REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

APPLICATION OF LIE SYMMETRIES TO DIFFERENCE EQUATIONS AND BOUNDARY VALUE PROBLEMS

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DOCTOR OF PHILOSOPHY THESIS
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A thesis submitted by Sümeyra ÇAĞLAK in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY** is approved by the committee on 30.07.2019 in Department of Mathematics, Program of Mathematics.

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Dedicated to my dear husband and my sweet daughter

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LIST OF SYMBOLS

$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The Euclidean space
$\bar{x}$	Transformed variable under the action of a Lie group
$\xi(x)$	Infinitesimal of a Lie group
X	Infinitesimal generator of a Lie group
prX	Infinitesimal generator of a Lie group that is prolonged to the space with derivatives
L_r	r-dimensional Lie algebra
∂y	The set of all first-order partial derivatives of the dependent variable y with respect to the independent variables
$\underset{+h}{S}$	Right discrete shift operator
$\underset{-h}{S}$	Left discrete shift operator
$\underset{+h}{D}$	Right finite-difference differentiation operator
$\underset{-h}{D}$	Left finite-difference differentiation operator
$\tilde{Z}$	The space of differential variables
$\underset{h}{Z}$	The space of finite-difference variables
$\underset{h}{\tilde{Z}}$	The product of the spaces $\tilde{Z}$ and $\underset{h}{Z}$

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ABSTRACT

Application of Lie Symmetries to Difference Equations and Boundary Value Problems

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Department of Mathematics
Doctor of Philosophy Thesis

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Lie groups of point transformations are applied to nonlinear hyperbolic partial differential equations in particular the sine-Gordon equation. Point symmetries of finite difference scheme for the sine-Gordon equation are obtained via methods developed from the existing literature. The method is extended to nonlinear partial difference equations and based on an algorithm that determines infinitesimal generators of the equation. Symmetries that leave the equation and the mesh invariant simultaneously is presented. It is shown that the sine-Gordon equation conserves the entire symmetry of the original differential form in its finite-difference model. Boundary value problems for differential and difference equations are also considered and the invariance of their boundary curves and boundary conditions under the Lie point symmetries of the associated equations is analyzed in both differential and difference forms. Symmetries affect the equations, mesh, boundaries and boundary conditions at the same time. The invariant discretization of the difference problem corresponding to boundary value problem for the sine-Gordon equation is studied.

Keywords: Lie group analysis, Finite difference schemes, Point symmetries, Hyperbolic equations, Nonlinear boundary value problems.

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ÖZET

Lie Simetrilerinin Fark Denklemlerine ve Sınır Değer Problemlerine Uygulanması

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Matematik Anabilim Dalı
Doktora Tezi

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Lie nokta dönüşüm grupları, doğrusal olmayan hiperbolik kısmi diferansiyel denklemlere özellikle sinüs-Gordon denklemine uygulandı. Sinüs-Gordon denkleminin sonlu fark şemasının nokta simetrileri, mevcut literatürden geliştirilen yöntemlerle elde edildi. Metot doğrusal olmayan kısmi diferansiyel denklemlere genişletildi ve denklemin sonsuz küçük üreteçlerini belirleyen algoritmaya dayandırıldı. Denklemi ve latisi aynı anda değişmez bırakan simetriler gösterildi. Sinüs-Gordon denkleminin, orijinal diferansiyel formun tüm simetrisini sonlu fark modelinde koruduğu gösterildi. Diferansiyel ve fark denklemleri için sınır değer problemleri de ele alındı ve hem diferansiyel hem de fark formlarında sınır eğrilerinin ve sınır koşullarının değişmezliği, ilgili denklemlerin Lie nokta simetrileri altında analiz edildi. Simetriler denklemleri, latisi, sınırları ve sınır koşullarını aynı anda etkiler. Sinüs-Gordon denklemi için sınır değer problemine karşılık gelen fark probleminin değişmez ayrıklaştırılması çalışıldı.

Anahtar Kelimeler: Lie grup analizi, Sonlu fark şemaları, Nokta simetrileri, Hiperbolik denklemler, Doğrusal olmayan sınır değer problemleri.

YILDIZ TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

1
Introduction

1.1 Literature Review

1.1.1 General Overview to Symmetry

Symmetry has a universal, amazing effect in nature since the beginning of life. It is an outstanding and powerful tool clarifying the regulations of the physical laws. A typical example of this feature is performing experiments in various places at several times. Since physical laws are invariant with respect to space translation and rotation and time translation, experiments can again be done [1].

Brian J. Cantwell expresses symmetry as an immense part of our conscious life and gives distinguished symmetry examples from nature in his excellent book [2]. The symmetries existing in physical laws rule the natural world. These symmetries provide us to understand of complex physical events and to formulate physical laws, simplify solving problems and improve our understanding of nature.

Since symmetry is a native and simple concept in nature, it has an important role in different branches of science especially in several areas of mathematics such as analysis, differential geometry, algebra, differential equations, numerical analysis; in chemistry, engineering and in many subjects of physics such as classical mechanics, quantum mechanics, atomic structure, high energy physics, scattering theory, shock waves and so on. In general, a symmetry is a transformation of an object leaving the object apparently unchanged. This feature may appear in all areas of life. Particularly, in the study of symmetries in mathematics, our objects are differential or difference equations and our transformations are point symmetries, contact symmetries, generalized symmetries, etc. preserving these equations or connecting them [3].

Symmetries are defined by qualities of their associated objects and this provides us to characterize and learn these objects. This might be observed trivial for symmetries of geometric objects. But in the course of symmetry analysis in mathematics, i.e., in the research of symmetries of differential or difference systems, symmetries inform critical data about structure of the equation such as solution set of the equations, integrability or linearization of the equations, etc. Inverse group classification is a remarkable example of this result. It states that a differential equation can be denoted in regard to differential invariants of the Lie symmetry group admitted by the equation [4].

1.1.2 Historical Development of Lie Groups

In the nineteenth century the Norwegian mathematician Sophus Lie proposed to improve a general theory to integrate ordinary differential equations (ODEs). The theory advanced by Evariste Galois and Niels Henrik Abel to solve algebraic equations led the way him to realize his aim. He developed the theory of continuous groups of transformations and applied it to differential equations during the period 1872–1899 [5, 6]. Now, we briefly give the outlines of history of Lie group theory [7, 8] that allow us to understand Lie's viewpoint.

1.1.2.1 Lie Group Theory and Abstract Algebra

In the beginning of the nineteenth century famous mathematicians like Lagrange, Gauss, and Abel examined a method for solving higher-order algebraic equations. They wanted to find an algebraic equation with reduced order having a solution set that consists of functions of the corresponding higher order equation. Galois developed a general criteria for solving algebraic equations. Galois theory was based on the notion of groups which are symmetries implicit in the solution set of algebraic equations commuting the roots of the equations. Camille Jordan who was studying with two doctorate students named Sophus Lie and Felix Klein wrote a book about Galois Theory in 1870. Lie began his mathematical studies when he was 26 at the University of Christiania. After he took course about Galois theory from Ludwig Sylow, he decided with Klein to investigate a procedure for solving differential equations like Galois theory for solving algebraic equations. In Galois theory functions of a Galois group commute roots of the corresponding polynomial; in Lie's theory transformations of a Lie group move solutions of the corresponding differential equation into solutions. Another connection between abstract algebra and Lie group theory is about solvability of equations. In abstract algebra a polynomial is solvable if and only if the Galois group corresponding to the polynomial is solvable. Similarly, in the study of differential equations if an equation admits a solvable Lie group then it is solvable. A system of ODEs acts like an auxiliary polynomial equation corresponding to the Galois group of a polynomial and is used to examine its algebraic solvability.

1.1.2.2 Lie Group Theory and Differential Geometry

Lie's interest was focused on geometry when he started university. He concerned with the studies of both Jean-Victor Poncelet and Julius Plucker about line geometry. He considered a tetrahedral line complex ς and a set T consisting of transformations that act on vertices of the tetrahedral. In Lie's group theory the elements of T correspond to transformations of a Lie group and vertices of tetrahedron ς correspond to orbits of the group. For the study of differential equations he examined a complex cone with vertex $a \in \varsigma$ and the set of all lines passing through the vertex point. Then the problem is to determine all surfaces having the feature that each point of surfaces intersects with the cone at the point a. This means that to find tangent vector field of surfaces which intersect with the cone at only one straight line. The geometrical meaning of this problem can be expressed by a first-order partial differential equation (PDE) as

$$f(x,y,z,a,b)=0,\ a=\frac{\partial z}{\partial x},\ b=\frac{\partial z}{\partial y}, \tag{1.1}$$

which will be later defined as *differential equation admitting (three-parameter x,y,z) group of transformations T*. Geometrically any transformation of T moves one surface into another.

Lie and the scholar of Plucker, Klein dealt with similar geometry problems. Klein played important role in Lie's academic performance. They studied together with Jordan on homogeneous curves invariant under a group that are called W-Curves. This work directed to Lie's attention to consider one-parameter subgroups of a group of projective transformations and led him to explore contact (tangent) transformations, that map surfaces in a space including points and their tangents. He was influenced by the studies of Gaston Darboux which combine the subjects differential geometry and differential equations. These studies contributed to develop Lie groups. Lie and Klein worked on geometry, symmetry, and group theory together for a while, then they turned to different areas shortly after.

Lie directed to his study from contact transformations to groups of transformations when he was professor in Norway and this brought about the idea of Lie groups. He worked on stability groups of differential equations for integrating the equations. Lie was inspired from Galois' hypothesis of commuting roots of algebraic equations, and considered a point transformation leaving a differential equation invariant if it commutes the solutions of the equation. This was the starting point of the theory of Lie groups. Lie found out that continuous symmetries of a differential equation could be used for solving the differential equation or reducing its order.

1.1.2.3 Studies about Lie Group Theory of Differential Equations

The method of symmetry analysis is mainly used to obtain exact solutions of differential equations. For an ODE symmetries can help to find its general solution reducing the order of the equation. For a PDE a symmetry group decreases the number of independent variables in the equation and this is sometimes resulted in reducing a PDE to an ODE. Symmetries also supply particular solutions called group invariant solutions. Group invariant solutions produce some special functions and satisfy boundary conditions for a boundary value problem (BVP). In many nonlinear systems of differential equations these solutions are only exact solutions and provide important information for mathematical and physical applications [9]. While obtaining group invariant solutions the invariants of a Lie symmetry group are described as new variables in the corresponding differential equation. For a PDE the number of independent variables $n \geq r$ is reduced by r if the equation admits an r-parameter symmetry group. Then any solution of the reduced differential equation forms a particular solution of the original differential equation. Elementary examples of Lie groups are translations, rotations, scalings.

A differential equation has infinitely many solutions by means of additive constants. If there is a proper initial condition then the solution is unique. Symmetries are transformations defined on the set of solutions of a differential equation and integration constants can be thought as parameter of the admitted symmetry group. Hence symmetries are used to map a solution of a differential equation to another solution. This leads to obtain new exact solutions from known ones or to produce series of exact solutions.

Another application of symmetries is to determine equivalent differential equations and connect them to each other. Equivalent differential equations that can be transformed to each other by point transformations, i.e., equivalent differential equations have the similar symmetry groups with respect to the same point transformations. An advantage of this property is to linearize nonlinear equations finding a linear differential equation similar to a nonlinear one. In fluid mechanics, the system of one-dimensional shallow-water equations, which is similar to a system of linear differential equations by a hodograph transformation is an apparent example of this situation [10].

Lie symmetries evaluate the entire set of point symmetries of a differential equation. There are two different approaches to determine all group of point transformations, i.e., point symmetries for differential equations. In the first approach every continuous or discrete point symmetry of the given differential equation corresponds to a bijective transformation of the associated Lie algebra. This condition results in a restriction on the general form of a transformation that can be a group of point symmetries of the equation. This restricted form of a general point transformation is substituted into the determining equations that evaluate the symmetries of a given differential equation and the continuous symmetries are separated into factors. Thus the group of discrete symmetries is determined [11]. There are some subalgebras which are invariant under any bijective transformation of the given Lie algebra. These algebras simplify the use of the first method. The second method computed complete point symmetries of a differential equation in a simpler way using information on the set of transformation group admitted by a differential equation.

In the construction of group-invariant or partially invariant solutions action of invariants of a Lie symmetry group is important. Invariants of a Lie symmetry group can be determined using the infinitesimal generators of the Lie symmetry group, solving a semi-linear characteristic system of differential equations that are obtained from the generators [4]. In the method of moving frame that is based on finite symmetry transformations, differential invariants of the symmetry group acting on differential terms in a finite order are obtained following the same way [12]. According to inverse group classification any differential equation can be symbolized with respect to the differential invariants of the symmetry group admitted by the equation. Thereby, the general type of differential equations represented by a symmetry group is found out using differential invariants.

The presence of conservation laws in physics and mathematics is of significant inference and application area of symmetries. In 1918 Noether [13] realized the classical relation between symmetries and conservation laws proving her well-known theorem. Conservation laws help us to know the characteristics of the corresponding differential equations and give essential information about the integrability of these equations.

Lie pointed out that the concept of infinitesimal operator which generates a one parameter group include all existing integration theories. Even though, the Lie's discovery was not used in the study of differential equations for a while and only the abstract part of Lie group theory developed. In his study about *The Theory of Differential Equations* published in 1906, Forsyth [14] wrote about theory of Lie groups and Bäcklund transformations. The first few chapters of the book by Cohen [15] describe the notion of a Lie's theory and the concept of invariance under a transformation group. However, studies about the theory of Lie groups were interrupted until after World War II. As nonlinear problems were studied more and more often, importance of symmetries was recognized and Lie's ideas began to attract attention again. The term Lie group was added to literature in 1928 by Hermann Weyl. In the United

States, the study of Birkhoff [16] and Sedov [17] related the subjects group invariance and dimensional analysis applying Lie groups to problems of fluid mechanics. Lie symmetry method for differential equations were exploited systematically to obtain general solutions of any type of problems including complex nonlinear ones of mathematical physics in Russian school led by Ovsiannikov [4]. During the last few decades, many scientists have interested in Lie's theory and numerous study has invested for the theory or application of symmetry analysis of differential equations [2, 4, 9, 10, 18–27].

For a differential equation the admitted Lie group is calculated by an algorithm which is based on infinitesimals of the group and consists of three steps: writing determining equations, splitting these equations with respect to arbitrary elements and solving these equations [4, 19, 20, 28]. Miller [29] studied on relation between invariant solutions and separation of variables. Applications of Lie symmetry method to numerical analysis can be found in studies of Shokin and Dresner [30–32]. The method is applied to control theory by Shaft [33], Ramakrishan and Schaettler [34]. The process of the method for applying to BVPs is given in books [35–37] and in papers [38, 39]. Kaya and Iskandarova [40, 41] studied Lie symmetry analysis of BVPs for some fractional differential equations. There are also many important articles [42–46] (and the references given therein) about symmetry analysis and classification of heat and wave equations.

Lie group theory is applied to differential equations in an algorithmic way. But the determination of Lie groups of differential equations by hand is time-consuming and there may be many errors. As the number of the symmetry variables, and the order of the differential equations increases, solving the related equations becomes more complicated. There are many packages that use software programs with symbolic manipulations, such as Mathematica, Macsyma, Maple, Reduce, Axiom, MuPAD [47–51] to perform symmetry analysis of differential equations.

1.1.2.4 Studies about Lie Group Theory of Difference Equations

Lie symmetry method is a powerful tool when studying differential equations in many respects such as reducing the order of an ODE, obtaining invariant solutions for PDEs, conservation quantities, or relating different differential equations by equivalence transformations. Constructing finite difference schemes that preserve symmetries of its continuous model has become an important application area in the last 30 years. Substantial efforts have been devoted to extending symmetry integration techniques of finite difference equations [26, 27, 52–55]. Traditionally, these symmetry integration methods rely on the infinitesimal generators of the Lie groups admitted by the equation.

Symmetry analysis of difference equations and meshes on which difference equations are written can be done using several methods. Yanenko [56] and Shokin [31], applied the Lie group theory to finite difference equations with the viewpoint of differential approximation. Thus, they have set down conditions to preserve continuous symmetries of a differential equation in its finite difference form. They performed numerical experiments and obtained difference schemes that are more accurate than non-invariant ones, after a frame transformation. Ames et al. [57] showed that, in some cases, the method of differential approximation produces accurate difference schemes like higher order numerical methods. Hoarau et al. [58] developed a "semi-invariant scheme" using the differential approximation that is invariant with respect to the symmetry group of the corresponding differential equation.

In the method of Dorodnitsyn, the set of all difference invariants of the corresponding Lie group is used to construct the invariant difference schemes. The starting point of this technique is a differential equation and its Lie symmetry group. Then a difference scheme- i.e., a difference equation and a mesh- that preserves all symmetries of the given differential equation is searched. The process of this method is presented in the papers of Dorodnitsyn and co-workers [59–63] and summarized in the book [26].

The method of Levi and his collaborators [53, 55, 64–68] starts from a difference equation and a mesh. Then a group of transformations leaving the mesh invariant for the given difference system is obtained. This supplies solving the equations, transforming solutions into solutions, classifying and identifying equations. This method can be applied for different types of meshes not necessarily uniform. Infinitesimal generators with coordinates of dependent and independent variables are used when only point symmetries are considered. But, if we want to find a difference model that conserves the entire set of symmetries of a differential equation in the continuous limit, we have to generalize the concept of point symmetries.

Fels and Olver [12, 69–71] improved an alternative method of equivariant moving frames essentially for symmetry preserving difference models. This approach is based on transformation of difference variables in the discrete space according to the symmetries of corresponding PDEs. In this case the considered symmetries leave invariant the final version of the transformed difference model. Ozbenli and Vedula [72] presented a numerical approach to develop higher-order accurate invariant numerical schemes with desired order of accuracy or to solve PDEs.

The method of Dorodnitsyn and Levi is based on infinitesimal criterion and needs to solve a system of PDEs while the method of equivariant moving frames needs to solve a system of algebraic equations. In both methods difference invariants which converge to differential invariants of the corresponding symmetry group in the continuous limit are produced. Both approaches are used to construct invariant ordinary and partial difference schemes in the literature.

1.2 Objective of the Thesis

In mathematical modeling of natural phenomena and problems that occur in engineering, physics, biology and various branches of science are generally nonlinear. Linear superposition principle is not proper for nonlinear differential equations to obtain exact solutions. Hence the well-known techniques like the Fourier method, the Laplace transform method can not be applied to nonlinear differential equations. Actually, to construct a general theory for solving nonlinear differential equations is not possible. However, various solution methods such as the method of compatible differential constraints, the inverse scattering transform method, sine–cosine method, tanh-method, the exp-function method, the method of additional generating conditions, similarity reductions method (Lie symmetry method), (G'/G)-expansion method, homogenous balance method and the transformed rational function method have improved for the study of nonlinear problems. Lie symmetry is the most commonly used method among them. Many classical methods such as separation of variables method, the method of integrating factor, self-similar solutions, the method of undetermined coefficients are particular cases of this technique. The method provides a comprehensive way to solve differential equations converting difficult nonlinear conditions to simple linear ones. Another advantage of this method is its applicability to any type of equations including ordinary or partial, linear or nonlinear, constant or variable coefficient. A deficiency of this method appears in the study of first-order ODEs. Because they admit infinite dimensional Lie groups and moreover there is not any standard technique to obtain an one-dimensional Lie group for them. As a result of this situation Lie symmetry method is applied to problems of biology or epidemiology less than problems of physics. This is because, in general, the problems in these fields are modelling of first-order ODEs while the problems in physics are of second-order. However, Nucci [73] applied Lie symmetry method to an epidemiologic model which describes human immunodeficiency virus (HIV) transmission in male homosexual/bisexual groups.

A striking feature of symmetries is to obtain group invariant solutions. Invariant solutions provide us to characterize the associated differential equations with respect to linearity, integrability or solvability of them. Thus Lie symmetry analysis has been employed to find new exact solutions in the study of PDEs by many researchers and there are numerous publications in this field. But there is not much work about Lie group study of partial difference equations. The main purpose of this thesis is consequently to apply Lie symmetry method to some difference models that approximate to PDEs particularly the sine-Gordon equation. This type of equation was chosen because it is a kind of wave equations that are one of the most practical models in many areas of science like fluid mechanics, plasma physics, hydrodynamics and general relativity. The studies about finding exact solutions and classification of wave equations have continued to be an important area of interest. Some of the well-known examples are shock waves, water waves, solitons and solitary waves.

The sine-Gordon equation is a nonlinear hyperbolic PDE consisting of the sine of the unknown function. This type of equation was first introduced by Bour [74] and rediscovered by Frenkel and Kontorova [75] in their work on crystal dislocations. Such equations were studied by many researchers in the 1970s due to the existence of soliton solutions. The sine–Gordon equations are of particular interest as a model field theory for elementary particles in quantum theory. However, it also has been used as a model in the theory of the splay waves in lipid membranes and magnetic flux on a Josephson line, solid state physics, nonlinear optics, Bloch-wall motion, stability of fluid motions and the motion of a rigid pendulum attached to a stretched wire.

In the present research we aim to analyze difference scheme for the sine-Gordon equation by means of Lie symmetries. We want to construct invariant finite difference schemes that conserve the symmetries of the differential equation being simulated into the finite difference simulation and to obtain point symmetries of the discrete sine-Gordon equation improving a methodology from the current one. We also propose to determine invariance of some BVPs for the sine-Gordon equation with respect to the Lie groups admitted by the governing equations.

1.3 Hypothesis

In order to practice Lie symmetry method on partial difference equations we concentrate on difference model of the sine-Gordon equation. We will use a process for obtaining Lie point symmetries acting on the difference equation and the mesh simultaneously. The procedure is in line with a variation of that used by Levi et al. [53] with some developments. In [53], a method is presented to evaluate point symmetries of an ordinary difference equation. We extend this method to nonlinear partial difference equation in particular the discrete sine-Gordon equation in the multidimensional case. We will construct the difference equation on a mesh given by discretely varying two equations. We will use infinitesimal generators with coordinates of dependent and independent variables to determine the point symmetries. The symmetry transformations left invariant the solution set of the difference scheme.

A PDE cannot express any natural model without additional (initial or boundary) conditions. However the studies about Lie symmetry analysis of BVPs (we assume initial value problems as a particular case of BVPs) is very limited. Applications of Lie symmetries on BVPs have some difficulties since every symmetry of a BVP must be a symmetry of the given equation, the domain and the boundary data at the same time. But the prescribed initial or boundary conditions are generally not invariant under the transformation groups of the corresponding equations. Within the framework of these criteria we will analyze the invariance of BVP for differential and discrete form of the sine-Gordon equation under the Lie group of point transformations admitted by the associated equation. The point symmetries act on the boundary conditions, the equations and the mesh. In order to preserve structure of boundary curves we will choose an unbounded domain for the problem in the differential case and a mesh which is a set of points lying in the plane and stretching in all directions with no boundaries for the discrete problem. We will use an invariance definition developed by Chernica [76] for a wide class of BVPs. The mentioned definition was used in the application of Lie symmetries to BVPs for PDEs (see, e.g., [76, 77] and the papers cited therein). We attempt to use the given definition for a nonlinear discrete problem.

When studying BVPs we ignore the solutions that do not satisfy the boundary conditions. Hence symmetry of boundary data and symmetry of domain are necessary conditions for symmetries of BVPs while symmetry of differential equation is not. As a result of this situation, a BVP may have different symmetries to those of the differential equation. The same idea is equally valid for BVPs of difference equations. In this context, we will examine symmetry preserving finite difference model of BVP for the sine-Gordon equation.

This thesis is organized as follows: The second chapter introduces basic theory of Lie groups and explains how Lie symmetry method is used in differential, difference equations and BVPs for PDEs. Section 3 investigates our main results which consist of applications of Lie symmetry method to the discrete sine-Gordon equation and some BVPs for this equation. It generalizes some approaches and definitions to the case of partial difference equations. The last chapter is devoted to results and discussion of the research.

2
Lie Symmetry Analysis of Differential and Difference Equations

The concept of symmetry is seen in all mathematical models represented by differential equations. The term of continuous groups developed by Sophus Lie provides to understand the symmetry underlying differential equations. Lie group theory for differential equations was motivated from Galois' group theory for algebraic equations. Lie group analysis is an effective way for integration, linearization and obtaining general solutions of differential equations.

Dimensional analysis provides us to determine basic dimensions and elementary quantities that can be seen in real world problems. If the considered problem expresses a BVP for a system of PDEs, then dimensional analysis can reduce the number of independent variables. There is a connection between dimensional analysis and finding solutions of BVPs for PDEs by means of invariance under a Lie group. For PDEs, reducing the number of independent variables via dimensional analysis is a particular case of reduction by invariance with respect to scaling (stretching) transformations group. The scaling transformations of dimensional analysis are examples of Lie groups.

In the study of Lie group analysis of differential equations there are much broader classes of transformations other than scalings. A Lie group admitted by a differential equation consists of transformations that take a solution of the equation into another solution. Lie groups are classified according to the space of variables on which they act. A Lie group of point transformations affects the dependent and independent variables of the corresponding differential equation. A Lie group of contact or tangent transformations affects dependent variables, independent variables and all first derivatives of the dependent variables. Furthermore, there are also higher-order symmetries or Lie-B*ä*cklund groups which are generalization of contact transformations group and affect independent variables, dependent variables and their derivatives to some fixed order. In contrast to point symmetries or contact symmetries, they act on an infinite-dimensional space. However, Lie's algorithm can be extended to determine these groups of transformations. Noether [13] showed that if there exist higher-order symmetries they can be applied to the theory of conservation laws.

The theory of Lie groups is based on infinitesimal generators. A Lie transformations group is described by infinitesimal generator of the group. Information of corresponding infinitesimal generator is essential for the application of Lie groups to differential or difference equations. A Lie group of transformations or an infinitesimal generator admitted by a differential equation can be computed by an algorithmic process which is called Lie's algorithm. Prolongation of a Lie transformations group and accordingly its infinitesimal generator is necessary to act on the terms with derivative. For a given linear or nonlinear differential equation, infinitesimal generators of the admitted group is determined solving a system of linear homogeneous equations (determining equations). These equations can be solved using symbolic manipulation programs [2, 48–51, 78]. For a multiparameter Lie group, infinitesimal generators corresponding each parameter construct a Lie algebra.

The theory of Lie group analysis of difference equations is also given by infinitesimal generator acting on the space differential and difference variables and almost same processes with differential equations are valid.

In this chapter, we give a brief overview for group analysis of differential and difference equations, which is presented in [4, 9, 10, 20, 22, 24, 26].

2.1 Basic Theory of Lie Groups

The notion of symmetry of objects in nature makes more clear symmetries of differential equations. In the previous chapter we have mentioned that symmetry of a geometrical object is a transformation which maps the object to itself, i.e., it leaves the appearance of the object the same. Invertibility and preserving of structure are the necessary conditions for a transformation to be a symmetry. Trivial symmetries which are transformations mapping each point to itself are seen in any geometrical object. In the plane, the set of horizontal lines, shown in Figure 2.1, has an infinite number of symmetries. Because rotating, shifting or flipping them about x-axis or y-axis or both, again gives the set of horizontal lines in the plane. The transformations $T_1 : (x, y) \rightarrow (e^{\alpha}x, y)$, $T_2 : (x, y) \rightarrow (\alpha x, y)$, $T_3 : (x, y) \rightarrow (x + \alpha, y)$ are trivial symmetries for this set of lines since they move each line into itself.

Discrete symmetries of geometrical objects are independent of some continuous

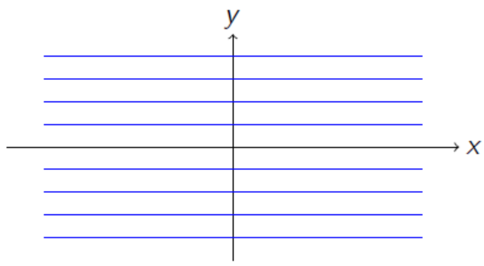

Figure 2.1 The family of lines $y = k$

parameters and they define non-continuous transformations for objects. For example, rotations of a square by multiples of right angles conserve the square's original appearance and form a discrete rotational symmetry. The unit circle $x^2 + y^2 = 1$ admits the reflections about the y-axis $T : (x, y) \to (x, -y)$ as a discrete symmetry because the transformed points are still on the unit circle, that is $x^2 + (-y)^2 = 1$. But trivial symmetries and discrete symmetries are not appropriate for our research. In this thesis, we study on symmetries which are dependent of some continuous parameters, for example the rotations of the unit circle about its centre by any number. These rotations form a continuous group of transformations. Now, we adapt the concept of symmetry to differential equations. The materials of this section are based on contents of [11, 20, 22, 24, 26].

2.1.1 Lie Group of Transformations

We first introduce the concept of group to better understand Lie groups. We generally study on n-dimensional Euclidean space in this chapter.

Definition 2.1 (Group)**.** A set of elements G with a binary operation ψ is called a *group* if the following four conditions (group axioms) are satisfied:

1. (*Closure property*) Composition of any two elements form a third element in the set. That is, for any $A, B \in G$, $\psi(A, B) \in G$.

2. (*Identity element*) There is a unique identity element $I \in G$ such that

$$\psi(A, I) = \psi(I, A) = A.$$

3. (*Inverse element*) For each element $A \in G$ the set contains a unique element A^{-1} such that

$$\psi(A, A^{-1}) = \psi(A^{-1}, A) = I.$$

4. (*Associative property*) For any $A, B, C \in G$:

$$\psi(A, \psi(B, C)) = \psi(\psi(A, B), C).$$

Moreover, a subset of G that forms a group with the same binary operation ψ is called a *subgroup* of G.

Definition 2.2 (One-parameter transformations group). Let $x \in D \subset \mathbb{R}^n$, i.e., $x = (x_1, x_2, \cdots, x_n)$ and

$$\overline{x} = X(x; \alpha) \tag{2.1}$$

be the set of transformations for each $x \in D$ and parameter $\alpha \in E \subset \mathbb{R}$, with $\psi(\alpha, \beta)$ defining a composition law of parameters $\alpha, \beta \in E$. Then this set forms a *one-parameter transformations group* if group axioms hold:

1. (*Closure property*) For each $\alpha \in E$, the transformations are one-to-one and onto.
2. (*Inverse element*) The set E with operation ψ constitutes a group structure.
3. (*Identity element*) There exists an identity element α_0 such that

$$X(x, \alpha_0) = x \tag{2.2}$$

   for each $x \in D$. That is, $\overline{x} = x$ when $\alpha = \alpha_0$ is the identity element.

4. (*Associative property*) If $\overline{x} = X(x; \alpha)$ and $\overline{\overline{x}} = X(\overline{x}; \beta)$, then

$$\overline{\overline{x}} = X(x; \psi(\alpha, \beta)).$$

Definition 2.3 (One-parameter Lie transformations group). A one-parameter transformations group describes a *one-parameter Lie transformations group* if the following conditions are satisfied:

5. α is a continuous parameter, i.e., E is an interval in $\mathbb{R}$.
6. X is infinitely differentiable with respect to $x \in D$ and it is an analytic function of $\alpha \in E$.
7. $\psi(\alpha, \beta)$ is an analytic function of α and β, $\alpha, \beta \in E$.

A one-parameter Lie group is parametrized with respect to analyticity of the group operation ψ such that the group operation is transformed to the ordinary sum in real space.

Some examples for one-parameter transformations group are given as:

- The translations group on a vector $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$,

$$\overline{x_i} = x_i + \alpha, \; i = 1, 2, \dots, n.$$

- The group of rotations on the plane,

$$\overline{x_1} = x_1 \cos\alpha + x_2 \sin\alpha, \; \overline{x_2} = -x_1 \sin\alpha + x_2 \cos\alpha.$$

- The group of scalings on the plane,

$$\overline{x_1} = \alpha x_1, \; \overline{x_2} = \alpha^2 x_2, \; 0 < \alpha < \infty.$$

- The group of projective transformations on the plane,

$$\overline{x_1} = \frac{x_1}{1 - \alpha x_1}, \; \overline{x_2} = \frac{x_2}{1 - \alpha x_1}.$$

Consider a point (x, y) in the plane and a one-parameter (α) Lie transformations group

$$T_\alpha : (x, y) \rightarrow (\overline{x}, \overline{y}) = (X(x, y; \alpha), Y(x, y; \alpha))$$

acting on this point. As the parameter α changes, the point (x, y) moves following a continuous path as shown in Figure 2.2. This curve is named *orbit of the group*.

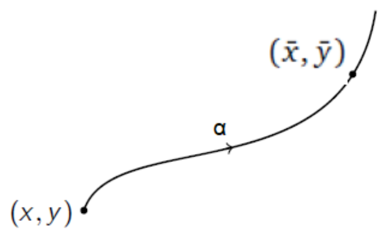

Figure 2.2 Orbit of a one-parameter Lie group

2.1.2 Group Generators and Lie Equations

Consider a one-parameter Lie transformations group (2.1). The expansion of (2.1) into Taylor's series in α about some neighborhood $\alpha = 0$, with initial condition (2.2) gives the *infinitesimal transformation*

$$\overline{x} = x + \alpha\xi(x) + O(\alpha^2) \tag{2.3}$$

where

$$\xi(x) = \frac{\partial X(x;\alpha)}{\partial \alpha}\Big|_{\alpha=0}. \tag{2.4}$$

The terms with order α^2 and higher are ignored to linearize $\overline{x}$ and consequently to obtain the slope of the curve which is the orbit of the group. Since the function $\xi(x)$ is the tangent vector to the curve on which the point x is mapped to the point $\overline{x}$, it is called the *tangent vector field* of Lie transformations group (2.1). The components of $\xi(x)$ are called the *infinitesimals* of (2.1).

The first-order differential operator

$$X = X(x) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i} \tag{2.5}$$

is called the *infinitesimal generator* of one-parameter Lie transformations group (2.1).

Given an infinitesimal transformation (2.3) or an infinitesimal generator (2.5) a group of transformations (2.1) is found by integrating the following system of ODEs which is called the *Lie equations*:

$$\frac{d\overline{x}}{d\alpha} = \xi(\overline{x}),\ \overline{x}|_{\alpha=0} = x. \tag{2.6}$$

There is another way to determine a Lie transformations group from its infinitesimal generator using the Taylor series expansion:

Theorem 2.1. *A one-parameter Lie transformations group of form (2.1) with infinitesimal generator (2.5) is expressed in terms of a power series*

$$\overline{x} = e^{\alpha X}x = x + \alpha X x + \frac{1}{2}\alpha^2 X^2 x + \cdots = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} X^k x, \tag{2.7}$$

where the operator $X^k = X^k(x)$ *is defined by* $X^k = XX^{k-1},\ k = 1, 2, \ldots$.

2.1.3 Invariance under a Lie Group

Definition 2.4. Let $f(x)$ be an infinitely differentiable function. Then it is called an *invariant function of Lie transformations group* (2.1) if the following equation holds:

$$f(\overline{x}) = f(x). \tag{2.8}$$

The following theorem shows a very simple way for determining invariance of a function under a Lie group using the infinitesimal generator of the group.

Theorem 2.2 (Invariance Condition). *For a function $f(x)$ to be invariant with respect* *to a Lie transformations group (2.1) necessary and sufficient condition is*

$$Xf(x) = 0. \tag{2.9}$$

Equation (2.9) is called invariance condition of the function $f(x)$ under a Lie group (2.1).

The following example shows simply how to find invariants of a Lie group and is given in [26].

Example. The group of Galilei transformations in two dimensional space is given by

$$\overline{x_1} = x_1 + \alpha x_2, \ \overline{x_2} = x_2. \tag{2.10}$$

The corresponding infinitesimal generator is

$$X = x_2 \frac{\partial}{\partial x_1}. \tag{2.11}$$

Then the group invariants are computed using equation (2.9) for operator (2.11) by

$$x_2 \frac{\partial F}{\partial x_1} = 0. \tag{2.12}$$

The solution of (2.12) gives the only invariant for group (2.10) as

$$F_1 = x_2.$$

2.1.4 Canonical Coordinates

Canonical variables of a Lie group provide to express the given Lie group and the corresponding infinitesimal generator in a simple form by shifting an element of the group.

Theorem 2.3. *Every one-parameter Lie transformations group (2.1) with the generator (2.5) can be reduced to the group of translations*

$$\overline{y_i} = y_i, \ i = 1, 2, \ldots, n-1, \tag{2.13}$$

$$\overline{y_n} = y_n + \alpha, \tag{2.14}$$

with the generator

$$Y = \frac{\partial}{\partial y_n},$$

introducing new coordinates $y = (y_1, y_2, \ldots, y_n)$ called canonical coordinates and defined *by the following first-order linear PDEs:*

$$X y_i(x) = 0,\ i = 1, 2, \dots, n-1, \tag{2.15}$$

$$X y_n(x) = 1. \tag{2.16}$$

Example. Consider the rotations group

$$\overline{x_1} = x_1 \cos\alpha - x_2 \sin\alpha, \tag{2.17}$$

$$\overline{x_2} = x_1 \sin\alpha + x_2 \cos\alpha \tag{2.18}$$

generated by the infinitesimal operator

$$X = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}.$$

Using equation (2.15), we get

$$y_1(x_1, x_2) = k$$

as the general solution of

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}.$$

That is

$$y_1 = \sqrt{x_1^2 + x_2^2}.$$

Now from formula (2.16), a particular solution of equation

$$\frac{dy_2}{dx_2} = \frac{1}{x_1} = \frac{1}{\sqrt{y_1^2 - x_2^2}}$$

gives the second canonical coordinate as

$$y_2 = \theta = \arcsin\frac{x_2}{y_1}.$$

Consequently canonical coordinates of rotations group (2.17), (2.18) are obtained as polar coordinates

$$(y_1, y_2) = (r, \theta) = (\sqrt{x_1^2 + x_2^2},\ \arcsin\frac{x_2}{y_1}). \tag{2.19}$$

These coordinates are written with respect to the rotations group (2.17), (2.18) in the normal form

$$\overline{r} = r,$$

$$\overline{\theta} = \theta + \alpha.$$

2.1.5 Group of Point Transformations and Prolonged Transformations

In mathematics, we can see prolongation of diffeomorphisms, flows, vector fields, group actions, and so on. Now, for our purpose, we introduce prolongation of a vector field.

Definition 2.5. A group of transformations of the form

$$\overline{x} = X(x, y; \alpha), \tag{2.20}$$

$$\overline{y} = Y(x, y; \alpha), \tag{2.21}$$

is called a *one-parameter Lie point transformations group* in the space of $n + m$ variables where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ represents n independent variables and $y = (y^1, y^2, \ldots, y^m) \in \mathbb{R}^m$ represents m dependent variables.

A Lie group of point transformations and its infinitesimal generator act on the differential terms by extending the group to a space of dependent variables, independent variables and derivatives of dependent variables with finite order.

We notate the entire set of first order partial derivatives of the dependent variable y with respect to the independent variable x by ∂y:

$$\partial y = \left(\frac{\partial y^1}{\partial x_1}, \frac{\partial y^1}{\partial x_2}, \ldots, \frac{\partial y^1}{\partial x_n}, \frac{\partial y^2}{\partial x_1}, \frac{\partial y^2}{\partial x_2}, \ldots, \frac{\partial y^2}{\partial x_n}, \ldots, \frac{\partial y^m}{\partial x_1}, \frac{\partial y^m}{\partial x_2}, \ldots, \frac{\partial y^m}{\partial x_n} \right). \tag{2.22}$$

This set consists of nm elements. Generally, for $k \geq 1$ we notate the entire set of kth-order partial derivatives of the dependent variable y with respect to the independent variable x by $\partial^k y$ which consists of the elements

$$y^{\mu}_{i_1 i_2 \ldots i_k} = \frac{\partial^k y^{\mu}}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}$$

with $\mu = 1, 2, \ldots, m$ and $i_j = 1, 2, \ldots, n,\ j = 1, 2, \ldots, k$.

Apparently, for transforming partial derivatives of the dependent variables, we need *prolongations* of transformations in a one-parameter Lie group (2.20), (2.21) on the space of dependent variables and independent variables (x, y) to the space of dependent variables, independent variables and partial derivatives of dependent variables $(x, y, \partial y, \partial^2 y, \ldots, \partial^k y)$ with $k > 2$. Consequently, the infinitesimal generator of Lie group (2.20), (2.21) is prolonged to infinitesimal generators in the space with derivatives.

For prolongation of a one-parameter transformation group to the space with first order derivatives, we consider the infinitesimal generator

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^k \frac{\partial}{\partial y^k} + \zeta_i^k \frac{\partial}{\partial y_i^k}, \tag{2.23}$$

where $i = 1, \dots, n,\ k = 1, \dots, m$ and ζ_i^k are some functions of $x_i,\ y^k$ and y_i^k.

The definition and geometric meaning of the derivatives are preserved if the following equations are invariant under the group of transformations (2.20), (2.21):

$$d^k y = y_i^k dx^i,\ k = 1, 2, \dots, m. \tag{2.24}$$

The invariance of the differential variables, i.e., equations (2.24) under the group of transformations defined by operator (2.23) requires the following *prolongation formulas*:

$$\zeta_i^k = D_i(\eta^k) - y_j^k D_i(\xi^j),\ i = 1, \dots, n,\ k = 1, \dots, m \tag{2.25}$$

where

$$D_i = \frac{\partial}{\partial x_i} + y_i^k \frac{\partial}{\partial y^k},\ i = 1, \dots, n, \tag{2.26}$$

is the total derivative operator with respect to the variable x^i.

The second-order prolongation formulas allow us to find the transformation of second derivatives and they are obtained by the same way as:

$$\zeta_{ji}^k = D_i(\zeta_j^k) - y_{sj}^k D_i(\xi^s),\ i, j = 1, \dots, n,\ k = 1, \dots, m. \tag{2.27}$$

By a similar process, prolongation formulas for finding coordinates of third and higher order derivatives are obtained. Hereafter, we represent the operator of a Lie group of transformations prolonged to the desired number of derivatives by the number of derivative or by "prX" :

$$X^{(k)} = \xi^i \frac{\partial}{\partial x_i} + \eta^k \frac{\partial}{\partial y^k} + \zeta_i^k \frac{\partial}{\partial y_i^k} + \zeta_{ij}^k \frac{\partial}{\partial y_{ij}^k} + \cdots. \tag{2.28}$$

From formulas (2.25) and (2.27) it is easy to see that the prolonged operator is linear and homogeneous with respect to the terms of the initial operator.

2.1.6 Multiparameter Lie Group of Transformations and Lie Algebras

In the previous sections, we only emphasize on one-parameter Lie transformations groups. But there are also multiparameter Lie transformations groups as an example classes of scalings in dimensional analysis. Similar to the analysis of one-parameter

Lie transformations group, infinitesimal generators are the basic elements in the analysis of multiparameter Lie transformations group. Every infinitesimal generator can be expressed in an exponential form which corresponds to a one-parameter Lie transformations group. Since this group is a subgroup of a multiparameter Lie transformations group, the theory of multiparameter Lie transformations groups is equal to the theory of infinitesimal generators of one-parameter subgroups. An r-parameter Lie transformations group can be constructed by composition of infinitesimal generators of one-parameter transformation groups.

Let

$$\overline{x} = X(x;\alpha) \tag{2.29}$$

be an *r-parameter Lie point transformations group* with variables $x = (x_1, x_2, \dots, x_n)$, parameters $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$, and the composition operation for parameters denoted by

$$\Theta(\alpha, \beta) = (\psi_1(\alpha, \beta), \psi_2(\alpha, \beta), \dots, \psi_r(\alpha, \beta)).$$

The composition operation is defined to satisfy the group axioms such that $\alpha = 0$ corresponds to the identity element $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$ and $\Theta(\alpha, \beta)$ is an analytic function in its domain.

For a multiparameter Lie transformations group, each parameter r corresponds to an infinitesimal generator. The set of infinitesimal generators of an r-parameter Lie transformations group (2.29) forms an r-dimensional vector space L_r called a *Lie algebra* or *r-dimensional Lie algebra* on which an operation called the *commutator* is defined.

Definition 2.6. For an r-parameter Lie transformations group (2.29) with infinitesimal generators X_m, $m = 1, 2, \dots, r$ the *commutator (Lie bracket)* of X_m and X_n is a first-order operator

$$[X_m, X_n] = X_m X_n - X_n X_m. \tag{2.30}$$

The commutator operation satisfies the following properties

- (*Bilinear*)

$$[aX_m + bX_n, X_p] = a[X_m, X_p] + b[X_n, X_p],$$

$$[X_m, aX_n + bX_p] = a[X_m, X_n] + b[X_m, X_p],$$

- (*Antisymmetric or skew-symmetric*)

$$[X_m, X_n] = -[X_n, X_m],$$

- (*Jacobi identity*)

$$[X_m, [X_n, X_p]] + [X_n, [X_p, X_m]] + [X_p, [X_m, X_n]] = 0.$$

Definition 2.7. The vector space of infinitesimal generators of multiparameter Lie group L_r is called a *Lie algebra* if the commutator $[X, Y] \in L_r$ when $X, Y \in L_r$.

Definition 2.8. A *subalgebra* of a Lie algebra L is a subset $J \subset L$ satisfying $[X_m, X_n] \in J$ when $X_m, X_n \in J$. Further, a subalgebra $J \subset L$ is called an *ideal* or *normal subalgebra* of L if $[X, Y] \in J$ for all $X \in J$, $Y \in L$.

Definition 2.9. A Lie algebra L_r is called an r-dimensional *solvable Lie algebra* if there exists a chain of subalgebras

$$\{0\} = L_0 \subset L_1 \subset \ldots \subset L_r$$

in such a way that $\dim(L_m) = m$ and L_{m-1} is a normal subalgebra of L_m for each m.

Solvable Lie algebras are important subjects in the study of higher-order ODEs.

2.1.7 Contact (Tangent) Transformations

Contact transformations have a wide application field in mechanics and the study of differential equations for a long time. They transforms the dependent variables, independent variables and the first derivatives in a differential equation.

For the case n independent variables $(x_1, x_2, \ldots, x_n)$ and a single dependent variable y the group of contact transformations is of the form

$$\begin{aligned} \overline{x_i} &= X_i(x, y, y_1; \alpha), \ i = 1, 2, \ldots, n, \\ \overline{y} &= Y(x, y, y_1; \alpha), \ \overline{y_i} = Y_i(x, y, y_1; \alpha), \end{aligned} \tag{2.31}$$

where y_1 represents the set of all partial derivatives y_i. The group of transformations (2.31) is not always equivalent to the group of point transformations prolonged for the first derivatives of a single dependent variable y. A group of point transformations is a subset of a group of contact transformations.

The invariance condition under a group of contact transformations is given by infinitesimal generator of the group similar to the case of point transformations group.

A second-order ODE admits infinitely many contact transformations while a first-order ODE admits infinitely many point transformations. In contrast to second-order ODEs, for third- and higher-order ODEs group of contact transformations can be computed splitting the determining equation.

Example. The third-order ODE $y''' = 0$ has 10-dimensional Lie transformations group generated by the operators

$$X_1 = \frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial y},\ X_3 = x^2\frac{\partial}{\partial y},\ X_4 = y\frac{\partial}{\partial y},\ X_5 = \frac{\partial}{\partial x},\ X_6 = x\frac{\partial}{\partial x},$$

$$X_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},\ X_8 = 2y'\frac{\partial}{\partial x} + y'^2\frac{\partial}{\partial y},$$

$$X_9 = (y - xy')\frac{\partial}{\partial x} - \frac{xy'^2}{2}\frac{\partial}{\partial y} - \frac{y'^2}{2}\frac{\partial}{\partial y'},$$

$$X_{10} = \left(xy - \frac{x^2y'}{2}\right)\frac{\partial}{\partial x} + \left(y^2 - \frac{x^2y'^2}{4}\right)\frac{\partial}{\partial y} + \left(yy' - \frac{xy'^2}{2}\right)\frac{\partial}{\partial y'},$$

where $X_1, X_2, \ldots, X_7$ are point transformations groups and X_8, X_9, X_{10} are contact transformations groups.

Higher-order symmetries or Lie-B$\ddot{a}$cklund groups are generalization of point or contact transformations groups with some additional properties. Since it is not related to our research, we are not focusing on the subject here.

2.2 Application of Lie Symmetries to Ordinary Differential Equations

Lie symmetries and integration have important roles in the study of ODEs. A kth-order ODE can be geometrically described as a surface in the $(k+2)$–dimensional space having elements that consist of the independent variable, the dependent variable and derivatives of the dependent variable with order k. Thus the solution set of the ODE can be considered as special curves on this surface. In this respect, a symmetry corresponds to a mapping which moves each solution curve into solution curves; a first integral corresponds to a quantity which is preserved on each solution curve. In other words, a symmetry is a one-parameter transformations group that acts on the elements of the $(k+2)$–dimensional space and maps one solution into another solution. A first integral is a quadrature which is described by a function of the independent variable, dependent variable and derivatives of dependent variable to order $k-1$ and which is constant on each solution.

Lie stated that the order of an ODE can be reduced by one if the equation admits a one–parameter Lie group of point transformation. Then, the solution of the original ODE arises from the solution of the reduced equation and a quadrature form. For a kth-order ODE if the corresponding symmetry group is an r–parameter solvable group of point transformations, then the order of the equation can be reduced by r splitting the equation to an $(k—r)$th-order equation and r quadratures. But the order of reduced equation is not $k-r$ according to the original independent and dependent variables. Conversely, in the integrating factor technique, the order of the reduced equation is $k-r$ with respect to the original independent and dependent variables.

For a first-order ODE, reducing by symmetries results in the quadrature of the equation. Actually this gives a first integral and corresponding integration factor. For a kth-order ODE, a first integral supplies a quadrature and reduces the order of the equation by one. In the study of ODEs, Lie symmetry reduction approach and integrating factor approach are complementary.

The order of an ODE can also be reduced by using the canonical coordinates or determining the differential invariants in the prolonged space.

For an ODE admitting a Lie transformations group, there exist particular types of solutions. These solutions are defined by the curves which are invariant under the corresponding Lie transformations group and called *invariant solutions*. Invariant solutions of first-order ODEs can be computed in an algebraic way. Invariant solutions of higher-order ODEs are computed in an algebraic way or via the solution of the first-order ODE for the invariant curves of the group.

In this section, we briefly illustrate how the symmetry method is applied to ODEs. The following materials are based on contents of [20, 22, 24].

Consider a one-parameter Lie group of point transformations

$$\overline{x} = X(x, y; \alpha), \;\; \overline{y} = Y(x, y; \alpha) \tag{2.32}$$

with the infinitesimal generator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{2.33}$$

An ODE

$$y' = \frac{dy}{dx} = f(x, y) \tag{2.34}$$

admits a one-parameter Lie group of point transformations (2.32) is called *point symmetry* if the form of the differential equation is preserved with respect to transformations of variables (2.32). That is, the equation

$$\frac{d\overline{y}}{d\overline{x}} = f(\overline{x}, \overline{y}) \tag{2.35}$$

is satisfied for the same function f as before changing variables. For higher-order equations the same definition is valid. Infinitesimal generator (2.33) of a Lie group admitted by a differential equation is called an *admitted operator* or *infinitesimal symmetry* of the equation. Invariance condition of ODE (2.34) under a Lie group generated by operator (2.33) is given by

$$X(y' - f(x, y)) = 0. \tag{2.36}$$

Consider the first order ODEs

$$\frac{dy}{dx} = f(x), \ \frac{dy}{dx} = g(y).$$

In the first equation, since $dy/dx = f(x)$, the slopes dy/dx of the solution curves $y = u(x, c_1)$ do not depend on the variable y. Hence any solution curve can be shifted in the y-direction into another solution curve through the mapping $(x, y) \mapsto (x, y + c_1)$. Similarly, for $dy/dx = g(y)$ the slopes dy/dx of the solution curves $y = v(x, c_2)$ are independent of x, so any of these curves can be shifted in the x-direction into any other by mapping $(x, y) \mapsto (x + c_2, y)$.

There are two ways to obtain the general solution of ODE (2.34) using the infinitesimals of an admitted point symmetry (2.32):

1. the method of canonical coordinates,
2. using an integrating factor.

2.2.1 The Method of Canonical Coordinates

Theorem 2.4. *A first-order ODE*

$$\frac{dy}{dx} = f(x, y) \tag{2.37}$$

admitting a Lie group as translations about its dependent variable is seperable.

Proof. Let $T : (x, y) \to (\overline{x}, \overline{y}) = (x, y + \alpha)$ be a symmetry of equation (2.37) with

$\alpha \in \mathbb{R}$. Since

$$f(x,y+\alpha)=f(\overline{x},\overline{y})=\frac{d\overline{y}}{d\overline{x}}=\frac{d\overline{y}(x,y)}{d\overline{x}(x,y)}=\frac{\overline{y}_x+y'\overline{y}_y}{\overline{x}_x+y'\overline{x}_y}=\frac{dy}{dx}=f(x,y),$$

the function f does not depend on y. So we have separable form of differential equation $\frac{dy}{dx}=f(x)$. ■

According to Theorem 2.4, if a Lie group of a differential equation of form (2.37) is equivalent to a translational symmetry- i.e., transformed to the group translations about the dependent variable- then solution of the equation can be obtained easily. A linear or nonlinear first-order ODE

$$y'=\frac{dy}{dx}=f(x,y) \tag{2.38}$$

with a symmetry group generated by the operator

$$X=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y} \tag{2.39}$$

is solved with the *method of canonical coordinates* following the steps:

1. Obtain canonical coordinates r,s by solving equations

$$\begin{aligned} Xr&=0,\\ Xs&=1, \end{aligned} \tag{2.40}$$

   for given infinitesimal generator (2.39).

2. Substitute the canonical variables r and s in equation (2.38) letting s be new dependent variable and r be new independent variable. Then equation (2.38) will take the integrable form

$$\frac{ds}{dr}=g(s) \quad \text{or} \quad \frac{ds}{dr}=h(r). \tag{2.41}$$

3. Integrate equation (2.41), substitute the expressions $r=r(x,y)$ and $s=s(x,y)$ in the solution $s=\phi(r,C)$, so the solution of equation (2.38) is obtained.

Example. [24] Consider the differential equation

$$y'=\frac{y}{x}+\frac{y^3}{x^3}. \tag{2.42}$$

Let us obtain solution of equation (2.42) using canonical coordinates. Since equation (2.42) is in the form of homogeneous differential equation, it admits the group of scaling transformation

$$\overline{x} = e^{\varepsilon} x, \ \ \overline{y} = e^{\varepsilon} y$$

with infinitesimal generator

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{2.43}$$

where

$$\xi(x,y) = \frac{\partial X(x,y;\varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \frac{\partial e^{\varepsilon} x}{\partial \varepsilon}\bigg|_{\varepsilon=0} = e^{\varepsilon} x|_{\varepsilon=0} = x,$$

$$\eta(x,y) = \frac{\partial Y(x,y;\varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \frac{\partial e^{\varepsilon} y}{\partial \varepsilon}\bigg|_{\varepsilon=0} = e^{\varepsilon} y|_{\varepsilon=0} = y.$$

Indeed

$$\frac{d\overline{y}}{d\overline{x}} = \frac{d(e^{\varepsilon} y)}{d(e^{\varepsilon} x)} = \frac{dy}{dx}.$$

Hence the transformation group $\overline{x} = e^{\varepsilon} x, \ \overline{y} = e^{\varepsilon} y$ leaves equation (2.42) invariant.

1. Solving the first-order linear PDEs (2.40) that defines canonical coordinates r,s for (2.43), we get

   $$r = \frac{y}{x}, \ \ s = \ln x.$$

2. Substituting the canonical coordinates r,s in equation (2.42) gives

   $$y' = \frac{dy}{dx} = \frac{d(xr)}{dx} = r + x\frac{dr}{dx} = r + x\frac{dr}{ds}\frac{ds}{dx} = r + xr'\frac{1}{x} = r + r'.$$

   Thus equation (2.42) becomes

   $$\frac{ds}{dr} = \frac{1}{r^3}. \tag{2.44}$$

3. Integration of equation (2.44) with canonical variables yields

   $$s = -\frac{1}{2r^2} + C.$$

   Finally, substituting $r = \frac{y}{x}$ and $s = \ln x$ gives

   $$y = \pm\frac{x}{\sqrt{C - \ln x^2}}.$$

2.2.2 Lie's Integrating Factor

Consider a first-order ODE formulated by

$$M(x,y)dx + N(x,y)dy = 0. \tag{2.45}$$

If the equation (2.45) admits a point transformations group with the infinitesimal generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

and $\xi M + \eta N \neq 0$, then the function

$$\mu = \frac{1}{\xi M + \eta N} \tag{2.46}$$

which is called *Lie's integrating factor* is an integrating factor for the equation (2.45).
Example. [24] The Riccati-type of equation

$$y' = \frac{2}{x^2} - y^2,\ (x \neq 0) \tag{2.47}$$

admits the group of scaling transformations

$$\overline{x} = e^{\alpha}x,\ \overline{y} = e^{-\alpha}y,$$

generated by the operator

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Equation (2.47) can be written in the form of equation (2.45) as

$$dy + \left(y^2 - \frac{2}{x^2}\right)dx = 0. \tag{2.48}$$

Substituting

$$\xi = x,\ \eta = -y,\ M = y^2 - \frac{2}{x^2},\ N = 1,$$

into (2.46) gives the integrating factor

$$\mu = \frac{x}{x^2y^2 - xy - 2}.$$

For the solution of equation (2.47), we multiply equation (2.48) with the integrating factor and obtain the exact differential equation

$$\frac{x}{x^2y^2 - xy - 2}dy + \frac{1}{x^2y^2 - xy - 2}\frac{x^2y^2 - 2}{x}dx = 0. \tag{2.49}$$

Now, we can solve this equation with known methods. Using the identity

$$\frac{x^2y^2-2}{x}=y+\frac{x^2y^2-xy-2}{x},$$

we rewrite equation (2.49) as

$$\frac{d(xy)}{x^2y^2-xy-2}+\frac{dx}{x}=0. \tag{2.50}$$

For the first term of this equation, we use $v=xy$ and write

$$\frac{d(xy)}{x^2y^2-xy-2}=\frac{dv}{v^2-v-2}=\frac{1}{3}\left(\frac{1}{v-2}-\frac{1}{v+1}\right)dv.$$

Integrating this term results with

$$\int\frac{1}{v^2-v-2}dv=\frac{1}{3}\ln\left(\frac{v-2}{v+1}\right).$$

Returning equation (2.50) with original variables gives

$$d\left(\frac{1}{3}\ln\left(\frac{xy-2}{xy+1}\right)+\ln x\right)=0.$$

As a last step, we integrate this equation and get the solution of equation (2.47) as

$$y=\frac{2x^3+c}{x(x^3-c)},$$

where c is an arbitrary constant. Since the functions $y=2/x$ and $y=-1/x$ satisfy differential equation (2.47), these are also solutions of the equation called invariant solutions.

2.2.3 Invariant Solutions

A Lie transformations group G admitted by an ODE provides a significant property for the equation. The symmetry transformations map any solution- i.e., integral curve- of the equation into another solution. That is, the transformations of the group replace the integral curves with each other leaving some of the integral curves the same. Such integral curves are called *invariant solutions*.

Invariant solutions shows a simple way to obtain general solutions to differential equations. This method is used for first order ODEs with two known infinitesimal generators.

Example. [24] (*General solution of an ODE obtained from invariant solutions*)

Consider the equation

$$y' = \frac{y}{x} + \frac{y^2}{x^3} \tag{2.51}$$

with two known infinitesimal symmetries

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}.$$

From the invariance condition $X_2(I) = 0$, the only independent invariant is found as y/x^2. Then, the invariant solution is obtained by taking $k = y/x^2$ or $y = kx^2$ with an arbitrary constant $k \neq 0$. In this case, the equation (2.51) reduces to

$$y' - \frac{y}{x} - \frac{y^2}{x^3} = 2kx - kx - k^2x = k(1-k)x = 0.$$

Hence $k = 1$, and in this case the invariant solution simply is

$$y = x^2. \tag{2.52}$$

Now, under the projective transformation

$$\overline{x} = \frac{x}{1-ax},\ \overline{y} = \frac{y}{1-ax} \tag{2.53}$$

generated by X_1, the invariant solution (2.52) is written in the form of new variables

$$\overline{y} = \overline{x}^2,$$

and substituting the expressions (2.53) for $\overline{x}$ and $\overline{y}$, the equation becomes

$$\frac{y}{1-ax} = \frac{x^2}{(1-ax)^2}.$$

Finally, denoting the parameter a by C, the general solution of equation (2.51) is obtained as

$$y = \frac{x^2}{1-Cx}. \tag{2.54}$$

2.2.4 Extension to Higher Order Ordinary Differential Equations

The order of a higher order ODE can be reduced step-by-step if there exists a multiparameter Lie transformations group admitted by the equation. Furthermore, if an ODE of order n has an r-parameter group, the order of the differential equation can be reduced to $n-r$ with the restriction that the corresponding Lie algebra is solvable.

Consider an ODE of order n of the form

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \ \ n \geq 2. \tag{2.55}$$

If $X(x, y)$ is the infinitesimal generator of a one-parameter group of (2.55) and $r(x, y)$, $s(x, y)$ are the canonical variables, we have that

$$X(r,s) = \frac{\partial}{\partial s}.$$

Writing (2.55) in terms of canonical variables gives

$$s^{(n)} = \Omega(r, s, s', ..., s^{(n-1)}), \quad s^{(k)} = \frac{d^k s}{dr^k} \tag{2.56}$$

for some function Ω. Since (2.56) is invariant under the group of translations in s-direction, we have that

$$\Omega_s = 0.$$

Therefore, (2.56) takes the form

$$s^{(n)} = \Omega(r, , s', ..., s^{(n-1)}).$$

Thus, for $v = ds/dr$ writing (2.55) in terms of canonical variables reduces the order of (2.55) to the following ODE of order $n-1$,

$$v^{(n-1)} = \Omega(r, v, ..., v^{(n-2)}), \quad v^{(k)} = \frac{d^{k+1} s}{dr^{k+1}}.$$

Consequently, solving this simplified equation, we can get the solution of the equation (2.55).

Example. [10] Consider the nonlinear second order ODE

$$y'' = \frac{y'^2}{y} + \left(y - \frac{1}{y}\right) y'. \tag{2.57}$$

The differential equation (2.57) admits one-parameter Lie group

$$\overline{x} = x + \varepsilon, \ \ \overline{y} = y$$

whose infinitesimal generator is

$$X(x, y) = \frac{\partial}{\partial x}. \tag{2.58}$$

Thus, the tangent vector field is $(\xi(x, y), \eta(x, y)) = (1, 0)$. Using this, we can find the set of new coordinates solving equations

$$Xr = 0, \ \ Xs = 1. \tag{2.59}$$

Solving these equations for operator (2.58) yields the canonical variables

$$r(x, y) = y, \ \ s(x, y) = x. \tag{2.60}$$

So, we have

$$\tilde{s} = \frac{ds}{dr} = \frac{dx}{dy} = \left(y'\right)^{-1}.$$

Choosing $v = y'$, i.e., $v = (\tilde{s})^{-1}$, differential equation (2.57) becomes

$$\frac{dv}{dr} = \frac{y''}{y'} = \frac{v}{r} + r - \frac{1}{r} \tag{2.61}$$

which directly leads to a linear differential equation. The general solution of (2.61) is

$$v(r) = r^2 - 2c_1 r + 1.$$

Then, we have that

$$s(r) = \int \frac{1}{r^2 - 2c_1 r + 1} dr.$$

Solving the last integral and writing resulted equation in terms of the original variables using (2.60), gives the general solution of (2.55)

$$y = \begin{cases} c_1 - \sqrt{c_1^2 - 1} \tanh\left(\sqrt{c_1^2 - 1}\,(x + c_2)\right), & c_1^2 > 1, \\ c_1 - (x + c_2)^{-1}, & c_1^2 = 1, \\ c_1 - \sqrt{1 - c_1^2} \tan\left(\sqrt{1 - c_1^2}\,(x + c_2)\right), & c_1^2 < 1, \end{cases}$$

where c_1 and c_2 are arbitrary constants.

Example. [26] This example shows how to find a Lie group admitted by an ODE. For the second-order ODE

$$y'' = \frac{1}{y^3}, \tag{2.62}$$

we look for the symmetry group generated by the operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

We need the prolongation of this operator to the first and second order derivatives

$$X^{(2)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta_1\frac{\partial}{\partial y'} + \zeta_2\frac{\partial}{\partial y''}, \tag{2.63}$$

where

$$\zeta_1 = D(\eta) - y'D(\xi),$$
$$\zeta_2 = D(\zeta_1) - y''D(\xi) = D^2(\eta) - 2y''D(\xi) - y'D^2(\xi),$$
$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + \cdots .$$

Applying prolonged operator (2.63) to equation (2.62) gives

$$D^2(\eta) - 2y''D(\xi) - y'D^2(\xi) = -\frac{3\eta}{y^4},$$

or equivalently,

$$\begin{aligned} &\eta_{xx} + 2\eta_{xy}y' + \eta_{yy}(y')^2 + \eta_y y'' - 2y''(\xi_x + \xi_y y') \\ &- y'(\xi_{xx} + 2\xi_{xy}y' + \xi_{yy}(y')^2 + \xi_y y'') + \frac{3}{y^4}\eta = 0. \end{aligned} \tag{2.64}$$

Substituting the points of equation (2.62) into (2.64) gives the determining equation

$$\begin{aligned} &\eta_{xx} + 2\eta_{xy}y' + \eta_{yy}(y')^2 + \eta_y\frac{1}{y^3} - 2\frac{1}{y^3}(\xi_x + \xi_y y') \\ &- y'(\xi_{xx} + 2\xi_{xy}y' + \xi_{yy}(y')^2 + \xi_y\frac{1}{y^3}) + \frac{3}{y^4}\eta = 0. \end{aligned} \tag{2.65}$$

Solving this equation gives the coordinates of infinitesimal generator that we seek. We consider equation (2.65) as a polynomial according to powers of y' and obtain the following overdetermined system of equations by equating coefficients:

$$\begin{aligned} \xi_{yy} = 0,\ \eta_{yy} - 2\xi_{xy} = 0, \\ 2\eta_{xy} - 3\frac{\xi_y}{y^3} - \xi_{xx} = 0, \\ \eta_{xx} + (\eta_y - 2\xi_x)\frac{1}{y^3} + \frac{3\eta}{y^4} = 0. \end{aligned} \tag{2.66}$$

The first two equations of (2.66) gives

$$\xi = f(x)y + g(x),\ \eta = f_x y^2 + h(x)y + p(x).$$

We substitute these results into the third and fourth equations of (2.66) and equate the coefficients with respect to powers of y to obtain the general solution

$$\xi(x) = Ax^2 + 2Bx + C,\ \eta(x,y) = (Ax + B)y,$$

where A, B, C are arbitrary constants. We obtain three-dimensional symmetry group by choosing zero any two of the constants as follows

$$X_1 = \frac{\partial}{\partial x},\ X_2 = 2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{2.67}$$

Thereby, equation (2.62) is invariant under one-parameter transformations groups generated by operators (2.67). The transformations of each group can be found solving corresponding Lie equations,

$$\begin{aligned} \overline{x} = x + \alpha,\ \overline{y} = y; \\ \overline{x} = e^{2\alpha}x,\ \overline{y} = e^{\alpha}y; \\ \overline{x} = \frac{x}{1-\alpha x},\ \overline{y} = \frac{y}{1-\alpha x}. \end{aligned} \tag{2.68}$$

Prolongations of transformations (2.68) to the first and second derivatives are obtained as

$$\begin{aligned} \overline{x} = x + \alpha,\ \overline{y} = y,\ \overline{y'} = y',\ \overline{y''} = y''; \\ \overline{x} = e^{2\alpha}x,\ \overline{y} = e^{\alpha}y,\ \overline{y'} = e^{-\alpha}y',\ \overline{y''} = e^{-3\alpha}y''; \\ \overline{x} = \frac{x}{1-\alpha x},\ \overline{y} = \frac{y}{1-\alpha x}, \\ \overline{y'} = \alpha y + (1-\alpha x)y',\ \overline{y''} = (1-\alpha x)^3 y''. \end{aligned} \tag{2.69}$$

By substituting transformations (2.69) into equation (2.62), we notice that the equation does not change, that is

$$\overline{y''} = \frac{1}{(\overline{y})^3}$$

at the point $(\overline{x}, \overline{y}, \overline{y'}, \overline{y''})$. Consequently, the symmetry group leaves invariant the equation, but moves the point by the transform $(x, y, y', y'') \rightarrow (\overline{x}, \overline{y}, \overline{y'}, \overline{y''})$.

2.3 Application of Lie Symmetries to Partial Differential Equations

In the case of PDEs, invariance of the given equation and its accompanying boundary conditions under a one-parameter transformations group leads to a reduction of one in the number of independent variables. The invariants of the group become the new variables.

The infinitesimal generator of a Lie point transformations group admitted by a given PDE is determined by an algorithm with respect to the infinitesimal criterion for the invariance of a PDE.

Invariance of a PDE under a Lie point transformations group supplies special solutions called invariant solutions or similarity solutions. These solutions are invariant under a subgroup of the Lie group admitted by the PDE and obtained solving PDEs with fewer independent variables than appear in the given PDE. Invariant solutions can be obtained for BVPs. In this case, the solutions are invariant under a subgroup of a Lie group admitted by the given PDE that leaves invariant the corresponding boundary curves and boundary conditions. Self-similar (auto model) solutions which form a subset of invariant solutions can be constructed from invariance under scaling groups.

Invariant solutions of a scalar PDE or a system of PDE can be obtained using an admitted point transformations group by two techniques:

- *The Invariant Form Method:* Firstly, the characteristic equations resulting from the invariance conditions (infinitesimal criteria) are solved and the invariant form is obtained. Then solution of the given PDE in this invariant form gives the invariant solution.

- *The Direct Substitution Method [35]:* Firstly, an independent variable is selected as a parameter. Then invariance conditions and necessary differential consequences are substituted into the given PDE. In this way, all derivatives with respect to the selected (parametric) independent variable are annihilated and one independent variable is reduced in the given PDE. Substituting solution of the reduced PDE into either invariance conditions or the given PDE gives the invariant solutions. In this method, invariant solutions are determined without explicitly solving characteristic equations that arise from invariance conditions.

  These two methods can be improved for obtaining invariant solutions of PDEs from an admitted multiparameter point transformations group.

Lie group of point transformations of a PDE corresponds to geometric movements in its solution set. Similar to the case ODEs, a Lie group of point transformations admitted by a PDE geometrically describes an integral curve of a vector field which is tangent to the surface defined by the given PDE. Point symmetries of PDEs also preserve the meaning of the derivatives in the prolonged (or jet) space that includes coordinates given by the independent variables, dependent variables and all their partial derivatives up to a fixed finite order. Superposition of all point symmetries admitted by a PDE form a Lie group which has an algebraic structure. Lie algebra of this group is given by an operation that is commutator of the infinitesimal generators of the point symmetries admitted by the given PDE.

In this section, we present constructing the solutions of PDEs with the view point of invariance under Lie point transformations groups. The following materials are based on content of [22].

2.3.1 Invariance for Partial Differential Equations

A symmetry group for PDEs is defined in the same way as ODEs.

Definition 2.10. Consider a kth-order PDE

$$F(x, y, \partial y, \partial^2 y, \dots, \partial^k y) = 0, \tag{2.70}$$

where $x = (x_1, x_2, \dots, x_n)$ denotes the n independent variables, y denotes the dependent variable, and $\partial^l y$ denotes the l-th order partial derivatives of y with respect to x with $\partial^l y / \partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_l} = y_{i_1 i_2 \dots i_j}$, $i_l = 1, 2, \dots, n$, for $l = 1, 2, \dots, k$. The PDE is invariant under one-parameter Lie transformations group

$$\begin{aligned} \overline{x} &= X(x, y; \varepsilon), \\ \overline{y} &= Y(x, y; \varepsilon), \end{aligned} \tag{2.71}$$

if and only if its k-th extension, defined by

$$\begin{aligned} \overline{x} &= X(x, y; \varepsilon), \\ \overline{y} &= Y(x, y; \varepsilon), \\ \partial \overline{y} &= \partial Y(x, y, \partial y; \varepsilon), \\ &\vdots \\ \partial^k \overline{y} &= \partial^k Y(x, y, \partial y, \dots, \partial^k y; \varepsilon), \end{aligned}$$

leaves invariant the surface (2.70). This case is expressed by transformations group

(2.71) is *a point symmetry admitted by PDE* (2.70).

A one-parameter Lie group of point transformations (2.71) leaves invariant the set of all solutions of PDE (2.70) if and only if (2.70) admits (2.71).

Theorem 2.5 (Infinitesimal Criteria for the Invariance of a PDE). *Let*

$$X = \xi_i(x,y)\frac{\partial}{\partial x_i} + \eta(x,y)\frac{\partial}{\partial y} \tag{2.72}$$

be the infinitesimal generator of Lie point transformations group (2.71) and

$$\begin{aligned} X^{(k)} =& \xi_i(x,y)\frac{\partial}{\partial x_i} + \eta(x,y)\frac{\partial}{\partial y} + \eta_i^{(1)}(x,y,\partial y)\frac{\partial}{\partial y_i} + \cdots \\ &+ \eta_{i_1 i_2 \dots i_k}^{(k)}(x,y,\partial y,\partial^2 y,\dots,\partial^k y)\frac{\partial}{\partial y_{i_1 i_2 \dots i_k}} \end{aligned} \tag{2.73}$$

be the k-th extended infinitesimal generator of (2.72), where $\eta_i^{(1)}$ and $\eta_{i_1 i_2 \dots i_j}^{(j)}$, $i_j = 1,2,\dots,n$ for $j = 1,2,\dots,k$ are the extended infinitesimals, in terms of $\xi(x,y) = (\xi_1(x,y),\xi_2(x,y),\dots,\xi_n(x,y))$, $\eta(x,y)$. Then one-parameter Lie point transformations group (2.71) is admitted by PDE (2.70), i.e., is a point symmetry of PDE (2.70) if and only if

$$X^{(k)}F(x,y,\partial y,\partial^2 y,\dots,\partial^k y) = 0 \quad \textit{when} \quad F(x,y,\partial y,\partial^2 y,\dots,\partial^k y) = 0. \tag{2.74}$$

2.3.2 Invariant Solutions for Partial Differential Equations

Definition 2.11. $y = \psi(x)$ is an *invariant solution* of PDE (2.70) obtained from its admitted point symmetry generated by operator (2.72) if and only if:

1. $y = \psi(x)$ is an invariant surface of (2.72);
2. $y = \psi(x)$ is a solution of (2.70).

As a consequence of this definition, we can say that for a solution of PDE (2.70) to be an invariant solution under point symmetry generated by operator (2.72), the necessary and sufficient conditions are

1. $X(y - \psi(x))$ when $y = \psi(x)$. This results with the equation

$$\xi_i(x,\psi(x))\frac{\partial \psi(x)}{\partial x_i} = \eta(x,\psi(x)); \tag{2.75}$$

2. $F(x, y, \partial y, \partial^2 y, \ldots, \partial^k y) = 0$ when $y = \psi(x)$, i.e.,

$$F(x, \psi(x), \partial \psi(x), \partial^2 \psi(x), \ldots, \partial^k \psi(x)) = 0. \tag{2.76}$$

Invariant solutions of PDEs can be obtained in two ways:

(I) Invariant Form Method

This procedure starts with solving first-order PDE (2.75). The corresponding characteristic equations for PDE (2.75) with $y = \psi(x)$ is

$$\frac{dx_1}{\xi_1(x,y)} = \frac{dx_2}{\xi_2(x,y)} = \cdots = \frac{dx_n}{\xi_n(x,y)} = \frac{dy}{\eta(x,y)}. \tag{2.77}$$

From the solution of the system of n first-order ODEs (2.77), we obtain n functionally independent constants as $r_1(x,y), r_2(x,y), \ldots, r_{n-1}(x,y), s(x,y)$ with $\partial s/\partial y \neq 0$. Then the general solution $y = \psi(x)$ of PDE (2.75) is given, implicitly, in the invariant form

$$s(x,y) = \Theta(r_1(x,y), r_2(x,y), \ldots, r_{n-1}(x,y)), \tag{2.78}$$

where Θ is an arbitrary differentiable function of $r_1(x,y), r_2(x,y), \ldots, r_{n-1}(x,y)$. Since $r_1(x,y), r_2(x,y), \ldots, r_{n-1}(x,y), s(x,y)$ are n group invariants of point symmetry (2.72), they form n canonical coordinates for the Lie point transformations group (2.71). Let $r_n(x,y)$ be the $(n+1)$-th canonical coordinate satisfying

$$X r_n = 1.$$

By substituting these canonical variables in PDE (2.70) a k-th order PDE with independent variables $r_1, r_2, \ldots, r_n$ and dependent variable s can be obtained and this new PDE would admit the one-parameter Lie translations group

$$\begin{aligned} \overline{r_i} &= r_i, \ \ i = 1, 2, \ldots, n-1, \\ \overline{r_n} &= r_n + \alpha, \\ \bar{s} &= s. \end{aligned}$$

Therefore, the variable r_n would not appear explicitly in the new PDE and hence solutions obtained implicitly in the form (2.78) would be solutions of the new PDE. These are invariant solutions of PDE (2.70). Consequently, such solutions are determined solving a reduced differential equation with $n-1$ independent variables $r_1, r_2, \ldots, r_{n-1}$ which are called similarity variables and a dependent variable s. The reduced differential equation is obtained rewriting PDE (2.70) in terms of invariant form (2.78). If $\partial \xi/\partial y \equiv 0$, then $r_i = r_i(x), \ i = 1, 2, \ldots, n-1$ and if $n = 2$ then the given PDE is reduced to an ODE.

(II) Direct Substitution Method

In this approach, we do not have to solve explicitly invariance condition (2.75) or characteristic equations (2.77). We start with the equation which is written from first-order PDE (2.75)

$$\frac{\partial y}{\partial x_n} = \frac{\eta(x,y)}{\xi_n(x,y)} - \sum_{i=1}^{n-1} \frac{\xi_i(x,y)}{\xi_n(x,y)} \frac{\partial y}{\partial x_i}, \tag{2.79}$$

with assumption $\xi_n(x,y) \neq 0$. Then substituting equation (2.79) and its differential relations directly into given PDE (2.70), for all terms in (2.70) including derivatives of y with respect to x_n gives a reduced differential equation. The reduced equation has order at most k including the dependent variable y, the $n-1$ independent variables $x_1, x_2, \ldots, x_{n-1}$, and the parameter x_n. Each solution of this reduced differential equation describes an invariant solution for PDE (2.70) with respect to the invariance under point symmetry (2.72) and provides invariance condition (2.79) and given PDE (2.70). If $n = 2$ then the given PDE is reduced to an ODE. Since this method does not deal with integration of the characteristic equations it is easier than the Invariant Form Method.

Determining Equations for Point Symmetries of a PDE

Consider a k-th order PDE ($k \geq 2$, $l \leq k$)

$$y_{i_1 i_2 \ldots i_l} = f(x, y, \partial y, \partial^2 y, \ldots, \partial^k y), \tag{2.80}$$

where $f(x, y, \partial y, \partial^2 y, \ldots, \partial^k y)$ is independent of $y_{i_1 i_2 \ldots i_l}$. PDE (2.80) admits the one-parameter Lie point transformations group generated by the operator

$$X = \xi_i(x,y) \frac{\partial}{\partial x_i} + \eta(x,y) \frac{\partial}{\partial y}. \tag{2.81}$$

Its k-th extension is formulated by (2.73), if and only if $\xi(x,y)$ and $\eta(x,y)$ satisfy the determining equation

$$\eta^{(l)}_{i_1 i_2 \ldots i_l} = \xi_j \frac{\partial f}{\partial x_j} + \eta \frac{\partial f}{\partial y} + \eta^{(1)}_j \frac{\partial f}{\partial y_j} + \cdots + \eta^{(k)}_{j_1 j_2 \cdots j_k} \frac{\partial f}{\partial y_{j_1 j_2 \cdots j_k}} \tag{2.82}$$

when y satisfies (2.80).

2.3.3 Examples

The following examples show some applications of Lie symmetries to PDEs.

Example. [24] (*Group transformations of solutions of heat equation*)

The linear heat equation

$$u_t - u_{xx} = 0 \tag{2.83}$$

is invariant under the *heat representation of the Galilean transformation*:

$$\overline{t} = t, \ \overline{x} = x + 2at, \ \overline{u} = ue^{-(ax+a^2t)} \tag{2.84}$$

generated by the operator

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}. \tag{2.85}$$

Any solution

$$u = \Phi(t, x)$$

of the heat equation can be mapped into a new solution by the group of transformations (2.84). Since the heat equation is invariant under this group, it is written in the form $\overline{u}_{\overline{t}} - \overline{u}_{\overline{xx}} = 0$. By using the new variables $\overline{t}, \overline{x}, \overline{u}$, the solution is written as

$$\overline{u} = \Phi(\overline{t}, \overline{x}).$$

Then substituting the expressions (2.84) for $\overline{t}, \overline{x}, \overline{u}$, and solving the resulting equation

$$ue^{-(ax+a^2t)} = \Phi(t, x + 2at)$$

for u, one obtains the new solution

$$u = e^{ax+a^2t}\Phi(t, x + 2at) \tag{2.86}$$

with the parameter a. For example, for the simple solution $u = x$, letting $\Phi(t, x + 2at) = x + 2at$ in (2.86), the new solution

$$u = (x + 2at)e^{ax+a^2t}$$

is obtained.

Example. [79] (*Reducing a PDE system to ODE system*)

For a two-dimensional incompressible laminar fluid flow, boundary layer equations are given by

$$\begin{aligned}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\ u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}+U\frac{\partial U}{\partial x},\\ u(x,0)=0,\ v(x,0)=0,\ u(x,\infty)=U(x).\end{aligned} \tag{2.87}$$

This example shows that the PDEs (2.87) admit scaling symmetry and how to reduce the equations to ODEs. The scaling transformation for the equations (2.87) can be written as

$$\overline{x}=e^{\alpha a}x,\ \overline{y}=e^{\alpha b}y,\ \overline{u}=e^{\alpha c}u,\ \overline{v}=e^{\alpha d}v,\ \overline{U}=e^{\alpha e}U. \tag{2.88}$$

Substituting (2.88) in (2.87), we have the equations in terms of new variables

$$\begin{aligned}\frac{\partial \overline{u}}{\partial \overline{x}}+e^{\alpha(b+c-a-d)}\frac{\partial \overline{v}}{\partial \overline{y}}=0,\\ \overline{u}\frac{\partial \overline{u}}{\partial \overline{x}}+e^{\alpha(b+c-a-d)}\overline{v}\frac{\partial \overline{u}}{\partial \overline{y}}=e^{\alpha(2b+c-a)}\frac{\partial^2 \overline{u}}{\partial \overline{y}^2}+e^{\alpha 2(c-e)}\overline{U}\frac{\partial \overline{U}}{\partial \overline{x}},\\ \overline{u}(\overline{x},0)=0,\ \overline{v}(\overline{x},0)=0,\ \overline{u}(\overline{x},\infty)=e^{\alpha(c-e)}\overline{U}(\overline{x}).\end{aligned} \tag{2.89}$$

In order to admit the group of scaling transformations (2.88), the equations (2.87) should be invariant under the group, i.e., the transformed equations (2.89) should be in the same form as the original equation. For the invariance condition, the parameters satisfy the equations

$$b=(a-c)/2,\ d=(c-a)/2,\ e=c.$$

Then, the group of scaling transformations admitted by the equations (2.87) is obtained as

$$\overline{x}=e^{\alpha a}x,\ \overline{y}=e^{\alpha(a-c)/2}y,\ \overline{u}=e^{\alpha c}u,\ \overline{v}=e^{\alpha(c-a)/2}v,\ \overline{U}=e^{\alpha c}U \tag{2.90}$$

with the infinitesimal generator

$$X=\alpha a\frac{\partial}{\partial x}+\alpha\frac{a-c}{2}\frac{\partial}{\partial y}+\alpha c\frac{\partial}{\partial u}+\alpha\frac{c-a}{2}\frac{\partial}{\partial v}+\alpha c\frac{\partial}{\partial U}. \tag{2.91}$$

From the invariance criterion and letting $c=ma$, where m is another parameter, we get the invariant variable and functions as

$$\xi=yx^{(m-1)/2},\ u=x^m f(\xi),\ v=x^{(m-1)/2}g(\xi),\ U=kx^m, \tag{2.92}$$

where k is constant. Putting these terms in (2.87), the PDE system is reduced to the

ODE system

$$\begin{aligned} mf + ((m-1)/2)\xi f' + g' &= 0, \\ mf^2 + ((m-1)/2)\xi f f' + g f' &= f'' + mk^2, \\ f(0) = 0,\ g(0) = 0,\ f(\infty) &= k. \end{aligned} \tag{2.93}$$

Example. [24] (*The Burgers Equation*)

The Burgers equation

$$u_t = u_{xx} + u u_x \tag{2.94}$$

has five linearly independent symmetries generated by infinitesimal operators

$$\begin{aligned} X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, \\ X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},\ X_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x+tu)\frac{\partial}{\partial u}. \end{aligned} \tag{2.95}$$

Consider the group of transformations generated by X_5

$$\overline{t} = \frac{t}{1-at},\ \overline{x} = \frac{x}{1-at},\ \overline{u} = u(1-at) - ax. \tag{2.96}$$

Using these transformations, any known solution $u = \psi(t,x)$ of the Burgers equation is mapped to the following one-parameter set of new solutions:

$$u = \frac{ax}{1-at} + \frac{1}{1-at}\psi\left(\frac{t}{1-at}, \frac{x}{1-at}\right).$$

Invariance of the Burgers equation with respect to the symmetry group of translation in time ($\overline{x} = x,\ \overline{t} = t + a,\ \overline{u} = u$) generated by X_1 yields the stationary solution

$$u = \psi(x)$$

which is an important solution in physical applications. Substituting this solution into the Burgers equation gives

$$\psi'' + \psi\psi' = 0. \tag{2.97}$$

Integrating (2.97) once yields

$$\psi' + \frac{\psi^2}{2} = C_1,$$

and integrating again by setting $C_1 = 0,\ C_1 = v^2 > 0,\ C_1 = -w^2 < 0$ the solutions

$$\begin{aligned} \psi(x) &= \frac{2}{x+C}, \\ \psi(x) &= v \tanh\left(C + \frac{v}{2}x\right), \\ \psi(x) &= w \tan\left(C - \frac{w}{2}x\right) \end{aligned} \tag{2.98}$$

are obtained.

2.4 Application of Lie Symmetries to Difference Equations

Lie symmetry method provides a convenient and practical way to solve and classify differential equations and also characterize their solution sets. Therefore it has been applied to differential equations for many years. However, application of Lie symmetries to discrete equations, i.e., difference equations or differential-difference equations has recently been introduced.

An ODE or PDE is discretized in order to solve it numerically by replacing differential derivatives with discrete derivatives. Some characteristic properties of the equation such as linearizability, Hamiltonian structure, integrability, conservation laws, point symmetries or generalized symmetries should be preserved in the discretization. Symmetry-preserving difference schemes, i.e., difference equations and meshes can have more accurate numerical results than standard schemes which do not conserve the geometrical properties of differential equations. Lie symmetries are used in the study of discrete equations on the basis of three subjects that spring from the questions:

- Which type of symmetries will be used?
- How will the symmetries be calculated?
- What can be done with these symmetries?

Various processes have been developed for applying Lie symmetry method to difference equations. The problem of construction invariant difference schemes which preserve symmetries of the original continuous equation was firstly investigated by Dorodnitsyn and coworkers [60, 62, 63, 80, 81]. In this approach, a differential equation and its Lie group are given objects. Lie symmetry groups admitted by differential models of many physical problems are already determined [4, 20]. The procedure then continues by discretizing the given differential equation while preserving its symmetry properties. A mesh is chosen and the infinitesimal generators are prolonged to all points of the mesh. By using these prolonged generators, invariants of the given group that act on the mesh are determined. These invariants are then combined to approximate the given differential equation to a difference equation. Hereby, an invariant difference scheme which consists of a difference equation and equations expressing the mesh points is obtained. In the continuous limits, the mesh equations generally converge to some trivial identities. The transformations of the group act on the equation and on the mesh. Actually, this method is an example of inverse group classification, using group of transformations that act on discrete space. For a given Lie group, inverse group classification provides to find all differential equations which admit the given group as a symmetry using differential invariants of the group. In the method of Dorodnitsyn, the maximal Lie symmetry group of a differential equation is chosen and then using difference invariants, a difference approximation of the differential equation is constructed. In the numerical experiments of difference schemes obtained by this approach, the mesh points may not remain fixed.

In a different approach improved by Levi and collaborators [53, 55, 64, 65, 67, 68, 82, 83], a system of difference equations on a fixed mesh is given. A Lie group of transformations is then determined which leaves the mesh invariant. Several methods exist in this approach according to the conditions on the transformations and the techniques used to find the symmetries. The obtained symmetries may not solve the given difference equation or corresponding differential equation. But they can be used to obtain a symmetry reduction, i.e., a reduction in the number of independent variables. The symmetries act on the equation and on the mesh. It is necessary to adapt the concept of point symmetries to difference equations in order to comprise all point symmetries of the corresponding differential equation in the continuous limit. Infinitesimal generators acting on the discrete space of dependent and independent variables are used to determine point symmetries. Even if the considered equations are nonlinear, symmetries are calculated solving a system of linear equations.

The method of Fels and Olver [12, 69–71, 84, 85] is an alternative approach to construct invariant difference schemes using moving frames. Firstly, Lie symmetry group of a given differential equation and a moving frame for this group are determined. A moving frame is a function that maps a unique element of the group transforming a given point to a point of a submanifold which is a cross-section to the orbits of the group. Since the chosen submanifold has a general property, moving frames are constructed easily and they are used to move an arbitrary function to an invariant function. In this way, existing difference schemes are mapped to a new scheme which is invariant under the Lie group admitted by the given equation. An advantage of this procedure is to use existing difference schemes. Hence existing numerical results can be used to implement invariant difference schemes obtained by this method.

In recent paper [86], Bihlo et al. introduced another approach to obtain invariant difference schemes that approximate multidimensional systems of differential equations. In this approach a system of differential equations and its symmetry group are considered as auxilary objects. Then the system of differential equation is expressed in terms of computational coordinates and the corresponding symmetry transformations are prolonged to the system of computational variables. An invariant difference scheme is constructed with the help of the prolonged transformations. Moreover, some differential equations that determine the location of mesh points are discretized using the difference invariants.

In this chapter we present some preliminaries and notations about transformation groups and prolongations in space of discrete variables that are used in the study of group analysis of difference equations and are given in [26].

2.4.1 Transformation Groups in Space of Discrete Variables

Finite-difference operators are defined on a finite subset of the countable set of mesh points and this provides the operators nonlocality property. The nonlocality of operators leads to some particular features that are absent in differential model such as right and shift differentiations with corresponding shift operators, uniform and nonuniform meshes and certain properties of the Leibniz rule. Because of the nonlocality of difference operators, transformations group can disrupt the proportions, orthogonality and some geometric properties of the mesh. Distortion of the mesh structure can affect difference equations. For example, if orthogonality of the mesh is distorted, geometric meaning of difference derivatives may not be preserved. Thus invariance criterion for the meshes should be given.

Now, we introduce some definitions and notations given in [26]. First consider one-dimensional case with independent variable x, dependent variable u. Let Z be the space of sequences $(x, u, u_1, u_2, ...)$ where $u_1, u_2, \ldots$ are differential variables, u_s is the s-th derivative. Here z represents a vector which consists of finitely many elements of $(x, u, u_1, u_2, ...)$ and z^i represents i-th coordinate of the vector. In space Z a transformation D is defined by the rule

$$D(x) = 1,\ D(u) = u_1,\ \ldots,\ D(u_s) = u_{s+1},\ s = 1, 2, \ldots.$$

The space of analytic functions $F(z)$ of finitely many variables z is represented by A.

Associating D with the action of the first-order linear differential operator

$$D = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \cdots + u_{s+1} \frac{\partial}{\partial u_s} + \cdots , \tag{2.99}$$

generalizes the differentiation operator to functions in A and $D(F(z)) \in A$.

Consider the sequences of formal power series

$$f^i(z,a) = \sum_{k=0}^{\infty} A_k^i(z) a^k, \ i = 1,2,\ldots, \tag{2.100}$$

with a parameter a, where $A_k^i(z) \in A$ and $A_0^i \equiv z^i$, z^i is the i-th coordinate of a vector in Z. The set of sequences

$$\left(f^1(z,a),\ f^2(z,a),\ldots,f^s(z,a),\ldots\right)$$

of formal power series (2.100) is represented by $\tilde{Z}$. The sequences $(x, u, u_1, u_2, ...)$ are included in space of such sequences and $Z \subset \tilde{Z}$.

In the space $\tilde{Z}$, the group of formal transformations generated by the total derivative operator (2.99) is constructed. Then the transformations of the group are established by the exponential operator $T_a = e^{aD}$,

$$\overline{z^i} = e^{aD}(z^i) = \sum_{s=0}^{\infty} \frac{a^s}{s!} D^{(s)}(z^i).$$

The point z is transformed to the point $\overline{z} \in \tilde{Z}$ with coordinates

$$\begin{aligned}
&\overline{x} = T_a(x) = x + a, \\
&\overline{u} = T_a(u) = \sum_{s=0}^{\infty} \frac{a^s}{s!} u_s, \\
&\overline{u_1} = T_a(u_1) = \sum_{s=0}^{\infty} \frac{a^s}{s!} u_{s+1}, \\
&\cdots\cdots\cdots \\
&\overline{u_k} = T_a(u_k) = \sum_{s=0}^{\infty} \frac{a^s}{s!} u_{s+k}, \\
&\cdots\cdots\cdots
\end{aligned} \tag{2.101}$$

The transformations (2.101) are the expansions of the function $u = u(x)$ to the Taylor series at the point $x + a$ and hence the transformations group (2.101) generated by operator (2.99) is called Taylor group. The definition and geometric meaning of the derivatives $(u_1, u_2, ...)$ are preserved under the Taylor group. That is, the Taylor group leaves invariant the system of equations

$$\begin{aligned} &du = u_1 dx, \\ &du_1 = u_2 dx, \\ &\cdots\cdots\cdots \\ &du_s = u_{s+1} dx, \\ &\cdots\cdots\cdots \end{aligned} \tag{2.102}$$

The Taylor group is a higher-order symmetry group and a useful tool for studying in the space of difference variables.

By setting an arbitrary parameter $a = h > 0$ and by using the generator (2.99) of the Taylor group, *right* and *left discrete shift operator* are obtained respectively as

$$\underset{+h}{S} = e^{hD} = \sum_{s=0}^{\infty} \frac{h^s}{s!} D^s, \tag{2.103}$$

$$\underset{-h}{S} = e^{-hD} = \sum_{s=0}^{\infty} \frac{(-h)^s}{s!} D^s, \tag{2.104}$$

where D is a derivative operator in $\tilde{Z}$.

Using shift operators $\underset{+h}{S}$ and $\underset{-h}{S}$, a pair of *right and left finite-difference differentiation operators* are obtained as

$$\underset{+h}{D} = \frac{1}{h}(\underset{+h}{S} - 1) = \sum_{s=1}^{\infty} \frac{h^{s-1}}{s!} D^s, \tag{2.105}$$

$$\underset{-h}{D} = \frac{1}{h}(1 - \underset{-h}{S}) = \sum_{s=1}^{\infty} \frac{(-h)^{s-1}}{s!} D^s. \tag{2.106}$$

The countable set

$$\underset{h}{w} = \{x_\alpha = \underset{+h}{S}^{(x)}\}, \ \alpha = 0, \pm 1, \pm 2, \ldots, \tag{2.107}$$

of values of independent variable x is called *uniform difference mesh*.

The *finite-difference* (or *discrete*) derivative of order s is denoted by $\underset{h}{u_s}$ and defined by a special form of formal power series:

$$\begin{aligned} &\underset{h}{u_1} = \underset{+h}{D}(u), \\ &\underset{h}{u_2} = \underset{-h+h}{DD}(u), \\ &\underset{h}{u_3} = \underset{+h-h+h}{DDD}(u), \\ &\ldots\ldots\ldots \end{aligned} \tag{2.108}$$

The sequences $\left(\underset{h}{u_1}, \underset{h}{u_2}, \underset{h}{u_3}, \ldots\right)$ of finite-difference derivatives are denoted by $\underset{h}{Z}$ and the product of the spaces $\underset{h}{Z}$ and $\tilde{Z}$ is denoted by $\underset{h}{\tilde{Z}}$,

$$\underset{h}{\tilde{Z}} = \left(x, u, u_1, u_2, \ldots; \underset{h}{u_1}, \underset{h}{u_2}, \ldots\right).$$

2.4.1.1 Formulation in the Multidimensional Case

The following materials are based on the content of [26]. Let Z be the space of sequences $(x, u, u_1, u_2, ...)$ where $x = \left\{x^i \mid i = 1, 2, ..., n\right\}$ are independent variables and $u = \left\{u^k \mid k = 1, 2, ..., m\right\}$ are dependent variables. The partial derivatives are given by $u_1 = \left\{u_i^k\right\}$ which is the set of mn first partial derivatives, $u_2 = \left\{u_{ij}^k\right\}$ which is the set of second partial derivatives, etc. The derivation is given by two operators

$$D_1 = \frac{\partial}{\partial x^1} + u_1 \frac{\partial}{\partial u} + u_{11} \frac{\partial}{\partial u_1} + u_{21} \frac{\partial}{\partial u_2} + ...,$$

$$D_2 = \frac{\partial}{\partial x^2} + u_2 \frac{\partial}{\partial u} + u_{12} \frac{\partial}{\partial u_1} + u_{22} \frac{\partial}{\partial u_2} + ...,$$

where

$$u_1 = \frac{\partial u}{\partial x^1}, u_{11} = \frac{\partial^2 u}{\partial \left(x^1\right)^2}, u_{21} = \frac{\partial^2 u}{\partial x^2 \partial x^1}, \ldots$$

in the case $n = 2$ and $x = \left(x^1, x^2\right)$. For simplicity we neglect the superscript k on u^k. The operators D_1 and D_2 generate two permuting Taylor groups [60] with finite transformations $\underset{a}{T^1} = e^{aD_1}$ and $\underset{a}{T^2} = e^{aD_2}$. The shift operators

$$\underset{\pm h}{S_1} = e^{\pm h_1 D_1} \equiv \sum_{s \geq 0} \frac{(\pm h_1)^s}{s!} D_1^s, \tag{2.109}$$

$$\underset{\pm h}{S_2} = e^{\pm h_2 D_2} \equiv \sum_{s \geq 0} \frac{(\pm h_2)^s}{s!} D_2^s \tag{2.110}$$

are obtained setting the arbitrary parameter values as $h_1, h_2 > 0$. Using the shift operators a couple of differentiation operators in discrete space are given by

$$\underset{+h}{D_i} = \pm\frac{1}{h}(\underset{\pm h}{S_i} - 1),\ i = 1,2. \tag{2.111}$$

A *uniform orthogonal difference mesh* is defined by the set of points

$$\left\{\underset{\pm h_1}{S^{\alpha}}\left(x^1\right), \underset{\pm h_2}{S^{\beta}}\left(x^2\right)\right\},\ \alpha, \beta = 0, 1, 2, ...,$$

in the $\left(x^1, x^2\right)$-plane and denoted by $\underset{h}{\omega}$.

2.4.1.2 Invariance Criterion for Uniform and Orthogonal Meshes

In the space of discrete variables $(x, u, u_1, u_2, \ldots; \underset{h}{u_1}, \underset{h}{u_2}, \ldots)$, under a Lie group of transformations G_1 generated by the operator

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \sum_{s\geq 1}\zeta^s\frac{\partial}{\partial u_s} + \sum_{m\geq 1}\underset{h}{\zeta^m}\frac{\partial}{\partial \underset{h}{u_m}}, \tag{2.112}$$

the transformations acting on the mesh space are defined as

$$\overline{h_+} = \underset{+h}{S}(\overline{x}) - \overline{x} = f(\underset{+h}{S}(z), a) - f(z, a), \tag{2.113}$$

$$\overline{h_-} = \overline{x} - \underset{-h}{S}(\overline{x}) = f(z, a) - f(\underset{-h}{S}(z), a). \tag{2.114}$$

Uniformity of a mesh is conserved by the action of right and left finite-difference differentiation operators.

Theorem 2.6. *[26] The group of transformations G_1 preserves the uniformity ($\overline{h_+} = \overline{h_-}$) of the mesh $\underset{h}{w}$ if and only if the condition*

$$\underset{+h}{D}\underset{-h}{D}(\xi(z)) = 0 \tag{2.115}$$

is satisfied at each point z of the given mesh.

The following theorem states the invariance criterion for orthogonal meshes.

Theorem 2.7. *[26] A Lie group of transformations G_1 leaves invariant an orthogonal mesh $\underset{h}{w}$ that is uniform or nonuniform if and only if the condition*

$$\underset{\pm h}{D_i}(\xi^j) = -\underset{\pm h}{D_j}(\xi^i),\ i \neq j, \tag{2.116}$$

is satisfied at each point z of the given mesh.

An orthogonal mesh $\underset{h}{w}$ directed to an angle α with the coordinate axes preserves its orthogonality if the following condition

$$\underset{+h}{D_2}(\xi^1)\cos\alpha - \underset{+h}{D_1}(\xi^1)\sin\alpha + \underset{+h}{D_2}(\xi^2)\sin\alpha + \underset{+h}{D_1}(\xi^2)\cos\alpha = 0$$

is satisfied.

2.4.2 Prolongation Formulas for Finite-Difference Derivatives

In this thesis, we consider only the prolongation of discrete variables in two-dimensional case. For other situations, the detailed information is given in [26].

In two dimensional case with dependent variable u, independent variables t, x and mesh variables h_1, h_2, we denote the spaces of differential variables, difference variables and the product of those spaces which is the space of sequences of formal power series by

$$\tilde{Z} = (t, x, u, u_t, u_x, u_{tx}, \ldots), \tag{2.117}$$

$$\underset{h}{Z} = (t, x, u, \underset{h}{u_t}, \underset{h}{u_x}, \underset{h}{u_{tx}}, \ldots, h_1, h_2), \tag{2.118}$$

$$\underset{h}{\tilde{Z}} = (t, x, u, u_t, u_x, \ldots, \underset{h}{u_t}, \underset{h}{u_x}, \underset{h}{u_{tx}}, \ldots, h_1, h_2), \tag{2.119}$$

where

$$u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}, \ \underset{h}{u_{ij}} = \underset{+h}{D_j}\underset{+h}{D_i}(u), \ldots, \ \underset{h}{\omega} = \underset{h}{\omega_1} \times \underset{h}{\omega_2} \tag{2.120}$$

and $\underset{h}{\omega_i}$ is the difference mesh in the i-th direction, respectively.

Transformations in $\underset{h}{\tilde{Z}}$ are defined by the sequence of series with analytic coefficients,

$$z^{j*} = \sum_{s\geq 0} A_s^j(z)a^s, A_0^j = z^j, \tag{2.121}$$

where z^j is a coordinate of the vector $(t, x, u, u_t, u_x, \ldots, \underset{h}{u_t}, \underset{h}{u_x}, \underset{h}{u_{tx}}, \ldots)$. These series form one-parameter groups generated by infinitesimal operators

$$X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \eta^k \frac{\partial}{\partial u^k} + \sum_{s\geq 1} \zeta_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}} + \sum_{l\geq 1} \zeta_{i_1 i_2 \ldots i_l} \frac{\partial}{\partial \underset{h}{u_{i_1 i_2 \ldots i_l}}}. \tag{2.122}$$

Prolongating the operator (2.122) for the variables h_1 and h_2 gives

$$X = \cdots + h_1 \underset{+h}{D_1}(\xi^t)\frac{\partial}{\partial h_1} + h_2 \underset{+h}{D_2}(\xi^x)\frac{\partial}{\partial h_2}. \tag{2.123}$$

For the first-order difference derivatives the coordinates of prolongation operator are given by formulas

$$\underset{h}{\zeta_t} = \underset{+h}{D_1}(\eta) - \underset{h}{u_t}\underset{+h}{D_1}(\xi^t) - \underset{+h}{S_1}(u_x)\underset{+h}{D_1}(\xi^x), \tag{2.124}$$

$$\underset{h}{\zeta_x} = \underset{+h}{D_2}(\eta) - \underset{+h}{S_2}(u_t)\underset{+h}{D_2}(\xi^t) - \underset{h}{u_x}\underset{+h}{D_2}(\xi^x). \tag{2.125}$$

If the considered mesh is invariantly uniform or invariantly orthogonal then the corresponding formulas for invariant meshes must be satisfied in addition to prolongation formulas (2.124)-(2.125).

2.5 Application of Lie Symmetries to Boundary Value Problems

Application of Lie groups to BVPs for ODEs is easier than the case for PDEs. For a BVP of an ODE, the order of the ODE is reduced by an admitted integrating factor or symmetry group (point or higher-order). In integrating factor reduction, the original variables and in symmetry reduction, differential invariants of the group are used to obtain a BVP for a lower order ODE.

A BVP for a scalar PDE, or system of PDEs, is invariant under a Lie transformations group if the group leaves invariant the boundary, the boundary conditions, and the equations of the BVP simultaneously. However, the concerned boundary conditions are usually not invariant under the symmetry of the considered PDEs. Hence studies about Lie symmetry analysis of BVPs are very limited. The solution of a BVP is an invariant solution of the corresponding Lie group of point transformations if the BVP is well-posed. But in the case of a linear BVP, point symmetries do not have to leave invariant the boundary conditions of the given problem. Moreover, when applying Lie symmetries to a linear nonhomogeneous PDE, invariance of the associated homogeneous PDE is sufficient. Because a homogeneous PDE is always invariant under a uniform scaling of its dependent variables. In this case, some of the boundary conditions may not be invariant (this called as incomplete invariance) under the symmetry admitted by the PDE. Invariant solutions and invariant forms are obtained from invariance of the corresponding homogeneous BVP. A superposition of invariant solutions or invariant forms can be used for the construction of solutions of the given BVP. In the case of invariance of a BVP under a multiparameter Lie point transformations group, general solution of the given BVP is obtained in an easier way.

In the study of Lie symmetry analysis of BVPs, invariance of boundary and boundary curve is essential but invariance of the associated equation is not necessary. It is because there are BVPs that have Lie symmetries different from the corresponding equation's symmetries. As a simple example for this situation we can consider the simplest $3rd-$ order ODE

$$y''' = 0,$$

which has seven Lie point symmetries:

$$X_1 = 1,\ X_2 = x,\ X_3 = y,\ X_4 = y',\ X_5 = xy',$$

$$X_6 = x^2,\ X_7 = 2xy - x^2y'.$$

Applying the initial condition $y''(0) = 0$ results in reduction order of given the ODE to

$$y'' = 0,$$

which has eight Lie point symmetries

$$X_1 - X_5,\ X_6 = yy',\ X_7 = xy - x^2y',\ X_8 = y^2 - xyy'.$$

It is generally more difficult to work with BVPs defined at free boundaries than with standard BVPs defined at fixed boundaries. However, application of the Lie symmetry method to BVPs with moving boundaries is sometimes more useful just for solving given BVPs. Because such boundaries can depend on invariant variables which reduce the given BVP to a BVP for a lower order differential equation. For this reason, when applying the Lie symmetry method to BVPs researchers studied BVPs with free boundaries rather than BVPs with fixed boundaries [18, 87, 88].

Invariance of a BVP under a Lie group is determined by infinitesimal generator of the group. In this direction, first definition of invariance for a BVP was given by Bluman [22].

Consider a k-th order ($k \geq 2$) scalar PDE denoted by

$$F(x, y, \partial y, \partial^2 y, \ldots, \partial^k y) = 0, \tag{2.126}$$

where $x = (x_1, x_2, \ldots, x_n)$ represents the coordinates corresponding to its n independent variables, y represents the coordinate corresponding to its dependent variable, and $\partial^j y$ represents the coordinates with components $\partial^j y / \partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_j} = y_{i_1 i_2 \ldots i_j}, i_j = 1, 2, \ldots, n$, for $j = 1, 2, \ldots, k$, corresponding to all j-th order partial derivatives of y with respect to x.

We rewrite PDE (2.126) with respect to the components of l-th order partial derivatives of y:

$$F(x, y, \partial y, \partial^2 y, \ldots, \partial^k y) = y_{i_1 i_2 \ldots i_l} - f(x, y, \partial y, \partial^2 y, \ldots, \partial^k y) = 0 \qquad (2.127)$$

where $f(x, y, \partial y, \partial^2 y, \ldots, \partial^k y)$ does not depend explicitly on $y_{i_1 i_2 \ldots i_l}$.

Now, consider a BVP for PDE (2.127) defined on a domain D in x-space $[x = (x_1, x_2, \ldots, x_n)]$ with boundary conditions

$$B_a(x, y, \partial y, \ldots, \partial^{k-1} y) = 0, \qquad (2.128)$$

described on boundary surfaces

$$\Omega_a(x) = 0, \ a = 1, 2, \ldots, s. \qquad (2.129)$$

We assume that BVP (2.127)-(2.129) has a unique solution. Consider an infinitesimal generator of the form

$$X = \xi_i(x) \frac{\partial}{\partial x_i} + \eta(x, y) \frac{\partial}{\partial y}, \qquad (2.130)$$

which defines a point symmetry acting on both (x, y)-space as well as on its projection to x-space.

Definition 2.12 (*Bluman's definition for invariance of BVPs*). The point symmetry X of form (2.130) is admitted by BVP (2.127)-(2.129) if and only if:

1. $X^{(k)} F(x, y, \partial y, \partial^2 y, \ldots, \partial^k y) = 0$ when $F(x, y, \partial y, \partial^2 y, \ldots, \partial^k y) = 0$;
2. $X \Omega_a(x) = 0$ when $\Omega_a(x) = 0, a = 1, 2, \ldots, s$;
3. $X^{(k-1)} B_a(x, y, \partial y, \ldots, \partial^{k-1} y) = 0$ when $B_a(x, y, \partial y, \ldots, \partial^{k-1} y) = 0$ on $\Omega_a(x) = 0, a = 1, 2, \ldots, s$.

This definition is applicable for BVPs with standard boundaries. But there are also BVPs with free boundaries or with boundary conditions defined at infinity. Because BVPs defined at free boundaries have moving surfaces such as $\Omega_b(x) = 0, b = 1, 2, \ldots, q$, where $\Omega_b(x)$ are unknown functions. These functions are considered as additional variables. Another major defect of this definition arises for the case BVPs in the unbounded domain. When the boundary conditions for $x = \infty$ is considered, the second axiom is meaningless. Thus Chernica et al. [76, 77], proposed a new definition of invariance for BVPs which extends Bluman's definition to all possible boundary conditions. They formulated the definition of invariance for BVPs at the case of operators of conditional symmetry describing what kind of transformations can be applied to transform boundary conditions at infinity to those containing no conditions at infinity.

Firstly, we give the definition of invariance for a BVP presented in [76]. For our purpose, we consider a BVP for a PDE with one dependent variable u and two independent variables t, x

$$u_t = f(x, u, \partial u, \partial^2 u, \dots, \partial^k u), \tag{2.131}$$

described on a domain $\omega \in \mathbb{R}^2$ with smooth boundaries. There exist three kinds of boundary conditions that can be seen in problems:

$$\omega_a(t,x) = 0: \ B_a(t, x, u, \partial u, \dots, \partial^{k_a} u) = 0, \ a = 1, 2, \dots, s, \tag{2.132}$$

$$W_b(t,x) = 0: \ B_b(t, x, u, \partial u, \dots, \partial^{k_b} u, W_b, \frac{\partial W_b}{\partial t}, \frac{\partial W_b}{\partial x}) = 0, \ b = 1, 2, \dots, r, \tag{2.133}$$

$$\gamma_c(t,x) = \infty: \ \Gamma_c(t, x, u, \partial u, \dots, \partial^{k_c} u) = 0, \ c = 1, 2, \dots, q, \tag{2.134}$$

where $k_a, k_b, k_c < k$ are given numbers, $\omega_a(t,x)$, and $\gamma_c(t,x)$ are known functions. The functions $W_b(t,x)$ define free boundary curves and one they have to be found. We assume that all functions arising in (2.131)-(2.134) are given such that a classical solution of this BVP exists.

Let us assume that PDE (2.131) admits a one-parameter Lie group generated by the operator

$$X = \xi_t(t,x,u)\frac{\partial}{\partial t} + \xi_x(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2.135}$$

For the application of this group to BVP (2.131)-(2.134), we prolong operator (2.135) to the space of variables (t, x, u, W):

$$prX = \xi_t(t,x,u,W)\frac{\partial}{\partial t} + \xi_x(t,x,u,W)\frac{\partial}{\partial x} + \eta(t,x,u,W)\frac{\partial}{\partial u} + \zeta(t,x,u,W)\frac{\partial}{\partial W}. \tag{2.136}$$

Definition 2.13 (*Chernica's first definition for invariance of BVPs*). BVP (2.131)-(2.134) is *invariant* under the Lie group $\tilde{X}$ (2.136) if and only if

1. PDE (2.131) is invariant with respect to the k-th prolongation of the Lie group generated by operator (2.135);
2. Each equation given by condition (2.132) is invariant with respect to the k^a-th prolongation of the Lie group generated by operator (2.135);

3. Each equation given by condition (2.133) is invariant with respect to the k^b-th prolongation of the Lie group with infinitesimal generator (2.136);
4. Each equation given by condition (2.134) is invariant with respect to the k^c-th prolongation of the Lie group with infinitesimal generator (2.135).

Although this definition is suitable for the broader classes of BVPs, there are some BVPs where no definition of invariance under a Lie group can be applied. Because the corresponding differential equation have a trivial symmetry only or is not invariant under any symmetry group. For this reason in [77], a new definition which is valid for more general types of symmetries including Q-conditional symmetry is developed.

Consider a BVP for PDE (2.127) with boundary conditions (2.128) and conditions defined at infinity:

$$\gamma_c(x) = \infty : \gamma_c(x, y, \partial y, \dots, \partial^{k_c} y) = 0, \; c = 1, 2, \dots, p_\infty, \tag{2.137}$$

where $k_c < k$ and p_∞ are given numbers and $\gamma_c(x)$ are specified functions that extends the domain on which the BVP is defined to infinity in some directions. We assume that all functions arising in (2.127), (2.128), (2.129), and (2.137) are given such that a classical solution of this BVP exists.

Let us assume that the operator

$$Q = \xi_i(x, y)\frac{\partial}{\partial x_i} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.138}$$

is a Q-conditional symmetry of PDE (2.127) satisfying the criterion

$$Q^{(k)} F(x, y, \partial y, \partial^2 y, \dots, \partial^k y)|_{F(x,y,\partial y,\partial^2 y,\dots,\partial^k y)=0} = 0, \tag{2.139}$$

where $Q^{(k)}$ is the k-th prolongation of Q and $Q(y) = 0$ with $Q(y) = \xi_i(x, y) y_{x_i} - \eta(x, y)$.

Let us consider for each $c = 1, 2, \dots, p_\infty$ the manifold:

$$M = \{\gamma_c(x) = \infty : \gamma_c(x, y, \partial y, \dots, \partial^{k_c} y) = 0\} \tag{2.140}$$

in the extended space of variables $x, y, y_x, \dots, y_x^{(k_c)}$. We assume that there exists a smooth bijective transformation of the form:

$$t = g(x), \; w = h(x, y) \tag{2.141}$$

where $h(x, y)$ is a smooth function, $g(x)$ is a smooth vector function that maps the manifold M into

$$M^* = \{\gamma_c^*(t) = 0 : \gamma_c^*(t, y, \partial y, \ldots, \partial^{k_c^*} y) = 0\} \tag{2.142}$$

of the same dimensionality in the extended space $t, w, w_t, \ldots, w_t^{(k_c)}(k_c^* \leq k_c)$ and $t = t_1, \ldots, t_n$.

Definition 2.14 (*Chernica's second definition for invariance of BVPs*). BVPs (2.127), (2.128), and (2.137) are Q-conditionally invariant under operator (2.130) if:

1. Criterion (2.139) is satisfied;
2. $Q(\Omega_a(x)) = 0$ when $\Omega_a(x) = 0, B_a|_{\Omega_a(x)=0} = 0, a = 1, \ldots, s$;
3. $Q^{(k)}(B_a(x, y, \partial y, \ldots, \partial^{k-1} y)) = 0$ when $\Omega_a(x) = 0$ and $B_a|_{\Omega_a(x)=0} = 0, a = 1, \ldots, s$;
4. There exists a smooth one-to-one and onto transform (2.141) that maps M into M^* with the same dimension;
5. $Q^*(\gamma_c^*(t)) = 0$ when $\gamma_c^*(t) = 0, c = 1, 2, \ldots, p_\infty$;
6. $(Q^*)^{(k_c^*)}(\gamma_c^*(t, y, \partial y, \ldots, \partial^{k_c^*} y)) = 0$ when $\gamma_c^*(t) = 0$ and $\gamma_c^*|_{\gamma_c^*(t)=0} = 0, c = 1, 2, \ldots, r$.

This definition is the same with Definition 2.12 if Q is a Lie symmetry transformation and there is not any boundary condition defined at infinity.

3
Main Results

Lie symmetry method represents a valuable and useful alternative to the study of difference equations as well as differential equations. This method has many practical applications. For example, discretization of differential equations preserving its characteristic properties such as conservation laws, Hamiltonian structure gives better numerical results. Lie symmetries have been applied to difference equations in several aspects; determining of symmetry group of a difference equation leaving the mesh on which the equation is written invariant, reducing the number of variables in equations, constructing invariant difference models for a given differential equation.

Symmetry analysis of BVPs is also a crucial application. A differential equation without boundary conditions does not represent any real phenomenon. Hence invariant BVPs under a symmetry group form a prominent, substantial models for the governing equations.

The sine-Gordon equation is a type of classical wave equations with a nonlinear sine source term. The sine-Gordon equation arises in differential geometry and in many branches of physics including relativistic field theory, Josephson junctions, dislocations in crystals, self-induced transparency in optics, charge-density waves in one-dimensional metals or mechanical trasmission lines. The equation has soliton solutions.

This chapter is devoted to our main results which consist of the application of Lie symmetries to the sine-Gordon equation. Throughout this research our original work is on the sine-Gordon equation and concluded with two significant results:

- A symmetry group for the discrete sine-Gordon equation leaving the mesh invariant is obtained.
- Invariance conditions for the discrete BVP of the sine-Gordon equation under the symmetry group of the related equation are determined.

3.1 Lie Point Symmetries of Difference Scheme for the sine-Gordon Equation

The studies about Lie symmetry analysis of difference equations spring from several approaches. Some of these approaches [12, 62, 80] deal with construction invariant difference schemes that conserve all symmetries of the original differential equation while others [53, 55, 64] deal with computing the symmetry group admitted by a difference equation.

In this section, we examine the difference equation for sine-Gordon equation with respect to Lie symmetry analysis [89]. Firstly, we construct a difference equation for the sine-Gordon equation according to the method of Dorodnitsyn. We use difference invariants and study on a five-point uniform and orthogonal mesh. The structure of the mesh is preserved satisfying invariance criteria for uniform and orthogonal meshes.

As a second step of our study, we investigate the set of point symmetries of the discrete sine-Gordon equation. We used a variation of Levi's procedure given in [53]. They proposed an algorithm to find symmetry group of an ordinary difference equation. Specifically, in our procedure we deal with a nonlinear PDE. Hence we extend Levi's method for partial difference equations. We study on a fixed mesh given by two equations. Applying infinitesimal generator criterion to the difference model of the sine-Gordon equation, we obtain a system of linear equations. Even if our main consideration is a nonlinear equation, in the process we work with linear equations. Since we only look for point symmetries, infinitesimal generator acting on the space of dependent and independent variables is used.

In this section we present the applications of these methods. In the first approach, we begin with a differential equation and then try to form an invariant difference scheme approximating the given equation. Let us consider the equation

$$u_{xt} = \sin u. \tag{3.1}$$

Equation (3.1) admits three-parameter point transformation group (e.g., see [20]) generated by operators

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t},\ X_3 = x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t}. \tag{3.2}$$

Equation (3.1) is equivalent to the equation which is called the *sine-Gordon equation*

$$v_{rr} - v_{ss} = \sin v, \tag{3.3}$$

via the point transformation

$$r = x + t, \ s = x - t, \ u(x,t) = v(r,s). \tag{3.4}$$

The operators

$$X_1 = \frac{\partial}{\partial r}, \ X_2 = \frac{\partial}{\partial s}, \ X_3 = s\frac{\partial}{\partial r} + r\frac{\partial}{\partial s} \tag{3.5}$$

describes symmetry group of equation (3.3).

For an invariant discretization of equation (3.1), a mesh on which the continuous limit of the equation exists should be determined so that it is invariant with respect to the group defined by the operators (3.2). Since the transformation group given by the operators (3.2) leaves invariant the dependent variable u, the invariance of the mesh can be considered independently from the invariance of the difference equation approximating (3.1). Thus, we can use the simplest orthogonal mesh that is uniform in both directions. The invariance conditions for uniform meshes (2.115) and for orthogonal meshes (2.116) hold for all operators of (3.2). The solutions of the difference equation approximating (3.1) satisfy these conditions.

Any of the variables $u_n^k, u_{n+1}^k, u_{n-1}^k, u_n^{k+1}, u_{n+1}^{k+1}, u_{n-1}^{k+1}, u_n^{k-1}, u_{n-1}^{k-1}, u_{n+1}^{k-1}$ can approximate the right-hand side of (3.2) at (x,t,u) since the given variables are invariants of the group (3.2). We prefer to use u_n^k. The function

$$I_{10} = h_x h_t \tag{3.6}$$

is also an invariant in the space of discrete variables. Now, second-order approximations to the mixed derivative u_{xt} are constructed as

$$u_{xt} \approx \left(\frac{u_{n+1}^{k+1} - u_{n-1}^{k+1}}{2h_x} - \frac{u_{n+1}^{k-1} - u_{n-1}^{k-1}}{2h_x} \right) \frac{1}{2h_t} + O(h^2) \tag{3.7}$$

which finally implies the equation

$$\frac{u_{n+1}^{k+1} - u_{n-1}^{k+1} - u_{n+1}^{k-1} + u_{n-1}^{k-1}}{4h_x h_t} = \sin u_n^k. \tag{3.8}$$

Using Taylor expansions, we show that difference equation (3.8) on an orthogonal mesh leads to a second-order approximation to the differential equation (3.1) preserving all the symmetry (3.2) of the equation in differential form :

$$\frac{u_{n+1}^{k+1} - u_{n-1}^{k+1} - u_{n+1}^{k-1} + u_{n-1}^{k-1}}{4h_x h_y} - \sin u_n^k = u_{xy} - \sin u + O(h_x^2 + h_y^2). \tag{3.9}$$

By the group of transformations (3.4), the difference equation (3.8) is taken into the equation

$$\frac{v_{n+1}^{k+1} - v_{n-1}^{k+1} - v_{n+1}^{k-1} + v_{n-1}^{k-1}}{4h_r h_s} = \sin v_n^k, \tag{3.10}$$

where h_r and h_s are mesh variables, and thus a symmetry preserving difference system, i.e., an invariant difference equation and a mesh for differential equation (3.3) are constructed. Since equation (3.1) is invariant under the point transformations group (3.4), mapping operators (3.2) into operators (3.5), equation (3.10) on the diagonal orthogonal mesh admits the complete group defined by operators (3.5).

Now, we present the second approach applying a method that extends the method of Levi et al. [53] used for ordinary differential equations to the partial differential equations sin particular to the sine-Gordon equation

$$u_{tt} - u_{xx} = \sin u. \tag{3.11}$$

We consider the following finite-difference scheme on the uniform mesh

$$F : \frac{\hat{u} - 2u + \check{u}}{\tau^2} - \frac{u_+ - 2u + u_-}{h^2} = \sin u, \tag{3.12}$$

$$\Omega : \hat{t} - 2t + \check{t} = 0, x_+ - 2x + x_- = 0 \tag{3.13}$$

for equation (3.11). Here difference equation (3.12) is formed on a five-point stencil, τ, h are mesh variables and the translations on the mesh variables given with $\hat{t} = t+\tau,\ \check{t} = t-\tau,\ x_+ = x+h,\ x_- = x-h,\ \hat{u} = u(t+\tau, x),\ \check{u} = u(t-\tau, x),\ u_+ = u(t, x+h),\ u_- = u(t, x-h)$. The infinitesimal generator prolonged to the finite-difference differentiation variables in the discrete subspace

$$(t, x, \hat{t}, \check{t}, x_+, x_-, u, \hat{u}, \check{u}, u_+, u_-)$$

is given by

$$prX = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \hat{\xi}^t \frac{\partial}{\partial \hat{t}} + \check{\xi}^t \frac{\partial}{\partial \check{t}} + \xi^x_+ \frac{\partial}{\partial x_+}$$
$$+ \xi^x_- \frac{\partial}{\partial x_-} + \hat{\eta} \frac{\partial}{\partial \hat{u}} + \check{\eta} \frac{\partial}{\partial \check{u}} + \eta_+ \frac{\partial}{\partial u_+} + \eta_- \frac{\partial}{\partial u_-}. \tag{3.14}$$

Applying this prolongation operator to difference scheme (3.12)-(3.13) to provide the invariance criterions $prXF = 0, prX\Omega = 0$ with $F = 0, \Omega = 0$, we get the system of equations

$$\eta(-2h^2 + 2\tau^2 - h^2\tau^2 \cos u) + (\check{\xi}^t - \hat{\xi}^t)[\tau(u_+ - 2u + u_-) + h^2 \tau \sin u]$$

$$+(\xi^x_+ + \xi^x_-)[h(\hat{u}-2u+\check{u})-h\tau^2\sin u]+(\hat{\eta}+\check{\eta})h^2-(\eta_+ + \eta_-)\tau^2=0, \tag{3.15}$$

$$\hat{\xi}^t-2\xi^t+\check{\xi}^t=0, \tag{3.16}$$

$$\xi^x_+-2\xi^x+\xi^x_-=0. \tag{3.17}$$

Firstly, we consider equation (3.16) to obtain infinitesimal transformation for the variable t. Using equation (3.12), we write the term $\hat{u}$ in tems of $\check{u}$ and u and substituting the obtained term in equation (3.16) gives

$$\xi^t\left(\hat{t},x,\frac{h^2(2u-\check{u})+\tau^2(u_+-2u+u_-)+h^2\tau^2\sin u}{h^2}\right)$$

$$-2\xi^t(t,x,u)+\xi^t(\check{t},x,\check{u})=0. \tag{3.18}$$

Differentiating equation (3.18) with respect to the variable $\check{u}$, we obtain the equation

$$-\hat{\xi}^t_{\hat{u}}+\check{\xi}^t_{\check{u}}=0,$$

which is resulted that $\hat{\xi}^t_{\hat{u}}$ and $\check{\xi}^t_{\check{u}}$ are not depend on t, that is

$$\hat{\xi}^t_{\hat{u}}=\check{\xi}^t_{\check{u}}=a(x).$$

From the last equation, we get the result that ξ^t is linear in u

$$\xi^t(t,x,u)=a(x)u+b(t,x).$$

Now, we seek for the coefficients of ξ^t. Substituting the last expression in equation (3.18) gives the equation

$$a(x)(\hat{u}-2u+\check{u})+b(\hat{t},x)-2b(t,x)+b(\check{t},x)=0. \tag{3.19}$$

Considering equation (3.19) as a polynomial and equating coefficients, we get $a(x)=0$ and

$$b(2t-\check{t},x)-2b(t,x)+b(\check{t},x)=0.$$

Differentiating the last equation according to the variables t and $\check{t}$ respectively, we obtain $b_{\hat{t}\hat{t}}(\hat{t},x)=0$ and integrating the differential equation gives

$$b(t,x)=b_1(x)t+b_0(x).$$

Consequently from the invariance of mesh in time variable, we find the coordinate in the prolongation operator for the time variable as

$$\xi^t = b_1(x)t + b_0(x). \tag{3.20}$$

We perform the same process for ξ^x and obtain the infinitesimal for the variable x as

$$\xi^x = d_1(t)x + d_0(t). \tag{3.21}$$

Now, we use ξ^t and ξ^x to find the last coordinate of the generator (3.14). Substituting the infinitesimals ξ^t and ξ^x in equation (3.15) gives

$$\begin{aligned}\eta(t,x,u)(-2h^2+2\tau^2-h^2\tau^2\cos u)-2\tau^2 b_1(x)(u_+-2u+u_-+h^2\sin u)\\ +2h^2 d_1(t)(\hat{u}-2u+\check{u}-\tau^2\sin u)\\ +\left[\hat{\eta}\left(\hat{t},x,\frac{h^2(2u-\check{u})+\tau^2(u_+-2u+u_-)+h^2\tau^2\sin u}{h^2}\right)\right.\\ \left.+\check{\eta}(\check{t},x,\check{u})\right]h^2-(\eta_++\eta_-)\tau^2=0.\end{aligned} \tag{3.22}$$

We differentiate this expression with respect to $\check{u}$ and u respectively to eliminate these terms and get

$$2h^2 d_1(t) + (-\hat{\eta}_{\hat{u}} + \check{\eta}_{\check{u}}) = 0, \tag{3.23}$$

$$\hat{\eta}_{\hat{u}\hat{u}}(\hat{t},x,\hat{u}) = 0. \tag{3.24}$$

Integrating equation (3.24) twice gives we get the infinitesimal which is linear in u

$$\eta = \alpha_1(x)u + \alpha_0(t,x).$$

Substituting this term into equation (3.22) and equating coefficients of u, $\cos u$ and $\sin u$ in the resulting equation gives

$$\alpha_0 = \alpha_1 = d_1 = b_1 = 0.$$

Thus, we obtain the solution of the determining system (3.15)-(3.17) as

$$\eta = 0, \xi^t = b_0(x), \xi^x = d_0(t), \tag{3.25}$$

where b_0 and d_0 are arbitrary functions of x and t, respectively. In particular choosing the arbitrary functions as x and t we obtain three-parameter group of point transformations

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} \tag{3.26}$$

for difference scheme (3.12)-(3.13). This shows that the difference equation (3.12) on the uniform and orthogonal mesh (3.13) preserves the point symmetries of its differential form (3.11). In the space of finite-difference variables, extending to all variables, the group generators can be written as

$$X_1 = \frac{\partial}{\partial t} + \frac{\partial}{\partial \hat{t}} + \frac{\partial}{\partial \check{t}},\ X_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial x_+} + \frac{\partial}{\partial x_-},$$

$$X_3 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} + x\frac{\partial}{\partial \hat{t}} + x\frac{\partial}{\partial \check{t}} + t\frac{\partial}{\partial x_+} + t\frac{\partial}{\partial x_-}.$$

3.2 Symmetry Analysis of the Nonlinear Discrete Boundary Value Problem for the Wave Equation

A BVP for a differential or difference equation admits a Lie group of transformations if the boundary curve, the boundary conditions and the corresponding equation are severally invariant under the given transformation group. However, in general invariance of all parts of a BVP are not satisfied at the same time. There are several definitions for invariance of a BVP. In this chapter, we examine the invariance of the sine-Gordon equation using the definition given in [77]. We consider both cases in differential and difference form. The crucial point of our work is to study on a discrete problem with difference equations. Actually, there is not much work about invariance of BVPs for difference equations. We apply transformation groups obtained in the last section to the boundary conditions. The groups and hence their generators are prolonged to the first order derivatives to act on boundaries with derivatives. We choose an unbounded domain to conserve invariance of the boundaries.

3.2.1 Lie Group Analysis of Boundary Value Problem for the sine-Gordon Equation

In this section, we analyze BVP for the nonlinear sine-Gordon equation in differential form with respect to invariance under Lie groups of point transformations of the related equation.

Let us consider the nonlinear hyperbolic problem

$$u_{tt} - u_{xx} = \sin u,\ t > 0,\ -\infty < x < \infty, \tag{3.27}$$

$$u(0,x) = \varphi(x), \tag{3.28}$$

$$u_t(0,x) = \psi(x). \tag{3.29}$$

The equation (3.27) admits three-dimensional Lie group [20] spanned by the operators

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}. \tag{3.30}$$

These operators generate the one-parameter Lie groups

$$T_1 : \bar{t} = t + \epsilon_1,\ \bar{x} = x,\ \bar{u} = u; \tag{3.31}$$

$$T_2 : \bar{t} = t,\ \bar{x} = x + \epsilon_2,\ \bar{u} = u; \tag{3.32}$$

$$T_3 : \bar{t} = t + x\epsilon_3,\ \bar{x} = x + t\epsilon_3,\ \bar{u} = u, \tag{3.33}$$

respectively. Since group the T_1 corresponds to translation on the variable t, the invariance of the boundary curve $t = 0$ is not preserved. Thus BVP (3.27)-(3.29) is not invariant with respect to the group T_1.

For the invariance of boundary condition (3.28) with respect to the symmetry group T_2, the equations

$$\bar{t}|_{t=0} = 0,\ [\bar{u} - \varphi(\bar{x})]|_{u-\varphi(x)=0} = 0 \tag{3.34}$$

must be satisfied. The first equation of (3.34) is an identity, while the second equation results

$$\varphi(x) = \varphi(x + \epsilon_2). \tag{3.35}$$

For the invariance of boundary condition (3.29), we need the first prolongation of the operator X_2. Using the prolongation formula for the first-order derivatives

$$\begin{aligned} X^{(1)} = X &+ (\eta_t + u_t\eta_u - u_t(\xi^0_t + u_t\xi^0_u) - u_x(\xi^1_t + u_t\xi^1_u))\frac{\partial}{\partial u_t} \\ &+ (\eta_x + u_x\eta_u - u_t(\xi^0_x + u_x\xi^0_u) - u_x(\xi^1_x + u_x\xi^1_u))\frac{\partial}{\partial u_x}, \end{aligned} \tag{3.36}$$

where ξ^0, ξ^1 are infinitesimals with respect to the variables t and x respectively, we get

$$X_2^{(1)} = \frac{\partial}{\partial x}. \tag{3.37}$$

Applying this operator to condition (3.29), we have

$$\bar{t}|_{t=0} = 0,\ [u_{\bar{t}} - \psi(\bar{x})]|_{u_t-\psi(x)=0} = 0 \tag{3.38}$$

which gives

$$\psi(x) = \psi(x + \epsilon_2). \tag{3.39}$$

BVP (3.27)-(3.29) is invariant under the group of transformations T_2 if and only if equations (3.35) and (3.39) are satisfied. These equations result that the functions $\varphi(x)$ and $\psi(x)$ are constant functions.

Following the same way, we obtain the invariance criterions of boundary condition (3.28) with respect to the symmetry group T_3 if the equations

$$t + x\epsilon_3 = 0 \text{ when } t = 0,$$

$$u - \varphi(x + t\epsilon_3) = 0 \text{ when } u - \varphi(x) = 0$$

are satisfied. The first equation results with $x = 0$ or $\epsilon_3 = 0$ that gives the trivial group. Hence we arrive at boundary condition (3.28) is invariant under the transformations group T_3 with the restriction

$$x = 0,\ \varphi(x) = \varphi(x + t\epsilon_3). \tag{3.40}$$

In order to examine invariance of boundary condition (3.29), we apply the first prolongation of the operator X_3, which is obtained from formula (3.36)

$$X_3^{(1)} = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial u_t} - u_t\frac{\partial}{\partial u_x}, \tag{3.41}$$

to (3.29) and get

$$x = 0, \psi(x) - \psi(x + t\epsilon_3) = u_x\epsilon_3. \tag{3.42}$$

Combining equations (3.40) and (3.42), we conclude that BVP (3.27)-(3.29) is invariant under the group of transformations T_3 with restriction $u_x(t,0) = 0$ and conditions that

- if $t = 0$ for all arbitrary functions $\varphi(x)$ and $\psi(x)$,
- if $t \neq 0$ then $\varphi(x)$ and $\psi(x)$ are constant functions.

Considering all situations examined above, one infers that BVP (3.27)-(3.29) admits two-parameter Lie group $T_2 \circ T_3$ that corresponds to symmetries $\overline{t} = t + x\epsilon_3$, $\overline{x} = x + t\epsilon_3 + \epsilon_2$, $\overline{u} = u$ if and only if $\varphi(x)$ and $\psi(x)$ are constant functions.

3.2.2 Lie Group Analysis of the Difference Scheme of Boundary Value Problem for the sine-Gordon Equation

In this section, we examine the Lie point symmetries of difference model for nonlinear problem (3.27)-(3.29).

In the previous section, for the sine-Gordon equation (3.27), we presented the five-point difference scheme

$$\frac{\hat{u}-2u+\check{u}}{h_1^2}-\frac{u_+-2u+u_-}{h_2^2}=\sin u \tag{3.43}$$

on the uniform and orthogonal mesh

$$\hat{t}-2t+\check{t}=0,\ x_+-2x+x_-=0. \tag{3.44}$$

Difference equation (3.43) on the set of a finite number of points (x_n^k, t_n^k) can be expressed as

$$E_1:\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{h_1^2}-\frac{u_{n+1}^k-2u_n^k+u_{n+1}^k}{h_2^2}=\sin u_n^k \tag{3.45}$$

on the uniformly spaced orthogonal mesh

$$E_2: t_n^{k+1}-t_n^k=h_1,\ E_3: x_n^{k+1}-x_n^k=0, \tag{3.46}$$

$$E_4: t_{n+1}^k-t_n^k=0,\ E_5: x_{n+1}^k-x_n^k=h_2. \tag{3.47}$$

In paper [55], Levi et al. mentioned certain independence criteria for difference schemes in two dimensional case. This criteria provides to calculate the values of (x,t,u) at all points starting from the reference point (x_n^k, t_n^k) and a given number of neighboring points and guarantees existence of solution of the system. In [55], Levi et al. imposed the following condition on the Jacobian

$$|J|=\left|\frac{\partial(E_1,E_2,E_3,E_4,E_5)}{\partial(t_n^{k+1},x_n^{k+1},t_{n+1}^k,x_{n+1}^k,u_n^{k+1})}\right|\neq 0 \tag{3.48}$$

for instance to move upward and to the right along the curves passing through (x_n^k, t_n^k) (with either k or n fixed). Difference scheme (3.45)-(3.47) satisfy certain independence criteria (3.48) by

$$t_n^k=h_1k+t_0,\ x_n^k=h_2n+x_0. \tag{3.49}$$

Now, using difference equation (3.43) for the BVP (3.27)-(3.29), we consider the following difference problem

$$\frac{\hat{u}-2u+\check{u}}{h_1^2}-\frac{u_+-2u+u_-}{h_2^2}=\sin u, \tag{3.50}$$

$$u_n^0=\varphi^h(x), \tag{3.51}$$

$$\frac{\hat{u}_n^1-u_n^0}{\tau^+}=\psi^h(x). \tag{3.52}$$

In this study, for simplicity we use the following notation $(t,x,u,\underset{h}{u_t},\underset{h}{u_x},\underset{h}{u_{tx}},\dots,h_1,h_2)$ for difference variables in two-dimensional case. With respect to these symbols we rewrite difference model (3.50)-(3.52) as

$$\underset{h}{u_{tt}}-\underset{h}{u_{xx}}=\sin u, \tag{3.53}$$

$$u(0,x)=\varphi^h(x), \tag{3.54}$$

$$\underset{h}{u_t}(0,x)=\psi^h(x). \tag{3.55}$$

Difference equation (3.53) admits three-parameter groups generated by the operators [89]

$$X_1=\frac{\partial}{\partial t},\ X_2=\frac{\partial}{\partial x},\ X_3=x\frac{\partial}{\partial t}+t\frac{\partial}{\partial x}. \tag{3.56}$$

These operators describe Lie symmetry groups which correspond to the translation about time variable, the translation in space axis and rotation, respectively. Since time translation violates invariance of the boundary surface $t=0$, difference scheme (3.53)-(3.55) does not admit symmetry group generated by X_1.

The invariance of boundary surface $t=0$ with respect to the transformation group generated by the operators X_2 is trivial. Boundary condition (3.54) is invariant under the symmetry of space translation X_2 if the equation

$$u-\varphi^h(x+\epsilon_2)=0 \text{ when } u-\varphi^h(x)=0$$

is satisfied. This results the condition

$$\varphi^h(x)=\varphi^h(x+\epsilon_2). \tag{3.57}$$

For the invariance of boundary condition (3.55) we require the first-order prolongation formulas in space of discrete variables. From (3.37) we know the coordinates for continuous derivatives in the prolongation of the operator X_2, and are zero. Using formulas (2.124)-(2.125), we obtain the coordinates of the first-order difference derivatives

$$\underset{h}{\zeta_t} = \underset{+h}{D_1}(0) - \underset{h}{u_t}\underset{+h}{D_1}(0) - \underset{+h}{S_1}(u_x)\underset{+h}{D_1}(1) = 0, \tag{3.58}$$

$$\underset{h}{\zeta_x} = \underset{+h}{D_2}(0) - \underset{+h}{S_2}(u_t)\underset{+h}{D_2}(0) - \underset{h}{u_x}\underset{+h}{D_2}(1) = 0 \tag{3.59}$$

for the operator X_2 with $\eta = 0, \xi^t = 0, \xi^x = 1$. In this case, $\underset{h}{X_2}^{(1)} = X_2$ where $\underset{h}{X_2}^{(1)}$ is the first prolongation of the operator X_2 in discrete space. Applying this prolongation to condition (3.55) we get the criterion

$$\psi^h(x) = \psi^h(x + \epsilon_2). \tag{3.60}$$

In consequence of criteria (3.57) and (3.60), one can say that difference scheme (3.53)-(3.55) is invariant with respect to the transformation group defined by the operator X_2 if and only if $\varphi^h(x)$ and $\psi^h(x)$ are constant functions.

Using the same procedure, we obtain the invariance criterion of condition (3.54) under the rotation group spanned by the operator X_3 as

$$t + x\epsilon_3 = 0 \text{ when } t = 0, \; u - \varphi^h(x + t\epsilon_3) = 0 \text{ when } u - \varphi^h(x) = 0$$

which results

$$x = 0, \varphi^h(x) = \varphi^h(x + t\epsilon_3). \tag{3.61}$$

We need to prolong operator (3.41) for first-order difference derivatives to analyze invariance of condition (3.55) under the symmetry group generated by this operator. Substituting $\eta = 0, \xi^t = x, \xi^x = t$ for the operator X_3 in formulas (2.124)-(2.125) we get the coefficients

$$\underset{h}{\zeta_t} = -u_x, \; \underset{h}{\zeta_x} = -u_t \tag{3.62}$$

and the prolongation operator

$$\underset{h}{X_3}^{(1)} = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial u_t} - u_t\frac{\partial}{\partial u_x} - u_x\frac{\partial}{\partial \underset{h}{u_t}} - u_t\frac{\partial}{\partial \underset{h}{u_x}}, \tag{3.63}$$

which generates the group $\overline{t} = t + x\epsilon_3$, $\overline{x} = x + t\epsilon_3$, $\overline{u} = u$, $\overline{u_t} = u_t - u_x\epsilon_3$, $\overline{u_x} = u_x - u_t\epsilon_3$, $\overline{\underset{h}{u_t}} = \underset{h}{u_t} - u_x\epsilon_3$, $\overline{\underset{h}{u_x}} = \underset{h}{u_x} - u_t\epsilon_3$.

Applying this operator to boundary condition (3.55) gives

$$t + x\epsilon_3 = 0 \text{ when } t = 0,\ \underset{h}{u_t} - u_x\epsilon_3 - \psi^h(x + t\epsilon_3) = 0 \text{ when } \underset{h}{u_t} - \psi^h(x) = 0,$$

and consequently,

$$x = 0,\ \psi^h(x) - \psi^h(x + t\epsilon_3) = u_x\epsilon_3. \tag{3.64}$$

From equations (3.61) and (3.64), we deduce that difference scheme (3.53)-(3.55) is invariant under the group of transformations X_3 with restriction on the first continuous derivative with respect to x variable, $u_x(t, o) = 0$, in two cases: if $t = 0$ for all arbitrary functions $\varphi^h(x)$ and $\psi^h(x)$; if $t \neq 0$ then $\varphi^h(x)$ and $\psi^h(x)$ are constant functions.

We omit the coordinates for the mesh variables in the prolongation operators $\underset{h}{X_2}^{(1)}$ and $\underset{h}{X_3}^{(1)}$. Indeed, infinitesimals of the operator X_2 are $\eta = 0$, $\xi^t = 0$, $\xi^x = 1$, and substituting these variables in $\underset{+h}{D_1}(\xi^t)$ and $\underset{+h}{D_2}(\xi^x)$ gives zero.

For the operator X_3, the infinitesimals are $\eta = 0$, $\xi^t = x$, $\xi^x = t$ and we calculate $\underset{+h}{D_1}(\xi^t)$ and $\underset{+h}{D_2}(\xi^x)$ for these variables as

$$\underset{+h}{D_1}(x) = \frac{1}{h}(\underset{h}{S_1} - 1) = (D_1 + \frac{h_1}{2!}D_1^2 + \cdots)(x) = 0, \tag{3.65}$$

$$\underset{+h}{D_2}(t) = \frac{1}{h}(\underset{h}{S_2} - 1) = (D_2 + \frac{h_1}{2!}D_2^2 + \cdots)(t) = 0, \tag{3.66}$$

where

$$D_1 = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{xt}\frac{\partial}{\partial u_x} + \ldots, \tag{3.67}$$

$$D_2 = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + \ldots. \tag{3.68}$$

4
Results and Discussion

Lie symmetry method is a beneficial and effective solution method for differential and difference equations. It has the ability to combine the existing techniques. Even if Lie symmetries are tools for solving differential and difference equations, they have many valuable applications. They can classify equations according to their linearity or integrability. Invariant solutions are constructed from invariance under Lie symmetry groups. These solutions are practical importance by means of obtaining new solutions from known ones. For ODEs, invariant solutions are obtained solving algebraic equations reduced from the given equation. For PDEs, these solutions are determined solving PDEs with reduced independent variables.

Lie symmetry analysis of discrete equations has come into prominence in recent years from several points of view. Similar to differential equations, the main operator is infinitesimal generators in the study of difference equations. Lie symmetries can be used to obtain more accurate and stable results for difference equations.

Invariance of BVPs relies on invariance of all boundary conditions of the given problem. Since it is a difficult process to leave invariant boundary curves and conditions under a Lie group at the same time, application of Lie groups to BVPs is not a highly studied topic. Some definitions and theorems are investigated to apply Lie symmetries to wider class of BVPs.

In this thesis, Lie symmetries are applied to some specific differential and difference models. The main model of our research is sine-Gordon equation which is nonlinear and has significant impacts in interdisciplinary studies of mathematics and physics.

We have improved a procedure from the recent literature to obtain Lie point symmetries of finite difference scheme for one-dimensional sine-Gordon equation. This process relies on an algorithm used for infinitesimal generators of the symmetry group. It is observed that the discrete sine-Gordon equation preserves the symmetries of its differential form. This is an essential conclusion from the view point of constructing invariant difference schemes.

We considered the boundary value problem for sine-Gordon equation in differential and difference form which is defined on an unbounded domain and mesh, respectively. We applied the invariance definition for boundary value problems given in [77] and obtained invariance conditions for the problems under the group of transformations admitted by continuous and discrete sine-Gordon equation. The symmetry operators act on the difference scheme, meshes, boundary conditions and preserve uniformness and orthogonality of the mesh. We used the prolongation formulas in discrete space that are formulated by Dorodnitsyn in [26] to analyze the invariance of the boundary conditions with derivative. We have observed that difference scheme (3.53)-(3.55) is invariant under the same restrictions with respect to the symmetry groups generated by (3.56) with differential form (3.27)-(3.29).

The present study has only investigated the point transformations of the considered models. Further studies will concentrate on the generalized symmetries of particularly difference models. The current research was limited by scalar differential or difference equations. One potential application of our technique would be systems of some differential or difference equations.

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Publications From the Thesis

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Papers

1. O. Yildirim and S. Caglak, "Lie point symmetries of difference equations for the nonlinear sine-gordon equation," *Physica Scripta*, vol. 94, no. 8, p. 085 219, 2019.

Conference Papers

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