jacobian matrix*

$$\begin{pmatrix} -3x^2 & 2y & 0 \\ -2x & 0 & 1 \end{pmatrix}. \tag{2.6}$$

Especially, for $p = (0,0,1)$, we can see that $rankJ_V(p) = 1 < 2 = min(2,3)$ and implies that $p = (0,0,1)$ is a singular point of V.

Definition 2.5. [20] If $V = V(f_1)$ is a variety defined by a single polynomial then a singular point p of V satisfies $\frac{\partial f}{\partial x_i}(p) = 0$. Namely, the gradient of f must be vanishes. i.e.

$$\nabla f = (\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p)) = (0, \dots, 0). \tag{2.7}$$

Example 4. *Let $f(x_1, x_2) = x_1^2 + x_2^3$ and $V = V(f_1)$. Then $\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}) = (0,0)$ and so $p_1 = (0,0)$ is a singular point of V.*

Definition 2.6. [20] A singular point $p \in V$ is an isolated singular point if there are no other singularities in a neighborhood of p_1, $U_{p_1} \subset V$.
i.e. $U_p \backslash \{p\}$ is smooth.
Otherwise, p is said to be a non-isolated singular point of V.

Example 5. *Let $f(x_1, x_2, x_3) = x_1^3 x_2^2 + x_2^2 x_3 + x_3^3$ and $V = V(f)$. Then $\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}) = (3x_1^2 x_2^2, 2x_1^3 x_2 + 2x_2 x_3, x_2^2 + 3x_3^2) = (0,0,0)$ and implies that $p_1 = (0,0,0)$ and $p' = (x,0,0)$ are a singular points of V.*
Here $p' = (x,0,0)$ is example of a non-isolated singularity by along $x - axis$.

2.3 Toric Geometry

In this section, we will basically utilize [21] and [22] for definitions and theorems.

2.3.0.1 Convex Polyhedral Cones

We will define steps by steps as follows;

$$\sigma \to \check{\sigma} \to S_\sigma \to R_\sigma \to X_\sigma \tag{2.8}$$

A lattice is named a discrete subgroup of N of $\mathbb{Z}^n$. If N is a lattice, for all $x \in N$, if an open set W satisfies $N \cap W = \{x\}$. For instance, $\mathbb{Z}^2$ is a lattice. Moreover for a subgroup of $\mathbb{R}^n$(isomorphic to $\mathbb{Z}^n$), $N_\mathbb{R} = N \otimes \mathbb{R}$ and N has a basis as $(v_1, \dots, v_n)$.
The set

$$\sigma = \{r_1 v_1 + r_2 v_2 + \dots + r_n v_n : \ r_1, \dots, r_n \in \mathbb{R}_{\geq 0}\} \tag{2.9}$$

is called the polyhedral cone and generated by $(v_1, \dots, v_n)$. The cone σ is convex if and only if for all $v_1, v_2 \in \sigma$, $\lambda v_1 + (1 - \lambda) v_2 \in \sigma$, where $\lambda \in [0,1]$.
Let σ a convex polyhedral cone. If σ does not contain a line which passes the origin,

then it is called strongly convex cone.
i.e. $\sigma \cup (-\sigma) = \{0\}$.

Definition 2.7. [22] Let $N \cong \mathbb{Z}^n \subset \mathbb{R}^n$ be a lattice and σ is a cone. If each generator of σ is an element of N then σ is called rational or simplicial cone.

Consider a cone generated by $(v_1, \ldots, v_k)$;

Definition 2.8. [22] If all coordinates of a vector $v \in \mathbb{Z}^n$ is coprime, then it is called primitive. If $(v_1, \ldots, v_k)$ is primitive, then σ is called regular and also there exist primitive vectors $(v_{k+1}, \ldots, v_n)$ such that $det(v_1, \ldots, v_n) = \pm 1$. Namely, this vector may be completed in a basis of the lattice.

The dimension of cone is the dimension of smallest vector space containing its. For example, consider $\sigma =< ((1,0),(0,1)) >$. It is a rational polyhedral cone and $dim(\sigma) = 2$.

The dual space of $N \cong \mathbb{Z}^n$ is $Hom_{\mathbb{Z}}(N, \mathbb{Z})$ that isomorphic to $(\mathbb{Z}^n)^*$ and showed by M. Let us M as a dual vector space of N. It is defined as;

$$N^* = M = Hom(N, \mathbb{Z}), \tag{2.10}$$

and we will take care the real vector space;

$$M_{\mathbb{R}} = M \bigotimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}e_1 \bigoplus \ldots \mathbb{R}e_n \cong \mathbb{R}^n. \tag{2.11}$$

If $e_1^*, \ldots, e_n^*$ are generators of the dual space M, then we have the following condition;

$$< e_i^*, e_j >= \delta_{ij}, \tag{2.12}$$

where δ_{ij} is Kronecker delta.

The set

$$\check{\sigma} = \{v \in M : < u, v > \geq 0, \ \forall u \in \sigma\} \tag{2.13}$$

is called the dual cone of σ and also is a convex cone.

Definition 2.9. [22] If σ is simplicial polyhedral cone, then $\check{\sigma}$ is simplicial polyhedral cone.

But the opposite is not always true. For example if $\sigma = \{0\}$, then $\check{\sigma} = M_{\mathbb{R}}$ is not a strongly convex cone.

Example 6. *Let $N \cong \mathbb{Z}^2$ a lattice, $\sigma \in N_{\mathbb{R}}$ and $\sigma =< u_1, u_2 >=< 2e_1 - e_2, e_2 >$. The dual space M is generated by $\pm e_1^*, \pm e_2^*$ and so the dual cone $\check{\sigma}$ is generated by v_1, v_2 such that $v_1 = a_1 e_1^* + a_2 e_2^*$, $v_2 = b_1 e_1^* + b_2 e_2^*$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.*
Then we have $< u_1, v_1 >=< 2e_1 - e_2, a_1 e_1^* + a_2 e_2^* >=< (2,-1), (a_1, a_2) >= 2a_1 - a_2 = 0$,
$< u_2, v_1 >=< (0,1), (a_1, a_2) >= a_2 \geq 0$,
$< u_1, v_2 >=< (2,-1), (b_1, b_2) >= 2b_1 - b_2 \geq 0$,
$< u_2, v_2 >=< (0,1), (b_1, b_2) >= b_2 = 0$. *Then, we get*
$2a_1 - a_2 = 0,\ a_2 \geq 0,\ 2b_1 - b_2 \geq 0,\ b_2 = 0$, *we can choose as $v_1 = (1,2)$ and $v_2 = (1,0)$ and so $\check{\sigma} =< e_1^* + 2e_2^*, e_1^* >$. Also $\sigma =< u_1, u_2 >=< 2e_1 - e_2, e_2 >$ has four faces as $\tau_1 = \{0\},\ \tau_2 = u_1,\ \tau_3 = u_2,\ \tau_4 = \sigma$.*

The set
$$\tau = \sigma \cap u^{\perp} = \{v \in \sigma :< v, u >= 0\} \tag{2.14}$$
is called a face of σ where $u \in \check{\sigma} \cap M$ and showed by $\tau \prec \sigma$. Besides, any cone is a face of itself. If the face has $(n-1)-$dimension, then it is named as facet.

The following properties can be written by [22];
1) Every face of convex polyhedral cone is a polyhedral convex cone.
2) Every face of face is a face of the cone.
3) Every intersection of faces of σ is a face of σ.
4) Let $\sigma \subset N_{\mathbb{R}}$, $(\sigma^v)^v = \sigma$.
5) If $\sigma_1 \subset \sigma_2$, then $\check{\sigma}_2 \subset \check{\sigma}_1$.
6) σ is strongly convex cone if and only if $\check{\sigma}$ has dimension n.

Let "$\star$" be an binary operation over non-empty set S such that $\star : S \times S \to S$. If "$\star$" is associative then it is called a semi-group. Moreover, if it is commutative, has zero element and satisfies the simplification law that is $s + t = s' + t \Rightarrow s = s'$ for all $s, s', t \in S$ then it is called a monoid.

Lemma 2.1. *[22] Let σ be a cone, and N be a lattice. Then $\sigma \cap N$ is a monoid.*

Definition 2.10. [22] Let T be a monoid. If there exist $a_1, \ldots, a_k \in T$ such that $\forall v \in T,\ v = \lambda_1 a_1 + \ldots + \lambda_k a_k$ where $\lambda_i \in \mathbb{Z}_{\geq 0}$ then T is called finitely generated monoid. Moreover $a_1, \ldots, a_k$ are called generators of the T.

Lemma 2.2 (Gordon's Lemma)**.** *[22] If σ is rational cone then $\sigma \cap N$ is a finitely generated monoid.*

We will utilize this lemma in order to polyhedral cone $\check{\sigma}$ and will indicate by S_σ the monoid $S_\sigma = \check{\sigma} \cap M$.

Generally, $S_\sigma = \check{\sigma} \cap M$ can be finitely generated monoid. Namely, by 2.2, if $\check{\sigma}$ is rational cone, then $S_\sigma = \check{\sigma} \cap M$ is finitely generated monoid. It is very important because we will obtain an affine toric variety from the relationship between the finite generators of S_σ.

2.4 Affine Toric Variety

In this section, we will basically utilize [21] and [22] for definitions and theorems. Now, we will present how to obtain an affine toric variety from a rational strong convex cone.

Definition 2.11. [22] $\mathbb{C}[z, z^{-1}] = \mathbb{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$ is called Laurent Polynomial ring.

Definition 2.12. [22] For $\lambda \in \mathbb{C}^*$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. The Laurent monomial is denoted by

$$\lambda \cdot z^\alpha = \lambda \cdot z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}. \tag{2.15}$$

There exists an isomorphism between the additative group $\mathbb{Z}^n$ and the multiplicative group of monic Laurent monomials as

$$\begin{aligned} \theta : \mathbb{Z}^n &\to \mathbb{C}[z, z^{-1}] \\ \alpha = (\alpha_1, \ldots, \alpha_n) &\longmapsto z^\alpha = \lambda z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}. \end{aligned} \tag{2.16}$$

By 2.16, we have

$$\chi : M \to \mathbb{C}[M] = \mathbb{C}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}] \tag{2.17}$$

by $\chi^{e_i^*} = X_i$ and $\chi^{-e_i^*} = X_i^{-1}$ where $\pm e_i^*$ is a generator of the dual space M. Hence, for all $v \in S_\sigma$, there is a generator $\chi^v \in \mathbb{C}[S_\sigma]$ correspond to v.
Because of the S_σ is finitely generated by 2.2, $\mathbb{C}[S_\sigma]$ has finite number of generators. Besides, $\forall v_1, v_2 \in S_\sigma$, we get

$$\chi^{v_1} \chi^{v_2} = \chi^{v_1 + v_2} \tag{2.18}$$

and so

$$\mathbb{C}[S_\sigma] = \{\sum \alpha_v \chi^v : v \in S_\sigma,\ \alpha_v \in \mathbb{C}\}. \tag{2.19}$$

In this place, $\chi^0 = 1$ is the constant polynomial corresponding to $0 \in S_\sigma = \check{\sigma} \cap M$.

Consider the map;

$$f : \mathbb{C}[Y_1, \ldots, Y_m] \to \mathbb{C}[S_\sigma] \tag{2.20}$$

by $\chi^{v_i} = Y_i$, where v_i is a generator of S_σ. It can be seen that the kernel of the map f is an ideal of the polynomial ring $\mathbb{C}[Y_1, \ldots, Y_m]$. If we indicate this by I, we obtain

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[Y_1, \ldots, Y_m]/I. \tag{2.21}$$

Because of θ and χ are describe likewise, the ideal I is besides kernel of θ. Namely, the ideal I is detected by relations between the generators of S_σ.

$Spec(R)$ is the set of all prime ideal of R. For example the spectrum of ring $\mathbb{C}[X_1, X_2] = \{(x-a, x-b)\}$ and also corresponding to the maximal ideal of its.

Definition 2.13 (Affine Toric Variety). [22] If σ is rational strongly convex cone $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^n$, then

$$X_\sigma = Spec(\mathbb{C}[S_\sigma]) \tag{2.22}$$

is called affine toric variety.

Theorem 2.3. *[22] Let $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^n$ and I is generated by the relations between the generators of the ring $\mathbb{C}[S_\sigma]$, then V_σ is the variety $V(I)$ in the space $\mathbb{C}^m$.*

Example 7. *Consider $\sigma =< e_1, e_2 >$, then $\check{\sigma}$ is generated by e_1^*, e_2^* and $S_\sigma =< (e_1^*, e_2^*) >$. Hence the generators of $\mathbb{C}[S_\sigma]$ are $\chi^{e_1^*} = X_1$ and $\chi^{e_2^*} = X_2$, and so $\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, X_2]$. If we pass to the variable Y_1, Y_2, we get $\mathbb{C}[S_\sigma] = \mathbb{C}[Y_1, Y_2]/I$, Here, note that $dim(V_\sigma) = dim(\mathbb{C}[Y_1, Y_2]) = 2$ and so there is no relation between generators of S_σ. Namely $I =< 0 >$, so*
$V_\sigma = Spec(\mathbb{C}[S_\sigma]) \cong V(< 0 >) = \mathbb{C}^2$.

Theorem 2.4. *[22] Let σ be a rational cone. Then the ring*

$$R_\sigma = \{f \in \mathbb{C}[z, z^{-1}] : supp(f) \subset \check{\sigma} \cap M\} \tag{2.23}$$

is a $\mathbb{C}$-algebra that has finite generators.

Example 8. *In $\mathbb{R}^2$, let $\sigma =< (2e_1 - e_2, e_2) >$ and implies that $\check{\sigma} =< (e_1^*, e_1^* + 2e_2^*) >$. But $e_1^* + e_2^*$ can not be generated by $\check{\sigma}$. Then we should add its as a new element of S_σ and so $S_\sigma =< (e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*) >$. Now, we construct an isomorphism θ between S_σ and $\mathbb{C}[z, z^{-1}]$ such that*

$$\begin{aligned} a_1 &\longmapsto u_1, \\ a_2 &\longmapsto u_2, \\ a_3 &\longmapsto u_3. \end{aligned} \tag{2.24}$$

By the isomorphism θ, we get $u_1 = z_1$, $u_2 = z_1 z_2$, $u_3 = z_1 z_2^2$. The $\mathbb{C}-$algebra R_σ can be represented as $R_\sigma = \mathbb{C}[S_\sigma] = \mathbb{C}[z_1, z_1 z_2, z_1 z_2^2] = \mathbb{C}[y_1, \ldots, y_n]/I_\sigma$ where the corelation

$a_1 + a_3 = 2a_2$ case to the corelation $u_1 u_3 = u_2^2$. Here I is generated by $\gamma_1\gamma_3 = \gamma_2^2$. Hence, we get affine toric variety
$X_\sigma = V(I_\sigma) = \{x = (x_1, x_2, x_3) \in \mathbb{C}^3;\ x_1 \cdot x_3 = x_2^2\}$.

Note that it has a singularity at origin, which is a quadratic cone and coresponding to the cone σ.

Definition 2.14. [22]

$$T_N \cong \mathbb{C}^* \times \mathbb{C}^* \times \ldots \mathbb{C}^* = (\mathbb{C}^*)^n \tag{2.25}$$

is called $n-$dimensional affine algebraic torus.

In general, for $\sigma \in N_{\mathbb{R}}$ we have $S_\sigma \subset S_{\{0\}}$ and implies that $\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_{\{0\}}]$ and so $X_{\{0\}} \subset X_\sigma$. Hence, for all $\sigma \in N_{\mathbb{R}}$, X_σ contains $n-$dimensional affine algebraic torus $T_N \cong (\mathbb{C}^*)^n$ as an open and dense subset. Because of this, X_σ is called toric variety.

2.5 Bifurcation Value

Let E, B be spaces and $f : E \to B$ be a map. For $t \in B$, each the disjoint sets $F_t = f^{-1}(t)$ is named the fiber over t. Also the space B is named the base space of the fibration, E is said to be the total space, a fibration denoted by (E_1, E_2, F, f).

Definition 2.15. Let a fibration (E_1, E_2, F, f), for each $t \in E_2$, there exists a neighbourhood $E_2 \supseteq U,\ t \in U$ such that $f^{-1}(U)$ is diffeomorphic to $U \times F$, then f is called locally trivial fibration.
Namely, each $f^{-1}(t),\ t \in B$ are diffeomorphic one to another.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial map. If an neighbourhood, $U \subset \mathbb{C}$, t_0 satisfying $f_| : f^{-1}(U) \to U$ is locally trivial $\mathbb{C}^\infty$- fibration then $t_0 \in \mathbb{C}$ is said to be a typical value of f.
Otherwise, t_0 is called a bifurcation value (or atypical value). The set of bifurcation values of f is denoted by $\mathscr{B}(f)$.
Besides, for $t_0 \in \mathbb{C}$, if there exists a compact set $K \subset \mathbb{C}^n$ and a neighbourhood $W \subset \mathbb{C}$ at t_0 satisfying under the condition $f_| : f^{-1}(W) \setminus K \to W$ is a locally trivial $\mathbb{C}^\infty$- fibration, then f is called topologically trivial at infinity at $t_0 \in \mathbb{C}$. Otherwise t_0 is a bifurcation value at infinity of f and denoted by $\mathscr{B}_\infty(f)$ that is bifurcation locus at infinity of f.
In general, $\mathscr{B}(f)$: the bifurcation locus of a polynomial f is the smallest subset $\mathbb{C}$ and union of the set of critical values at infinity $\mathscr{B}_\infty(f)$ and critical values. Namely,

$$\mathscr{B}(f) = f(\mathrm{Sing}\, f) \cup \mathscr{B}_\infty(f). \tag{2.26}$$

The following example relates $f(\mathrm{Sing} f)$ to a part of $\mathscr{B}(f)$.

Example 9. *Consider* $f : \mathbb{C}^2 \to \mathbb{C}$ *with* $f(x_1,x_2) = x_1^2 + x_2^2$. *Here* $(0,0)$ *is a singular point of* f *and so* $f((0,0)) = 0 \in f(\mathrm{Sing} f)$. *Thereby the fiber* $f^{-1}(0)$ *and* $f^{-1}(t)$, *for* $t \neq 0$ *are different as topologically and implies that* $0 \in \mathscr{B}_\infty(f)$.

The following example relates $\mathscr{B}_\infty(f))$ to a part of $\mathscr{B}(f)$.

Example 10. *[1] Let us take the polynomial* $f : \mathbb{C}^2 \to \mathbb{C}$, *with* $f(x_1,x_2) = x_1^2 x_2 - x_1$ *has no singular points, otherwise it has the critical values at infinity for* $t_1 = 0$. *To more explain, let* $t_1 = 0$,
$f^{-1}(0) = \{(x_1,x_2) \in \mathbb{C}^2 : x_1^2 x_2 - x_1 = 0\} \cong \mathbb{C} \sqcup \mathbb{C}^* \ (disjoint\ union)$, *but* $t_1 \neq 0$,
$f^{-1}(t_1) = \{(x_1,x_2) \in \mathbb{C}^2 : x_1^2 x_2 - x_1 = t\} \cong \mathbb{C}$. *We can see that for* $t_1 \neq 0$, $f^{-1}(t_1)$ *and* $f^{-1}(0)$ *are topologically different. Because they have different connected component numbers that is a topological invariant.*

2.6 Tame Polynomial

Let a polynomial $f : \mathbb{C}^n \to \mathbb{C}$. If a compact neighborhood K of the critical points of f satisfying under the condition $\|\partial f\|$ is bounded away from zero on $\mathbb{C}^n \setminus U$, then f is named "tame polynomial". Namely, when f is "tame", then $B_\infty(f) = \emptyset$.

Definition 2.16. [1] Tame polynomial can be characterized below by Milnor number

$$\mu(f) = dim_{\mathbb{C}}(\mathbb{C}[x_1,\ldots,x_n]/(f_1,\ldots,f_n)) \tag{2.27}$$

where $f_j = \frac{\partial f}{\partial x_j}$ for $1 \leq j \leq n$.

Proposition 2. *[1] A polynomial is "tame" if and only if* $\mu(f) < \infty$ *and* $\mu(f^w) = \mu(f)$ *for all sufficient small* $w \in \mathbb{C}^n$.

Example 11. *Let* $f(x,y) = 2x^3 + x - y^2$, *then* $\frac{\partial f(x,y)}{\partial x} = 6x^2 + 1$, $\frac{\partial f(x,y)}{\partial y} = -2y$, *and implies that* $\mu(f) = dim_{\mathbb{C}}(\mathbb{C}[x,y]/(6x^2+1,-2y))$ *then* $\mu(f) = 2$, *because for* $I =< 6x^2+1, -2y >$ *the set* $\{1, x\}$ *can not be generated by* I.

As well as, $\mu_p(f)$ is a topological invariant and a useful for singularity theory.

Example 12. *Let* $f(x_1,x_2) = x_1^2$, *then* $\frac{\partial f(x_1,x_2)}{\partial x_1} = 2x_1$, $\frac{\partial f(x_1,x_2)}{\partial x_2} = 0$, *and implies that* $\mu(f) = dim_{\mathbb{C}}(\mathbb{C}[x_1,x_2]/(2x_1,0))$ *then* $\mu(f) = \infty$, *because for* $I =< 2x_1 >$ *the set* $\{1, x_2, x_2^2, \ldots, x_2^n, x_2^{n+1}, \ldots\}$ *can not be generated by* I.

In this thesis, we study the topological map $f : \mathbb{C}^n \to \mathbb{C}$ such that f is not tame, namely $\mathscr{B}_\infty(f) \neq \emptyset$. If f is not tame, namely $\mathscr{B}_\infty(f) \neq \emptyset$, there exists a curve $X(t) \in \mathbb{C}^n$, $\forall t$ such that it approaching to the critical point of f at infinity.

2.7 Asymptotic Critical Value

Lemma 2.3. *(Curve selection lemma) [3]*
Let $f_1,\ldots,f_s,g_1,\ldots,g_s,h_1,\ldots,h_r \in R[x_1,\ldots,x_n]$ be polynomial functions with real coefficients. Let $U=\{x\in R^m:\ f_i(x)=0, i\in[1,q]\},\ \ W=\{x\in R^m:\ g_i(x)>0, i\in[1,s]\}$. Suppose that there exists a sequence $\{x^k\}\subseteq U\cap W$ such that
$lim_{k\to\infty}\|x^k\|=\infty$ for all $j\in[1;r]$, $lim_{k\to\infty}h_j(x^k)=0$. Then there exists a real analytic curve $p:(0,\epsilon)\to U\cap W$ with $lim_{t\to 0}\|Q(t)\|=\infty,\ \ lim_{t\to 0}h_j(Q(t))=0$ for $j\in[1;r]$, where $Q(t)=at^{\alpha}+a_1t^{\alpha+1}+\ldots$ with $a\in R^m\setminus\{0\}$ and $\alpha<0$.

Here we will consider h_j as a gradient and we will use effectively this lemma.

Definition 2.17. [3] Let Δ is a closed face of convex hull of $supp(f)\cup\{0\}$ in $\mathbb{R}^n$. For $f_\Delta(z)=\sum_{\nu\in\Delta}a_\nu z^\nu$, if the system of equations

$$\frac{\partial f_\Delta}{\partial z_1}(z)=\ldots=\frac{\partial f_\Delta}{\partial z_n}(z)=0 \tag{2.28}$$

has no solution in $(\mathbb{C}^*)^n$, then f is called non-degenerate on Δ. Moreover, if f is non-degenerate on every compact face Δ of convex hull of $supp(f)\cup\{0\}$ in $\mathbb{R}^n$, then f is called Newton non-degenerate.

Definition 2.18. [3] If the intersection of $supp(f)$ with each coordinate axis is non-empty, then f is called convenient.

To give our main study, the following theorem in [3] should be investigated.

Theorem 2.5. *[3] Suppose that f is not convenient, Newton non-degenerate and $f(0)=0$. Then the following inclusion*

$$\mathscr{B}(f)\subseteq\Sigma_f\cup\{0\}\cup\bigcup_{\gamma\in\mathbb{B}}\Sigma_\gamma \tag{2.29}$$

holds, where $\Sigma_\gamma=\{f_\gamma(z^0):\ \ z^0\in(\mathbb{C}^*)^n$ and $grad\ (f_\gamma(z^0))=0\},\ \ \Sigma_f=f(\mathrm{Sing}f)$ and $\mathbb{B}$ is the set of bad faces (see 3.1).

Proof. Let $p(t)$ an analytic curve by aid of the curve selection lemma to choose a curve $p(t)$ such that $lim_{t\to 0}p(t)=\infty$ and $lim_{t\to 0}f(p(t))\in\mathbb{C}$ and we denote

$$lim_{t\to 0}f(p(t))\in\Sigma_f\cup\{0\}\cup\bigcup_{\gamma\in\mathbb{B}}\Sigma_\gamma. \tag{2.30}$$

To indicate this, see the expansion
$p(t)=at^{\alpha}+a_1t^{\alpha+1}+a_2t^{\alpha+2}+\ldots$

$f(p(t)) = bt^{\beta} + b_1 t^{\beta+1} + b_2 t^{\beta+2} + \dots$
$grad\ f(p(t)) = ct^{\gamma} + c_1 t^{\gamma+1} + c_2 t^{\gamma+2} + \dots$
and investigate each case step by step.

• If $grad\ f(p(t)) = 0$ then $lim_{t\to 0} f(p(t)) \in \Sigma_f$.

• If $grad\ f(p(t)) \neq 0$ and

$p(t) = (p_1(t), p_2(t), \dots, p_n(t)) = (w_1^0 t_1^{\nu} + w_1^1 t^{\nu_1+1} + \dots, \dots, w_k^0 t_k^{\nu} + w_k^1 t^{\nu_k+1} + \dots, 0, \dots, 0)$.

Let $\iota_\nu : \overline{supp(f)} \cap \mathbb{R}^k \to \mathbb{R}$, γ a face of $\overline{supp(f)}$ such that the linear function $\iota_\nu(x) = \Sigma_{j=1}^k \nu_i x_i$ takes the minimal value, say d, on γ and let $m \in (-\infty, 0)$ be such that $m < min_x\{\iota_\nu(x) : x \in \overline{supp(f)}\}$.
By the aid of this, we can write

$$f(p(t)) = f_\gamma(w^0)t^d + f_{\gamma_1}(w^0)t^{d+1} + \dots \tag{2.31}$$

and

$$\frac{\partial f}{\partial z_j}(p(t)) = \frac{\partial f_\gamma}{\partial z_j}(w^0)t^{d-\nu_j} + \dots \tag{2.32}$$

where $w^0 = (w_1^0, w_2^0, \dots, w_k^0, 1, \dots, 1)$ and γ is a bad face such that $\iota_\nu(x)$ takes a minimal value over its.
• If $d > 0$, we get $lim_{t\to 0} f(p(t)) = 0 \in \{0\}$.
• If $d = 0$ and $\nu_k \leq 0$ we get a contradiction!
• If $d = 0$ and $\nu_k > 0$, we get a bad face such that

$$lim_{t\to 0} f(p(t)) = f_\gamma(w^0) \in \Sigma_\gamma. \tag{2.33}$$

We will focus this case, because we want to construct a curve $p(t)$ such that the critical values under some condition are a asymptotic critical values.
• If $d < 0$, we attain a contradiction, due to nondegenerate condition. ■

In other words, we aim to construct a curve satisfies two conditions, approaching to infinity and $lim_{t\to 0}\|X(t)\|\|grad\ \ f(X(t))\| = 0$, respectively and implies that we get $lim_{t\to 0} f(p(t)) = f_\gamma(w^0) \in \Sigma_\gamma$.

Jelonek and Kurdyka [5, 6] introduced the concept of asymptotic critical value of polynomial mapping that is defined as;

Definition 2.19. [5] $\mathcal{K}_\infty(f) = \{y \in \mathbb{C}$: there is a sequence $X(t)$, $lim_{t\to 0}\|X(t)\| = \infty$

and $lim_{t\to 0}\|X(t)\|\|grad\ f(X(t))\| = 0$ such that $lim_{t\to 0}\|f(X(t))\| = y\}$.

This is weaker condition than $\mathscr{B}_\infty(f)$ and when $t \notin \mathscr{K}_\infty(f)$, then f satisfies a Malgrange's condition at this point. As we have, $\mathscr{B}_\infty(f) \subset \mathscr{K}_\infty(f)$. To control the set $\mathscr{B}_\infty(f)$, the set of "asymptotic critical value of f" ,$\mathscr{K}_\infty(f)$, can be used. Here $\mathscr{K}_\infty(f)$ is also a superset of $\mathscr{B}_\infty(f)$.

Example 13. *[15] Consider the polynomial*
$f : \mathbb{R}^2 \to \mathbb{R},\ f(x_1, x_2) = x_2(x_1^2 x_2^2 + 3x_1x_2 + 3)$ *and the curve*
$Q(t) : (0,1) \to \mathbb{R}^2, \quad t \to (\frac{-3}{2t}, t)$ *then we get* $lim_{t\to 0}\|Q(t)\| = \infty$ *and* $lim_{t\to 0}\|Q(t)\|\|grad\ (f(Q(t)))\| = 0$. *Since* $lim_{t\to 0} f(Q(t)) = 0$, *we obtain* $0 \in \mathscr{K}_\infty(f)$.

As a result, we focus on to give a method to construction curve that approaching to $\mathscr{K}_\infty(f)$, namely

$$lim_{t\to 0} f(X(t)) \in \mathscr{K}_\infty(f)). \tag{2.34}$$

3
THE METHOD OF REAL CURVE CONSTRUCTION

3.1 Preliminary steps for the method

In this section, we will give a method to construction a real curve $X(t)$ approaching to $t_0 \in \mathcal{K}_\infty(f)$. Namely

$$lim_{t\to 0} f(X(t)) \in \mathcal{K}_\infty(f) \tag{3.1}$$

Let us give some basic notions of toric geometry following [4] and [23]. Consider

$$f(x) = \Sigma_{\nu\in\mathbb{N}} a_\nu z^\nu, \;\; supp(f) = \{\nu \in \mathbb{N}^n : a_\nu \neq 0\} \tag{3.2}$$

and $\Delta(f)=$ the convex hull closure of $supp(f)$ in $\mathbb{R}^n$ and is called Newton polyhedron of f. The convex hull of $supp(f)\cup\{0\}$ in $\mathbb{R}^n$ is denoted by $\widetilde{\Gamma}\_(f)$. Also, $(\widetilde{\Gamma}\_(f))^*$ is called the dual of $\widetilde{\Gamma}\_(f)$ and K be a unimodular simplicial subdivision of $(\widetilde{\Gamma}\_(f))^*$.

Δ^a is denoted by a face of $\widetilde{\Gamma}\_(f)$ determined by the condition $\langle a, y\rangle \leq \langle a, x\rangle$ where $x \in \widetilde{\Gamma}\_(f)$ and $y \in \Delta^a$ for $a \in (\mathbb{R}^n)^*$.
Also, define

$$f_\gamma(x) = \sum_{\alpha\in\gamma} a_\alpha x^\alpha, \tag{3.3}$$

where $\gamma \subset \Delta(f)$ is a face of the Newton polyhedron of f. Let $\sigma \in K$ be a unimodular simplicial cone with $dim(\sigma)=k$. We can define an algebraic torus of dimension $n-k$ related to the cone σ as

$$\Phi[\sigma] = (\mathbb{C}^*)^n/\{(t^{b_1},\dots,t^{b_n}); t\in(\mathbb{C}^*)^k, (b_1,\dots,b_n)\in\sigma\}. \tag{3.4}$$

Besides, let us consider a disjoint union of tori is defined as by

$$M_{\overline{\sigma}} = \cup_{\sigma'\subset\sigma}\Phi[\sigma'] \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \ni (u_1,\dots,u_k,u_{k+1},\dots,u_n) \tag{3.5}$$

with $\overline{\sigma} = \cup_{\sigma' \subset \sigma} \sigma^{'}$ where $\sigma^{'}$ run over all subcones of σ.

In this study, firstly, we will present a change variable to restrict over the chart 3.4, and so we look for the singularities of f over this chart. For this purpose, we focus on the following set;

$$u = (u', u'') \in \mathbb{C}^n_k = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}, \tag{3.6}$$

where $u' \in \mathbb{C}^k$ called affine part and $u'' \in (\mathbb{C}^*)^{n-k}$ called toric part.

In order to investigate the the bifurcation value of $f(x)$, [4, p. 2.4] passes from $\mathbb{C}^n$ to $\mathbb{C}^n_k := \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$. To examine the topology of $f^{-1}(t) \in \mathbb{C}^n$, [4] states that it is enough to study the question on $(\mathbb{C}^*)^{n-k}$. We will use this approach to determine the asymptotic critical value set $\mathscr{K}_\infty(f)$.

We recall the method to construct a curve (3.1) proposed by [24]. Namely, the critical value of f_γ, for γ : bad face (Definition 3.1), on the toric part $(\mathbb{C}^*)^{n-k} \subset \mathbb{C}^n_k$ represents an asymptotic critical value and it can be obtained by the curve $X(t)$ (3.1) constructed step by step in our study.
Let us define the special face on which f will be restricted.

Definition 3.1. [23] The face $\gamma \subset \Delta(f)$ is called *bad* if it verifies the following two conditions;
(i) The affine subspace of dimension $= dim\,\gamma$ spanned by γ contains the origin,
(ii) ($\pm$ condition for the bad face) There exists an hyperplane $H \subset \mathbb{R}^n$, $\gamma = H \cap \Delta(f)$ defined by an equation $\sum_{j=1}^n p_j x_j = 0$ where there exists $i \neq j$ verifying $p_i p_j < 0$.

Let us see the definitions given so far in the following example,

Example 14. *Let $f(x,y) = x^4y^3 + (x^3y+1)^2 - 1$, then*
$supp(f) = \{(4,3),(6,2),(3,1)\} = \{v_1, v_2, v_3\}$,
$\widetilde{\Gamma}_{-}(f) = supp(f) \cup \{0\} = \{(4,3),(3,1),(6,2),(0,0)\}$,
$(\widetilde{\Gamma}_{-}(f))^* = \{<(-1,3),(-3,4),(1,2)>\}$.
Besides, γ : bad face is generated by $\gamma =< (3,1),(6,2) >$ because it passes to the origin and the equation of bad face is $x - 3y = 0$.
We can choose $p_1 = 1$ and $p_2 = -3$ and so $p_1p_2 = -3 < 0$. Finally, since $(-1,3)\perp\gamma$, we get $\Phi[\gamma] = (\mathbb{C}^*)^2/(t_1^{-1}, t_1^3) = (t_2, t_2^{-2})$.

Our purpose is to detect the singularities of $f(x)$ over the bad face by changing the variable. Because of this, we will give a condition for the bad face.

Let $a_1, \cdots, a_k$ be a unimodular basis of a $k-$ dimensional cone σ,
i.e., $\sigma = \Sigma_{i=1}^k t_i a_i, t_i \geq 0$. In this case, we may take $m_1, \cdots, m_n \in \mathbb{R}^n$ as a basis

of the dual cone $\sigma^* = \{x \in \mathbb{R}^n; \langle x, a\rangle \geq 0, \forall a \in \sigma\}$ such that $\langle a_i, m_j\rangle = \delta_{ij},\ i \in \{1,\ldots,k\},\ j \in \{1,\ldots,n\}$ where δ_{ij} is Kronecker Delta. Throughout the thesis, we use the notation $i \in [r_1; r_2] \Leftrightarrow i \in \{r_1, \cdots, r_2\}$ for two integers $r_1 < r_2$.
The basis $a_1, \cdots, a_k$ can be extend to an $n-$dimensional basis $a_1, \cdots, a_n$ by means of supplementary vectors $a_{k+1}, \cdots, a_n$ in the condition of $\mid det(a_1, \cdots, a_n) \mid = 1$. We show $\sigma^* = \{\sum_{i=1}^n \lambda_i m_i; \lambda_1, \ldots, \lambda_k \geq 0\}$ and
$V_{\sigma^*} = \{\lambda_{k+1} m_{k+1} + \ldots + \lambda_n m_n, \lambda_j \in \mathbb{R}, j = k+1, \cdots, n\}$, respectively.

Suppose that γ is a bad face and a $n-$dimensional cone σ verifying

$$\gamma \subset \sigma^* = \{x \in \mathbb{R}^n; \langle \alpha, x\rangle \geq 0, \forall \alpha \in \sigma\} \tag{3.7}$$

has a basis $(a_1, \ldots, a_k)$ such that $\gamma = \{v \in \Delta(f); \langle a_i, v\rangle = 0,\ i = 1, \ldots, k\}$. The existence of this basis can be reached from Definition 3.1 (ii).

Note that we put conditions on bad face γ such that $\forall\ v \in \gamma$, $< a_i, v >= 0\ \ for\ \ i \in 1, \ldots, k$ and $< a_i, v > \geq 0\ \ for\ \ i \in \{k+1, \ldots, n\}$ to get a polynomial if we restrict f over the monomials γ.

Now, we will give a method to detect unimodular matrixes used for changing variable.

Let us define an unimodular matrix

$$W = ({a_1}^T, \ldots, {a_n}^T) = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}, W^{-1} = M = ({\mu_1}^T, \ldots, {\mu_n}^T) = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix}, \tag{3.8}$$

where $(m_1, \ldots, m_n)$ is a basis of σ^* and $\sigma^* = \Sigma_{i=1}^k \mathbb{R}_{\geq 0} m_i + \Sigma_{j=k+1}^n \mathbb{R} m_j$. In particular, we will choose the cone σ so that $\{m_1, \cdots, m_n\} \subset (\mathbb{R}^*)^n$. This is possible thanks to the conditions of 3.1.

Under the change of variables

$$(x_1, \cdots, x_n) = (u^{w_1}, \cdots, u^{w_n}), \tag{3.9}$$

we get a rational function by (3.9)

$$f^W(u) = \sum_{\alpha \in supp(f)} a_\alpha u^{\alpha \cdot W}. \tag{3.10}$$

Namely, for $\alpha \in \Delta(f)$, each monomial of $f(x)$ can be written as

$$x^{\alpha} = u^{\alpha \cdot W} = u_1^{<a_1,\alpha>} \cdot \ldots \cdot u_k^{<a_k,\alpha>} \cdot u_{k+1}^{<a_{k+1},\alpha>} \cdot \ldots \cdot u_n^{<a_n,\alpha>}. \tag{3.11}$$

Especially if $v \in \gamma$, then $\langle a_i, v\rangle = 0 \ for \ i \in [1;k]$ and $\langle a_i, v\rangle \geq 0 \ for \ i \in [k+1;n]$. Hence we get $u^{v \cdot W} = u_{k+1}^{<a_{k+1},v>} \cdot \ldots \cdot u_n^{<a_n,v>}$.

By this condition, the toric variety is obtained from $f_{\gamma}^{W}(x)$ lies in the toric part $(\mathbb{C}^*)^{n-k} \subset \mathbb{C}_k^n$.

Now we are in the $u-$ space and look at its singularity by restricting the function over γ.

Now we investigate the singularity of $f_{\gamma}^{W}(u)$ by means of the logarithmic gradient

$$\vartheta_u f^W(u) = (\vartheta_{u_1} f^W(u), \cdots, \vartheta_{u_n} f^W(u)), \tag{3.12}$$

with $\vartheta_{u_j} = u_j \frac{\partial}{\partial u_j}$, $j \in [1;n]$, for $u^* = (0, u''_*) \in \mathbb{C}_k^n$.

If $\vartheta_u f_{\gamma}^{W}(u^*) = 0$, then $u^* = (0, u''_*) \in \mathbb{C}^n{}_k$ is called a critical point. We give the notation $u' = (u_1, \cdots, u_k)$, $U'' = (U_{k+1}, \cdots, U_n) = (u_{k+1} - u_{k+1}^*, \cdots, u_n - u_n^*)$, respectively. The local expansion of the Laurent polynomial
$f^W(u)$ at $u = u^* = (0, u''_*) \in \mathbb{C}_k^n$ is given by

$$f^W(u) = \sum_{\beta \in supp_{u^*}(f^W)} a_{\beta}^*(u - u^*)^{\beta} = \sum_{\beta \in supp_{u^*}(f^W)} a_{\beta}^* u'^{\beta'} U''^{\beta''}, \tag{3.13}$$

for $supp_{u^*}(f^W) := \{\beta \in \mathbb{Z}^n; a_{\beta}^* \neq 0\}$.

After the expansion, negative power can occur. In this case, we will expand its Laurent polynomial series.
The expression matching to the term $\alpha.W \in (\mathbb{Z}_{\geq 0}^k \setminus \{0\}) \times \mathbb{Z}_{<0}^{n-k}$ in (3.10) will produce a series in (3.13) with $(\beta', \beta'') \in (\mathbb{Z}_{\geq 0}^k \setminus \{0\}) \times (\mathbb{Z}_{\geq 0})^{n-k}$ with respect to the rule

$$\frac{1}{u_j} = \frac{1}{u_j^*} \sum_{\ell \geq 0} (-\frac{U_j}{u_j^*})^{\ell}. \tag{3.14}$$

Lemma 3.1. *[23] The Laurent polynomial $f^W(0, u'') = f_{\gamma}^{W}(u) = \sum_{\alpha \in \gamma} a_{\alpha} u^{\alpha.W}$ is a polynomial (with positive power terms) , when it restrict over u'' variables.*

Now let us see this process again with the following example. We received the method in [25] to determine the unimodular matrix W.

Example 15. *For 14, $a_1 = (-1,3) \perp \gamma$ and for $a_2^{(1)} = (1,-2)$, $< v, a_2^{(1)} > \geq 0$, where $v \in \gamma$. Hence we can take*
$W = (a_1^t, (a_2^{(1)})^t) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$, $W^{-1} = M = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ *and detect* $M^T = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$.
Also, changing variables $v_3 \cdot W = (0,1)$ and $v_1 \cdot W = (5,-2)$ and so
$f^W(u) = u_1 u_2^{-2} + (u_2+1)^2 - 1$. *For the expansion of the Laurent polynomial, calculate* $\frac{1}{u_2^2} = (\frac{-1}{1-(1+u_2)})^2 = [1 + \Sigma_{j=1}(1+u_2)^j]^2$. *So, we have*
$\vartheta_u f^W(u) = (\vartheta_{u_1} f^W(u), \vartheta_{u_2} f^W(u)) = (5u_1^5 u_2^{-2}, -2u_1^5 u_2^{-2} + 2u_2(u_2+1))$. *For* $\vartheta_u f_\gamma^W(u^*) = 0$, *then* $u^* = (0,-1)$ *and so* $f^W(0,u'') = f_\gamma^W(u) = \sum_{\alpha \in \gamma} a_\alpha u^{\alpha.W} = (u_2+1)^2 - 1$. *For* $U_1 = u_1$ *and* $U_2 = u_2 + 1$, *then* $f^W(U_1,U_2) = U_1^5 (U_2 - 1)^{-2} + (U_2)^2 - 1$.

Here, we will some basic definitions to illustrate sufficient conditions of 3.19.
We define Δ^* as a convex hull of $\cup_{i=1}^n \Delta_{u^*}(\langle \mu_i, \vartheta_u f^W(u) \rangle)$.
Here the polyhedron $\Delta_{u^*}(\langle \mu_i, \vartheta_u f^W(u) \rangle)$ is defined as a convex hull of $supp_{u^*}(\langle \mu_i, \vartheta_u f^W(u) \rangle)$ obtained after the expansion as in (3.13).

Proposition 3. *([23, Proposition 3.1]) Assume that $\vartheta_u f_\gamma^W(u^*) = \vartheta_u f^W(0,u''_*) = 0$. Then we can detect a facet Γ of the polyhedron Δ^* verifying $dim\,(\Gamma \cap \mathbb{R}^{n-k}) = n-k-1$ is determined by the aids of a vector $q \in \mathbb{Z}^n$ such that*

$$\Gamma = \{\beta \in \Delta^*; \langle \beta, q \rangle \leq \langle \tilde{\beta}, q \rangle \text{ for every } \tilde{\beta} \in \Delta^*\}. \tag{3.15}$$

Namely, the inequality $\langle \beta, q \rangle \leq \langle \tilde{\beta}, q \rangle$ is valid with each
$\tilde{\beta} \in \Delta_{u^*}(\langle \mu_i, \vartheta_u f^W(u) \rangle)$, $i \in [1;n]$, for any $\beta \in \Delta^*$. Also, we will denote ρ ,is an integer, by the following

$$\rho = min_{\tilde{\alpha} \in \Delta^*} \langle \tilde{\alpha}, q \rangle . \tag{3.16}$$

It is clear that 3.16 equal to $\langle \alpha, q \rangle$ for $\alpha \in \Gamma$ since the definition of 3.15

In this situation, we define a special curve in $u-$space

$$Q(t) = (u'(t), u''(t)) = (c' t^{q'} + h.o.t., u''_* + c'' t^{q''} + h.o.t.), \tag{3.17}$$

where $q = (q', q'')$ is detected in Proposition 3, $u''_* \neq 0$, as $u''_* \in (\mathbb{C}^*)^{n-k}$ and $c' t^{q'} = (c'_1 t^{q'_1}, \cdots, c'_k t^{q'_k})$, etc.

By the means of all aforementioned, we will look for conditions to verify 3.19.

Definition 3.2. ([5, 6, 23]) Let us consider a curve $x = X(t)$ verifies the following two conditions;

$$lim_{t\to 0}||X(t)|| = \infty, \tag{3.18}$$

$$lim_{t\to 0} x_i \frac{\partial f(X(t))}{\partial x_j} \to 0, \tag{3.19}$$

for each pair $(i, j) \in [1; n]^2$. The value of $lim_{t\to 0} f(X(t))$ is called the asymptotic critical value of f. $\mathcal{K}_\infty(f)$ is called the set of asymptotic critical values of f.

Besides [15], if the image value of f, which is not asymptotic critical is called *t−regular value of f*. If limit $lim_{t\to 0} f(X(t)) = p_0$ exists for the curve (3.18), the negation of the condition (3.19) is known as Malgrange condition for the fiber $f^{-1}(p_0)$. It means that $\exists\epsilon > 0$ such that $lim_{t\to 0}||X(t)||||grad\ f(X(t))|| > \epsilon$.

In order to construct a curve $||X(t)|| \to \infty$ aforementioned, it is sufficient to take only one torus chart $\Phi[\sigma]$.

To construct a special curve $X(t)$ a image of the curve $Q(t)$ in $u-$ space by the map (3.9). Further we impose the following condition on (3.15), $q = (q', q'')$ and (3.8):

$$\exists i \in [1; n] \ such\ that \ \ < (q', 0), w_i >< 0, \tag{3.20}$$

in the view of $X_i(t) = c_i t^{<(q',0),w_i>}(1 + \ h.o.t)$.
That is our first condition for our method and is called (μ) condition.

Also, $X(t)$ is denoted by the image of the curve $Q(t)$ defined in (3.17) by the aids of the map (3.9).

Lemma 3.2. *[23] The condition (μ) of 3.20 is adequate to exist a curve* $||X(t)|| \to \infty$ *with finite limit* $lim_{t\to 0} f(X(t)) = lim_{t\to 0} f^W(Q(t))$.
$lim_{t\to 0}\ \vartheta_u f^W(Q(t)) = 0$ *verifies and the limit* $lim_{t\to 0} f^W(Q(t))$*is corresponding to a critical value of the polynomial* $f^W_\gamma(u)$.

Proof. we can get $x_i(t) = c_i t^{\langle (q',0),w_i \rangle}(1 + h.o.t.)$ since (3.17) and $x_i = u^{w_i}$.
Because of this definition of curve (3.17), it is clear that the value
$lim_{t\to 0} f(X(t)) = lim_{t\to 0} f^W(Q(t))$ is existed. ■

For the second condition, we shall see the following relation between the logarithmic gradient vectors.

$$\begin{pmatrix} \vartheta_{x_1} f(x) \\ \vartheta_{x_2} f(x) \\ \vdots \\ \vartheta_{x_n} f(x) \end{pmatrix} = M^T \begin{pmatrix} \vartheta_{u_1} f^W(u) \\ \vartheta_{u_2} f^W(u) \\ \vdots \\ \vartheta_{u_n} f^W(u) \end{pmatrix}. \tag{3.21}$$

Let $\vec{\ell} = (\ell_1, \cdots, \ell_n) \in (\mathbb{R}^*)^n$ be a vector in general position with non-zero components (actually corresponding to coefficients of the curve) and denote $< \vec{\ell}, u^W >= \sum_{j=1}^n \ell_j u^{w_j}$, then

$$< \vec{\ell}, x > \begin{pmatrix} \partial_{x_1} f(x) \\ \partial_{x_2} f(x) \\ \vdots \\ \partial_{x_n} f(x) \end{pmatrix} =< \vec{\ell}, u^W > \begin{pmatrix} \frac{\mu_1}{u^{w_1}} \\ \frac{\mu_2}{u^{w_2}} \\ \vdots \\ \frac{\mu_n}{u^{w_n}} \end{pmatrix} \begin{pmatrix} \vartheta_{u_1} f^W(u) \\ \vartheta_{u_2} f^W(u) \\ \vdots \\ \vartheta_{u_n} f^W(u) \end{pmatrix}. \tag{3.22}$$

On comparing the orders in t of the L.H.S. and R.H.S. for $x = X(t)$ in (3.22), it is sufficient to investigate a curve $Q(t)$ satisfying

$$\min_{i \neq j} \langle (q', 0), w_i - w_j \rangle + ord\left(< \mu_j, \vartheta_u f^W > (Q(t))\right) > 0, \tag{3.23}$$

for every $j \in [1; n]$. Also, we define the new integer as follow;

$$L_0 = max_{i \neq j} < (q', 0), w_i - w_j >, \tag{3.24}$$

and will use it to control powers in the series $Q(t)$.

Example 16. *To see more clearly 3.22, let us try to get in two variables.*
Firstly, let us see

$$\begin{pmatrix} \vartheta_{x_1} f(x) \\ \vartheta_{x_2} f(x) \end{pmatrix} = M^T \begin{pmatrix} \vartheta_{u_1} f^W(u) \\ \vartheta_{u_2} f^W(u) \end{pmatrix}. \tag{3.25}$$

Namely, let us change parameters for Jacobean matrix.
Since $x_1 = u^{w_1}$, $x_2 = u^{w_2}, u_1 = x^{m_1} = x_1^{m_{11}} x_2^{m_{12}}, u_2 = x^{m_2} = x_1^{m_{21}} x_2^{m_{22}}$ *then*
$\vartheta_{x_1} u_1 = x_1 \frac{\partial u_1}{\partial x_1} = x_1 \cdot m_{11} \cdot x_1^{m_{11}-1} \cdot x_2^{m_{12}} = m_{11} x_1^{m_{11}} x_2^{m_{12}} = m_{11} \cdot u_1$ *and similarly*
$\vartheta_{x_1} u_1 = m_{21} \cdot u_2$, $x_1 \frac{\partial}{\partial x_1} = x_1 \left(\frac{1}{\frac{\partial x_1}{\partial u_1}} \cdot \frac{\partial}{\partial u_1} + \frac{1}{\frac{\partial x_1}{\partial u_2}} \cdot \frac{\partial}{\partial u_2} \right) =$

$x_1[\frac{\partial u_1}{\partial x_1}\cdot\frac{\partial}{\partial u_1}+\frac{\partial u_2}{\partial x_1}\cdot\frac{\partial}{\partial u_2}] = m_{11}\cdot u_1\frac{\partial}{\partial u_1}+m_{21}\cdot u_2\frac{\partial}{\partial u_2} =$
$m_{11}\vartheta_{u_2}+m_{21}\vartheta_{u_2} =< \mu_1,(\vartheta_{u_1},\vartheta_{u_2})>$, *as desired. Moreover, let* $\vec{\ell}=(\ell_1,\ell_2)\in(\mathbb{R}^*)^2$ *be a vector with non-zero components and denote* $\left\langle\vec{\ell},u^W\right\rangle=\sum_{j=1}^{2}\ell_j u^{w_j}$*. Then we have*

$$\left\langle\vec{\ell},x\right\rangle\begin{pmatrix}\partial_{x_1}f(x)\\ \partial_{x_2}f(x)\end{pmatrix}=\left\langle\vec{\ell},u^W\right\rangle\begin{pmatrix}\frac{\mu_1}{u^{w_1}}\\ \frac{\mu_2}{u^{w_2}}\end{pmatrix}\begin{pmatrix}\vartheta_{u_1}f^W(u)\\ \vartheta_{u_2}f^W(u)\end{pmatrix}. \tag{3.26}$$

$$(\ell_1u^{w_1}+\ell_2u^{w_2})\cdot\begin{pmatrix}\frac{1}{u^{w_1}}\begin{pmatrix}m_{11}\\ m_{21}\end{pmatrix}\\ \frac{1}{u^{w_2}}\begin{pmatrix}m_{12}\\ m_{22}\end{pmatrix}\end{pmatrix}=(\ell_1+\ell_2u^{w_2-w_1})\cdot\begin{pmatrix}m_{11}\\ m_{21}\end{pmatrix}$$
$$+\ (\ell_1u^{w_1-w_2}+\ell_2)\cdot\begin{pmatrix}m_{12}\\ m_{22}\end{pmatrix}=\ell_1\left[\begin{pmatrix}m_{11}\\ m_{21}\end{pmatrix}+u^{w_1-w_2}\begin{pmatrix}m_{12}\\ m_{22}\end{pmatrix}\right]+$$
$$\ell_2\left[u^{w_1-w_2}\begin{pmatrix}m_{11}\\ m_{21}\end{pmatrix}+\begin{pmatrix}m_{12}\\ m_{22}\end{pmatrix}\right]=$$
$\frac{<\ell,u^w>}{u^{w_1}}=\frac{<(\ell_1,\ell_2),(u^{w_1},u^{w_2})>}{u^{w_1}}=\frac{\ell_1u^{w_1}+\ell_2u^{w_2}}{u^{w_1}}=\ell_1+\ell_2u^{w_2-w_1}$.

If we write in the last equation

$$Q(t)=(u'(t),u''(t))=(c't^{q'}+h.o.t.,u''_*+c''t^{q''}+h.o.t.) \tag{3.27}$$

and focus on $ord((\ell_1+\ell_2u^{w_2-w_1})_{|Q(t)})$*, we arrive the equation* $\min_{i\neq j}\left\langle(q',0),w_i-w_j\right\rangle(Q(t))$.

Also for the last part of the 3.26, we arrive the equation $ord\left(\left\langle\mu_j,\vartheta_u f^W\right\rangle(Q(t))\right)$*. As a result, the logarithmic gradient goes to zero, the following integer that is the power of t,* $\min_{i\neq j}\left\langle(q',0),w_i-w_j\right\rangle+ord\left(\left\langle\mu_j,\vartheta_u f^W\right\rangle(Q(t)),\right)$ *should be positive.*

Hence, we give the following theorem.

Theorem 3.1. *[23] Consider a polynomial* $f\in\mathbb{C}[x_1,\cdots,x_n]$ *and its Newton polyhedron* $\Delta(f)$*, has maximal dimension* n*. Suppose that* γ *is one of its bad faces is given in Definition 3.1.*
(i) Under the conditions (μ) of 3.20, we obtain a curve $X(t)$ *verifying (3.18), (3.19) of Definition 3.2 so that a critical value of the polynomial* $f_\gamma^W(u)$ *is equal to* $lim_{t\to0}f(X(t))$*.*
(ii) $X(t)$ *is found as an image by the map (3.9) of a curve* $Q(t)$ *whose coefficients* $\mathbf{c}\in\mathscr{C}$ *verify* $(L_0-\rho+1)\mid\mathbb{J}\mid-$*tuple of algebraic equations for* ρ (3.16), L_0 *(3.24). (iii) The curve* $Q(t)$ *aforementioned in (ii) has a parametric representation (3.17) of parametric length* $L_0-\rho+2$*, in other words, we may suppose whose parametrization coefficients verifies* $(c'(j),c''(j))=0$ *for* $j>L_0-\rho+1$.

By the choice made in Proposition 3, (3.15), (3.16), we get the expansion of $< \mu_j, \vartheta_u f^W > (Q(t))$, in t with the following form

$$g^j_\rho(\mathbf{c})t^\rho + g^j_{\rho+1}(\mathbf{c})t^{\rho+1} + h.o.t. \tag{3.28}$$

We define the index set

$$\mathbb{J} = \{j \in [1;n];\ min_{i\neq j} < (\acute{q}, 0), w_i - w_j >< 0\}. \tag{3.29}$$

The algebraic function $g^j_\rho(\mathbf{c})$ depends on all n variables $(c'(0), c''(0)) \in \mathbb{C}^n \subset \mathscr{C}$ for each $j \in \mathbb{J}$ in view of the choice of $q \in \mathbb{Z}^n$.

As $| \mathbb{J} |< n$ the system of algebraic equations $g^j_\rho(\mathbf{c}) = 0, \forall j \in \mathbb{J}$ has non-trivial solutions in $\mathbb{C}$.

Here for each $j \in \mathbb{J}$, the vector with polynomial entries $g^j_\rho(\mathbf{c})$ is depended on all n variables $(c'(0), c''(0)) \in \mathbb{C}^n \subset \mathscr{C}$ selection of $q \in \mathbb{Z}^n$ is obtained in Proposition 3.

The vector with polynomial entries $g^j_{\rho+1}(\mathbf{c})$ efficiently is depended on $(c'(0), c''(0), c'(1), c''(1)) \in \mathbb{C}^{2n} \subset \mathscr{C}$, so the system of equations $g^j_{\rho+1}(\mathbf{c}) = 0, \forall j \in \mathbb{J}$ which has further non-trivial solutions in $\mathbb{C}$.

Thereby, we may acquire non-trivial solutions to $(L_0 + 1 - \rho) | \mathbb{J} |$ −tuple of algebraic equations

$$g^j_\rho(\mathbf{c}) = g^j_{\rho+1}(\mathbf{c}) = \cdots = g^j_{L_0}(\mathbf{c}) = 0, \forall j \in \mathbb{J}, \tag{3.30}$$

for L_0 (3.24).

In order to demonstrate this, it is enough to indicate that $g^j_{\rho+\ell}(\mathbf{c})$ efficiently is attached to $(c'(\ell), c''(\ell))$ are absent in $g^j_{\rho+\tilde{\ell}}(\mathbf{c})$ for $\tilde{\ell} \in [0; \ell - 1]$.

In summary we get the following theorem [23, Theorem 3.7] tells us that every critical value of polynomial

$$f^W_\gamma(u) = \sum_{\alpha\in\gamma\cap supp(f)} a_\alpha u^{\alpha.W} \tag{3.31}$$

with γ bad face is an asymptotic critical value under certain conditions.
Namely, the limit $lim_{t\to 0} f^W(Q(t))$ corresponds to a critical value of the polynomial $f^W_\gamma(u)$.

Corollary 3.1. *[23] Under supposition of Theorem 3.2, we get the following inclusion*

$$\bigcup_{\gamma} f_\gamma(\text{Sing}\, f_\gamma \cap (\mathbb{C}^*)^{dim\,\gamma}) \subset \mathscr{K}_\infty(f), \tag{3.32}$$

where γ runs among bad faces of $\Delta(f)$ when a cone σ verifying condition (μ) of 3.20 may be constructed.

Proof. Theorem 3.2 tells us $f_\gamma^W(\text{Sing}\, f_\gamma^W \cap (\mathbb{C}^*)^{dim\,\gamma}) \subset \mathscr{K}_\infty(f)$. It is enough to show that $f_\gamma^W(\text{Sing}\, f_\gamma^W \cap (\mathbb{C}^*)^{dim\,\gamma}) = f_\gamma(\text{Sing}\, f_\gamma \cap (\mathbb{C}^*)^n)$ for $f_\gamma(x) = \sum_{\alpha \in \gamma \cap supp(f)} a_\alpha x^\alpha$.

From Lemma 3.1, $f_\gamma^W(u)$ is a polynomial depending effectively on toric variables u'' and independent of affine variables u' (the condition (i) of the Definition 3.1). This means that $\vartheta_{u_1} f_\gamma^W(u) = \cdots = \vartheta_{u_k} f_\gamma^W(u) = 0$. Thus, for $u''_* \in \text{Sing}\, f_\gamma^W \cap (\mathbb{C}^*)^{dim\,\gamma}$, the vanishing of the logarithmic gradient vector holds: $\vartheta_u f_\gamma^W(0, u''_*) = 0$. By using the map $u''(x) = (x^{m_{k+1}}, \cdots, x^{m_n})$ induced by the inverse to (3.9), we see $f_\gamma(x) = f_\gamma^W(0, u''(x))$. Taking the relation (3.21) into account, we see that this entails
$\vartheta_x f_\gamma(x_*) = 0$ for $x_* \in (\mathbb{C}^*)^n$ that satisfies $u''(x_*) = u''_*$.
Conversely, if $\vartheta_x f_\gamma(x_*) = 0$ for $x_* \in (\mathbb{C}^*)^n$, by (3.21), we see $\vartheta_u f_\gamma^W(0, u''_*) = 0$ for $u''_* = u''(x_*)$ the image of the map (3.9). ∎

3.2 Real Curve Construction

In this section, we investigate the construction of real curve (3.1) for a real polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}$. Our aim is to solve equations (3.30) in the real space for $\ell \in [0; L_0 - \rho + 1]$.

Theorem 3.2. *[24] We apply notions of precedent sections to the real polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}$. If $card(\mathbb{J}) = 1$, for the index set* (3.29)*, then the real curve* (3.1) *can be constructed that approaches a real asymptotic critical value of f.*

Proof. As $\mathbb{J} = \{j\}$ with some $j \in [1; n]$, for each equation
$g^j_{\rho+\ell}(\mathbf{c}) = 0,\ \ell \in [0; L_0 - \rho + 1]$, we shall try to solve algebraic equations in variables $c_j(0), c_j(1), \cdots, c_j(\ell)$ that can be solved in real while other variables can be chosen in an arbitrary manner, especially chosen to be real. We explain the proof step by step.
Firstly, let us consider the equation $g^j_\rho(\mathbf{c})$ in a variable $c_j(0)$:
$g^j_\rho(\mathbf{c}) = b^0_0 + b^0_1(c_j(0)) + b^0_2(c_j(0))^2 + \ldots + b^0_{p_0}(c_j(0))^{p_0},\ \ p_0 \geq 1$, where all b^0_i are real numbers for $i \in [0; p_0]$. Note that $g^j_\rho(\mathbf{c})$ has a non-zero term b^0_0. The coefficients $b^0_1, \ldots, b^0_{p_0}$ depend also on real $c_i(0),\ i \in [1; n] \backslash \{j\}$ that can be consider as real free variable. This means that there is sufficient freedom to change the constant b^0_0 so

that the equation $g^j_\rho(\mathbf{c})$ has a real root as an equation in $c_j(0)$.

Secondly, for the equation $g^j_{\rho+1}(\mathbf{c}) = 0$ which can be considered as an equation that depends on only $c_j(0), c_j(1)$. However we found the variable $c_j(0)$ as real number from the first step. So, the equation consists of
$g^j_{\rho+1}(\mathbf{c}) = b^1_0 + b^1_1(c_j(1)) + b^1_2(c_j(1))^2 + \ldots + b^1_{p_1}(c_j(1))^{p_1}, \quad p_1 \geq 1$. By an argument analogous to see the existence of real $\{c_i(0)\}^n_{i=0}$ satisfying $g^j_\rho(\mathbf{c}) = 0$, we conclude that there exist real $\{c_i(0), c_i(1)\}^n_{i=0}$ satisfying $g^j_{\rho+1}(\mathbf{c}) = 0$.

It can bee seen that for each step the equation $g^j_{\rho+\ell}(\mathbf{c}) = 0,\ \ \ell \in [0; L_0 - \rho + 1]$ can be solved in terms of $c_j(\ell)$ that can be chosen real. In this way we can construct a real curve $X(t)$, as desired. Thus the proof is completed. ■

As a result of this theorem, for example, if we take all exponents of $f(x)$ even, the infimum of this function will be the element of $\mathscr{K}_\infty(f)$.

4
EXAMPLES

Examples in this section are received from work [23]. We will give two examples that illustrate Theorem 3.2.

4.1 Non-isolated Case

Example 17. (Non-isolated singularity on a two dimensional bad face) [23]

For a polynomial $f(x) = x^{v_1} + (x^{v_2} - x^{v_3} + 1)^2 + (x^{v_2} - x^{v_3} + 1)^3 + x^{v_4} - 2$
with $v_1 = (2, 1, 1), v_2 = (2, 2, 1), v_3 = (1, 2, 1), v_4 = (3, 1, 1)$.

We shall restrict the monomial of f over the bad face to look for singularities.

Note that non-isolated singularities at infinity case have not been studied in [26].
By changing the variable, we will construct a unimodular matrix W and M.

We state that

$M = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = ({\mu_1}^T, {\mu_2}^T, {\mu_3}^T) = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$, is unimodular. So we may take

$M^{-1} = W = ({a_1}^T, {a_2}^T, {a_3}^T) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -3 & 2 \end{pmatrix}$. Here, γ of $\Delta(f)$ is the only bad face located on the plane spanned by v_2, v_3. By aid the of the above matrix W, we get $f^W(u) = -2 + u_1 + (u_2 - u_3 + 1)^2 + (u_2 - u_3 + 1)^3 + \frac{u_1 u_2}{u_3}$.
Note that $f_\gamma^W(0, u_2, u_3) = (u_2 - u_3 + 1)^2 + (u_2 - u_3 + 1)^3$ is a polynomial and has non-isolated singularities along a line $u_2 - u_3 + 1 = 0$. For example, we may select, $u^* = (0, -1/3, 2/3)$.

Since the negative power occurs, we shall expand the Laurent series.
For $U_2 = u_2 + 1/3, U_3 = u_3 - 2/3,\quad f^W(u)$ has the following expansion

$f^W(u) = -2 + u_1 + (U_2 - U_3)^2 + (U_2 - U_3)^3 + \frac{3u_1}{2}(U_2 - 1/3)(1 - \frac{3U_3}{2} + (\frac{3U_3}{2})^2 + \cdots)$, in the neighbourhood of u^*.
The facet given in Γ, (3.15) is determined by the calculating of $\Delta(\langle \mu_i, \vartheta_u f^W(u) \rangle)$ for $i = 1, 2, 3$.

A direct computation indicates $\langle \mu_3, \vartheta_u f^W(u) \rangle = \frac{u_1}{16} - 2U_2 + 2U_3 + h.o.t.$

Now, we detect the facet 3.15 that is a face closest to the origin of
$\Delta(\langle \mu_i, \vartheta_u f^W(u) \rangle)$ $i = 1, 2, 3$.

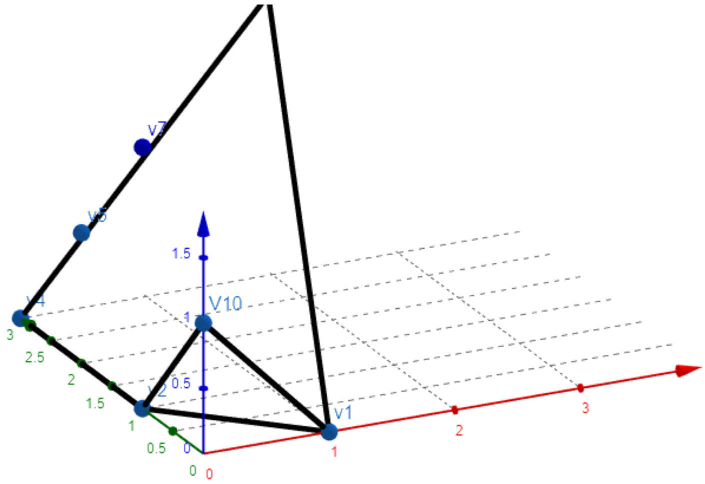

Figure 4.1 The facet Γ

We see that detect the facet Γ locates on the plane including $(1,0,0),(0,0,1),(0,1,0)$ and $q = (1,1,1)$, $(q', 0) = (1,0,0)$ and compute $L_0 = 3$ and $\rho = 1$, respectively. Note that a curve $Q(t)$ (3.17) with real coefficients of parametric length 4. Compared to [7], the length is so less than that one.

In other words,
$u_1 = \sum_{j=0}^{3} c_1(j)t^{j+1}, u_2 = -1/3 + \sum_{j=0}^{3} c_2(j)t^{j+1}, u_3 = 2/3 + \sum_{j=0}^{3} c_3(j)t^{j+1}$
that verifies $-3 + ord \langle \mu_3, \vartheta_u f^W(Q(t)) \rangle > 0$, can be constructed. Here one can obtain $\mathbb{J} = \{3\}$.

However, by the aids of the method of [7, Theorem 3.5.], the real curve with needed property has parametric length $16 \times 15^2 + 1 = 3601$.

Indeed, providing that we put these terms into $\langle \mu_3, \vartheta_u f^W(u) \rangle$, the expansion with initial term proportional to t^1, ($\langle q, \alpha \rangle = 1$ for $\alpha \in \Gamma$) is obtained as follows

$\langle \mu_3, \vartheta_u f^W(u) \rangle (Q(t)) = \{c_1(0)/2 - 2c_2(0) + 2c_3(0))\}t +$

$1/4\{2c_1(1)+6c_1(0)c_2(0)-4c_2(0)^2-8c_2(1)+3c_1(0)c_3(0)+8c_2(0)c_3(0)-4c_3(0)^2+8c_3(1)\}t^2+$
$1/8\{4c_1(2)+12c_1(1)c_2(0)+24c_2(0)^3+12c_1(0)c_2(1)-16c_2(0)c_2(1)-16c_2(2)+6c_1(1)c_3(0)-18c_1(0)c_2(0)c_3(0)-72c_2(0)^2c_3(0)+16c_2(1)c_3(0)-9c_1(0)c_3(0)^2+72c_2(0)c_3(0)^2-24c_3(0)^3+6c_1(0)c_3(1)+16c_2(0)c_3(1)-16c_3(0)c_3(1)+16c_3(2)\}t^3+\cdots$

Since $L_0=3$, we shall solve the coefficient equations of t, t^2, t^3 is an equation system. The coefficients of t, t^2, t^3 depends on $((c_1(0), c_2(0), c_3(0)), ((c_1(0), c_2(0), c_3(0), c_1(1), c_2(1), c_3(1)), ((c_1(0), c_2(0), c_3(0), c_1(1), c_2(1), c_3(1), c_1(2), c_2(2), c_3(2))$, respectively. Hence, we get the system of algebraic equations including $(c_i(j))_{i=1,2,3,j=0,1,2}\in\mathbb{C}^9$. Here we shall detect each $(c_i(j))_{i=1,2,3,j=0,1,2}\in\mathbb{C}^9$ which yield the system of algebraic equations.

In order to do this, we may take $(c_1(3), c_2(3), c_3(3))\in\mathbb{C}^3$ as arbitrary non-zero vector.

The change variable $x_1=u_2u_3^{-1}, x_2=u_1^{-1}u_2, x_3=u_1^2u_2^{-3}u_3^2$, are obtained by the image of curve $Q(t)$, which verifies (3.18) , (3.19) of Definition 3.2 and $lim_{t\to 0}f(X(t))=-2\in\mathcal{K}_\infty(f)$.

As a result, we construct the curve $X(t)$ approaches to the surface $\{x; f(x)=-2\}$ as $t\to 0$.

The curve $X(t)=(x_1(t), x_2(t), x_3(t))$ which is asymptotically approaching to the surface $\{x; f(x)=-2\}$ and consists of the following parameters by the found coefficients $(c_i(j))_{i=1,2,3,j=0,1,2}\in\mathbb{C}^9$:

$$x_1(t)=\frac{t^4+t^3+t^2+t-\frac{1}{3}}{t^4+\frac{131t^3}{256}-\frac{t^2}{4}+\frac{3t}{4}+\frac{2}{3}} \tag{4.1}$$

$$x_2(t)=\frac{t^4+t^3+t^2+t-\frac{1}{3}}{t^4+t^3+t^2+t} \tag{4.2}$$

$$x_3(t)=\frac{\left(t^4+\frac{131t^3}{256}-\frac{t^2}{4}+\frac{3t}{4}+\frac{2}{3}\right)^2\left(t^4+t^3+t^2+t\right)^2}{\left(t^4+t^3+t^2+t-\frac{1}{3}\right)^3}. \tag{4.3}$$

Solving this by the computer program Mathematica, we get the above explicit form. Note that this curve is one of many curves with the same characteristic.

In Figure 5.2, we pose two parts of the curve corresponding to the asymptotes as $t\to 0^+,\ t\to 0^-$.

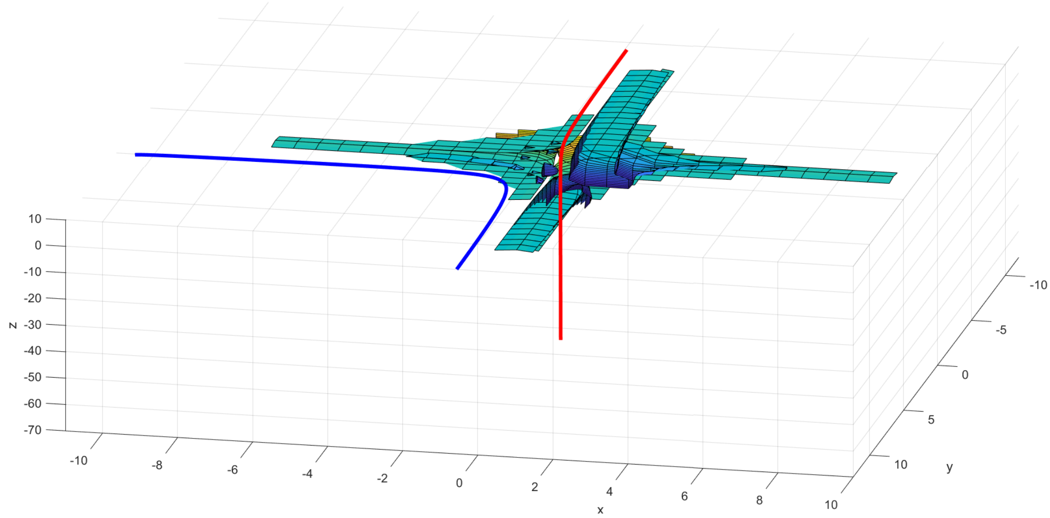

Figure 4.2 Branches of the curve $X(t)$

In Examples 17 and 18 figures explaining algebraic surfaces and rational parametric curves are drawn utilizing the computer program MATLAB.

4.2 Isolated Case

Example 18. (Isolated singularities at infinity) [23]
For a polynomial $f(x) = -3x^{v_0} + x^{v_1} + x^{v_2} + x^{3v_0}$ with $v_0 = (2,2,1), v_1 = (1,0,1)$, $v_2 = (0,1,1)$. By changing the variable, we will construct a unimodular matrix W and

$$M.\ W = (a_1{}^T, a_2{}^T, a_3{}^T) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -1 & -1 \\ 2 & 2 & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = (\mu_1{}^T, \mu_2{}^T, \mu_3{}^T) = \begin{pmatrix} -1 & -2 & -1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

A face γ of $\Delta(f)$ is only bad face and is located on the cone $\{t.v_0; t > 0\}$.
$f^W = u_1{}^3 u_2{}^2 u_3{}^2 + u_2 + u_3{}^3 - 3u_3$. To look for singularities over the algebraic torus, we restrict f over bad face.
The singular points of $f_\gamma^W(u) = u_3{}^3 - 3u_3$ are $u_3^* = \pm 1$ and critical values are ∓ 2. By the aids of [26], we pose that, in this case the bifurcation set $\mathscr{B}(f) \subset \mathscr{K}_\infty(f)$ contains $\{\pm 2\}$.

We will plain constructing a curve $X(t)$ which verifies (3.18) , (3.19) of Definition 3.2 and also the limit condition $lim_{t\to 0} f(X(t)) = -2$. Likewise, another curve may be also constructed, verifying $lim_{t\to 0} f(X(t)) = 2$.

Let us compute $\langle \mu_j, \vartheta_u f^W(u) \rangle$, $j = 1,2,3$ and detect the facet Γ that is the closest face

of the origin. For instance, $\left\langle \mu_3, \vartheta_u f^W(u)\right\rangle$ consist of the following
$3{u_1}^3{u_2}^2(U_3+1)^2 - u_2 + 3U_3(U_3+1)(U_3+2)$, with $U_3 = u_3 - 1$.
The facet Γ consists on the plane including $(3,2,0),(0,1,0),(0,0,1)$. By calculating, we get $q = (-1,3,3)$,i.e., $(q',0) = (-1,3,0)$ and implies that
$\langle (q',0), w_1\rangle = -1, \langle (q',0), w_2\rangle = -1, \langle (q',0), w_3\rangle = 4$. Likewise, $\mathbb{J} = \{3\}$, note that $card(\mathbb{J}) = 1$ and $L_0 = max_{i\neq j}\left\langle (q',0), w_i - w_j\right\rangle = 5$. Hence (3.17) consist the following parts
$u_1 = c_1(0)t^{-1} + c_1(1) + h.o.t., u_2 = c_2(0)t^3 + c_2(1)t^4 + h.o.t, u_3 = 1 + c_3(0)t^3 + c_3(1)t^4 + h.o.t.$ Indeed, providing that we put these terms into $\left\langle \mu_3, \vartheta_u f^W(u)\right\rangle$, the expansion with initial term t^3 ($\langle q, \alpha\rangle = 3$ for $\alpha \in \Gamma$) is obtained as follows;
$\{c_2(0) + c_1(0)^3 c_2(0)^2 + 6c_3(0)\}t^3 +$
$\{3c_1(0)^2 c_1(1) c_2(0)^2 + c_2(1) + 2c_1(0)^3 c_2(0) c_2(1) + 6c_3(1)\}t^4 +$
$\{3c_1(0)c_1(1)^2 c_2(0)^2 + 3c_1(0)^2 c_1(2) c_2(0)^2 + 6c_1(0)^2 c_1(1) c_2(0) c_2(1) + c_1(0)^3 c_2(1)^2 + c_2(2) + 2c_1(0)^3 c_2(0) c_2(2) + 6c_3(2)\}t^5 + h.o.t.$
The coefficients of t^3, t^4, t^5 depend on
$(c_1(0), c_2(0), c_3(0)), (c_1(0), c_2(0), c_3(0), c_1(1), c_2(1), c_3(1))$,
$(c_1(0), c_2(0), c_3(0), c_1(1), c_2(1), c_3(1), c_1(2), c_2(2), c_3(2))$, respectively. Namely, under the condition $-5 + ord\left\langle \mu_3, \vartheta_u f^W\right\rangle (Q(t)) > 0$, the real curve can be constructed. Even though we obtained $Q(t)$ whose minimum parametric length is 4, we have known that after the method of [7, Theorem 3.5.], the rational curve with needed properties has a length 3601.

We obtain the curve $X(t)$ as the image of the curve $Q(t)$ by the map $x_1 = u_1 u_3, x_2 = (u_1^2 u_2 u_3)^{-1}, x_3 = u_1^2 u_2^2 u_3$, as desired.
The curve $X(t) = (x_1(t), x_2(t), x_3(t))$ asymptotically approaching to the surface $\{x; f(x) = -2\}$ as follows:

$$x_1(t) = \left(t^2 + t + \frac{1}{t} + 1\right)\left(t^6 - \frac{8t^5}{3} - t^4 - \frac{t^3}{3} + 1\right), \tag{4.4}$$

$$x_2(t) = \frac{1}{\left(t^2 + t + \frac{1}{t} + 1\right)^2 \left(t^6 - \frac{8t^5}{3} - t^4 - \frac{t^3}{3} + 1\right)\left(t^6 + t^5 + t^4 + t^3\right)}, \tag{4.5}$$

$$x_3(t) = \left(t^2 + t + \frac{1}{t} + 1\right)^2 \left(t^6 - \frac{8t^5}{3} - t^4 - \frac{t^3}{3} + 1\right)\left(t^6 + t^5 + t^4 + t^3\right)^2. \tag{4.6}$$

Solving this in the computer program Mathematica, we get the above explicit form.

In Figure 5.3, we pose two parts of the curve corresponding to the asymptotes as $t \to 0^+,\ t \to 0^-$.

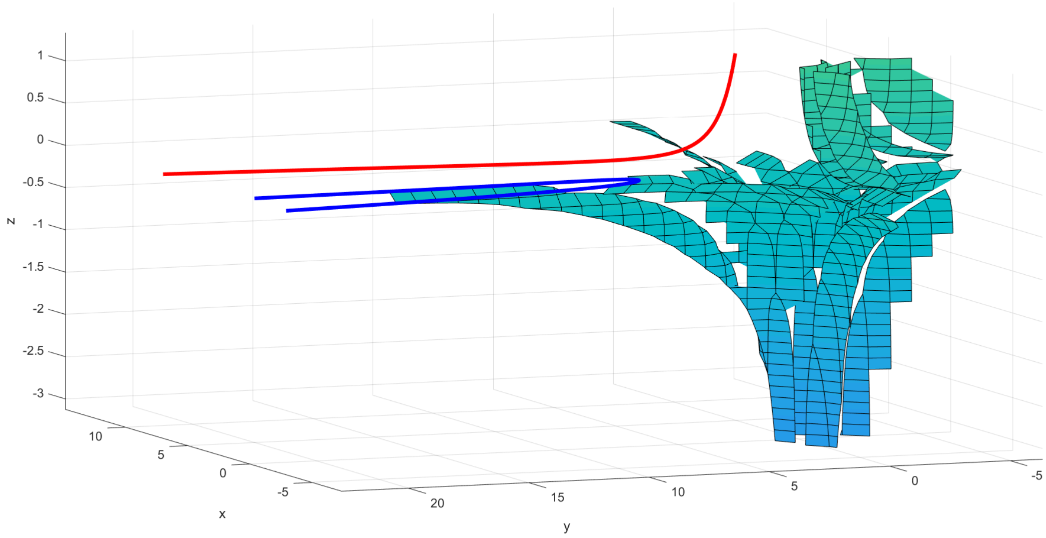

Figure 4.3 Branches of the curve $X(t)$

5
RESULTS AND DISCUSSION

Consider $f : \mathbb{C}^n \to \mathbb{C}$ a polynomial mapping and for $x \in \mathbb{C}^n$, $t \in \mathbb{C}$ such that $f(x) = t_x$. We focus on the problem of how the fiber $f^{-1}(t)$ changes topologically, when $t \in \mathbb{C}$ changes.

Namely, when $f^{-1}(t_0)$ is not diffeomorphic to other fibers $f^{-1}(t)$ for $t_0, t \in \mathbb{C}$? The t is element of the bifurcation locus of f which is denoted by $\mathscr{B}(f)$, if the fiber $f^{-1}(t)$ is not topologically equivalent to $f^{-1}(s)$ for any value $s \in \mathbb{C}$ near enough to $t \in \mathbb{C}$.

The bifurcation locus of a polynomial map f is the smallest subset of $\mathbb{C}$ such that f is a locally trivial $\mathbb{C}^{\infty}$- fibration over $\mathbb{C} \setminus \mathscr{B}(f)$.

Also, $\mathscr{B}_{\infty}(f)$ comprise values for which f is not a locally trivial fibration at infinity (i.e. outside a large ball) and is called the critical value of f at infinity.

In conclusion, the equality $\mathscr{B}(f) = f(\text{Sing } f) \cup \mathscr{B}_{\infty}(f)$ holds. Here $f(\text{Sing } f)$ is the critical value of polynomial at this point.

In general, the bifurcation locus $\mathscr{B}(f)$ is not equal to its bifurcation set $f(\text{Sing } f)$ which is showed by Broughton, in 1998 [1]. But, describing exactly the critical value set at infinity $\mathscr{B}_{\infty}(f)$ is a difficult task.

In the literature, some special sets called supersets including $\mathscr{B}_{\infty}(f)$ are defined to approach to $\mathscr{B}_{\infty}(f)$. We focus on the asymptotical critical value of polynomial mapping that is a superset and denoted by $\mathscr{K}_{\infty}(f)$.

The asymptotical critical value of polynomial mapping defined as

$\mathscr{K}_{\infty}(f) = \{y \in \mathbb{C} :$ there is a sequence $X(t)$, $lim_{t\to 0}\|X(t)\| = \infty$ and $lim_{t\to 0}\|X(t)\|\|grad\, f(X(t))\| = 0$ such that

$lim_{t\to 0}\|f(X(t))\| = y\}$ and $\mathscr{B}_{\infty}(f) \subset \mathscr{K}_{\infty}(f)$. It was introduced firstly with $\mathscr{K}_{\infty}(f)$ by Z. Jelonek and K. Kurdyka [5].

In this thesis, we firstly present an effective method to construct curves approaching the asymptotic critical value set of the polynomial map.

To this end, we suggest a way to construct rational curves with parametric representation with very few coefficients. At this moment, we pose that the asymptotic critical value set contains the critical value of a polynomial associated with the bad face of the Newton polyhedron. For this purpose, we use toric geometry as a tool, which has been introduced into the study of this question by A.Némethi and A.Zaharia [3]. Hence, we will give a method to the construction curve that approaching $t_0 \in \mathcal{K}_\infty(f)$ such that $lim_{t\to 0} f(X(t)) \in \mathcal{K}_\infty(f)$.

Secondly, we give a method to construct a curve approaching an asymptotic critical value of a real polynomial map, corresponding to detect real coefficients of the parametric representation of the curve.

As a result, if we take all exponents of $f(x)$ even, the infimum of this function will be the element of $\mathcal{K}_\infty(f)$. Finally, we hope that the study can be applied to optimization problems.

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PUBLICATIONS FROM THE THESIS

Papers

1. S. Tanabé, A. Gunduz, B. ERSOY, “On real curve construction for asymptotic critical value set of a polynomial map,” Comptes rendus de l’Académie bulgare des Sciences, vol. 74, no. 5, 2021.

Conference Papers

1. S. Tanabé, A. Gunduz, B. ERSOY, “Detection of Asymtotic Critical Values of Polynomial Mapping”, ICOMAA 2019 (Özet Bildiri/Sözlü Sunum)(Yay n No:6861504)

2. S. Tanabé, A. Gunduz, B. ERSOY, “On the real curve method for asymptotic critical value set of polynomial map”, ICOLES 2020 (Özet Bildiri/Sözlü Sunum)(Yay n No:6861515)

Projects

1. "Cebirsel Varyeteler İle İlgili Periyot İntegralleri", -Tübitak 1001, Araştırmacı: O. Kaya, Yürütücü:S. Tanabé, Bursiyer: A. Gündüz, 01/03/2017 - 01/03/2020, Grant No. 116F130