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LEGENDRE WAVELET OPERATIONAL MATRIX METHOD FOR SOLVING SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Selvi ALTUN

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LIST OF SYMBOLS

D	Operational matrix of derivative
$\psi_{nm}(t)$	Legendre wavelet
ϕ	Scaling function
a	Dilation parameter
b	Translation parameter
I_a^α	Riemann-Liouville fractional integral
D_a^α	Riemann-Liouville fractional derivative
${}^C D^\alpha$	Caputo fractional derivative
$L_p[a,b]$	Lebesque space
$\Gamma(n)$	Gamma function
$\beta(m,n)$	Beta function
$(X,\|.\|)$	Normed space
$(.,.)$	Inner product
L_m(x)	Well-known Legendre polynomial
$P_m(t)$	Shifted Legendre polynomial

LIST OF ABBREVIATIONS

a.e.	Almost everywhere
BVP	Boundary value problem
DE	Differential equation
et al.	In addition
etc.	And other similar things
FDE	Fractional order differential equation
HDE	High order differential equation
IVP	Initial value problem
i.e.	That is
LPOMM	Legendre polynomial operational matrix method
LWOMM	Legendre wavelet operational matrix method
LWPT	Legendre wavelet-polynomial transformation
MRA	Multiresolution of analysis
PLSM	Polynomial least squares method

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ABSTRACT

LEGENDRE WAVELET OPERATIONAL MATRIX METHOD FOR SOLVING SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Department of Mathematical Engineering

Ph.D. Thesis

Adviser: Assoc. Prof. Dr. Aydın SEÇER

This thesis introduces a new numerical approach to solve high order and fractional order differential equations of the linear and non-linear forms and systems of such equations utilizing the Legendre wavelet operational matrix method. We first formulated the operational matrix and its fractional derivatives in some special conditions by using some significant features of Legendre wavelets and shifted Legendre polynomials. Then, the high order and fractional order differential equations and systems of such equations were transformed to a system of algebraic equations by using these operational matrices. At the end of each chapter of the thesis, the introduced tecnique is tested on several illustrative examples. Comparing the methodology with several recognized methods demonstrates that the most important advantages of the introduced method are the understandibility of the calculations and its accuracy.

Key words: Legendre wavelet, operational matrix, fractional order differential equations, the system of fractional order differential equations, Caputo fractional derivative

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ÖZET

KESİRLİ MERTEBEDEN DİFERANSİYEL DENKLEM SİSTEMLERİNİN ÇÖZÜMÜ İÇİN LEGENDRE DALGACIĞI OPERASYONEL MATRİS METODU

Selvi ALTUN

Matematik Mühendisliği Anabilim Dalı

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Bu tezde, Legendre dalgacığı operasyonel türev matris metodu kullanılarak, yüksek mertebeden ve kesirli mertebeden diferansiyel denklemlerin ve denklem sistemlerinin doğrusal ve doğrusal olmayan formlarının nümerik çözümleri için yeni bir yaklaşım geliştirilmiştir. Öncelikle, operasyonel matris ve kesirli türevi bazı özel koşullar altında, Legendre dalgacığı ve Legendre polinomlarının bazı önemli özellikleri kullanılarak formüle edilmiştir. Sonrasında, yüksek mertebeden ve kesirli mertebeden diferansiyel denklem ve denklem sistemleri bu operasyonel matrisler yardımıyla cebirsel denklem sistemlerine dönüştürülmüştür. Tezde önerilen metot her bölümün sonunda yeterli sayıda aydınlatıcı örnekle test edilmiştir. Sonuç olarak, bazı bilinen metotlarla karşılaştırması gösteriyor ki, Legendre dalgacığı operasyonel türev matrisi metodunun en büyük avantajı sadeliği ve hesaplamalardaki anlaşılabilirliğidir.

Anahtar Kelimeler: Legendre dalgacığı, operasyonel matris, kesir mertebeli diferansiyel denklemler, kesir mertebeli diferansiyel denklem sistemleri, Caputo kesir türevi

CHAPTER 1

INTRODUCTION

1.1 Literature Review

Differential and integral operators are the basis of mathematical models and they are also used as a means of understanding the working principles of natural and artificial systems. Therefore, differential and integral equations are of great importance both theoretically and practically. Such equations have a wide range of applications, including in the physical sciences, such as physics and engineering, as well as in social science. The system of differential equations, as differential equations, are often used in issues such as theory of elasticity, dynamics, fluid mechanics, oscillation, and quantum dynamics [21], [22], [57].

Interest in differential and integral operators has led to the exploration of fractional differential and integral operators by examining these issues further in depth. Owing to a question, the origin of fractional calculus arose in a message from Leibniz to L'Hospital in 1695. Over the years, a variety of definitions that satisfy the idea of fractional derivative have been found by several great mathematicians, but Riemann-Liouville and Caputo fractional derivatives are most commonly utilized definements in the world of fractional calculus. Altough the theory about Riemann-Liouville definition was constituted very well, this consept has some troubles with using to real-world problems. To make a success of these troubles, Caputo derivative was established.

For three centuries, analysis of the fractional calculus has been restraint to the discipline of pure theoretical mathematics, but this topic has received attention in recent years because of its ability to simplify numerous physical, engineering and economics phenomena, such as the fluid-dynamic traffic model, damping laws, continuum and statistical mechanics, diffusion process, solid mechanics, control theory, colored noise, viscoelasticitiy, electrochemistry and electromagnetic, among others.

Because variety of solution of fractional differential equations can not be found analytically, numerical and approximate methods are needed. There are a lot of tecniques that have been studied by many researchers to solve FDEs and the system of such equations numerically. Several of these tecniques are the Adomian decomposition method presented in [35] by Song *et al.*, collocation method, operational matrix method improved in [18], [19], [24] and [26], perturbation-iteration method introduced in [28] by Şenol *et al.*, computational matrix method illustrated in [27] by Khader *et al.*, differential transform method demonstrated in [43] by Ertürk *et al.*, variational iteration method, Laplace transform method given in [41] by Gupta *et al.*, fractional complex transform method studied in [44] by Ghazanfari *et al.* etc. Also numerical solutions of these equations and the system of such equations were presented by using the Bernstein operational matrix method [29], Genocchi operational matrix method [49], Jacobi operational matrix method [34], Chebyshev wavelet operational matrix method [30], polynomial least squares method [47], Legendre wavelet-like operational matrix method [48] and Genocchi wavelet-like operational matrix method [50].

The orthogonal functions and polynomial series are very important field in science and engineering. Block-pulse fuctions, sine-cosine functions, Jacobi, Legendre, Hermite, Genocchi, Laguerre and Chebyshev polynomials are the most commonly utilized among these functions. What makes these functions important is that they permit the undertaking problem to be reduced to a system of algebraic equations and the approximation of analytic functions. The problem is solved by truncating series of orthogonal basis functions and utilising operational matrix and its derivatives.

The operational matrix of derivatives D is defined as:

$$\frac{d\psi(t)}{dt} \cong D\psi(t) \tag{1.1}$$

in which $\psi(t) = [\psi_1, \psi_2, ..., \psi_N]$ and ψ_i $(i = 1, 2, ..., N)$ are orthogonal basis functions, orthogonal on a certain interval $[a,b]$. The matrix D can be uniquely identified on the basis of the specific orthogonal functions [9-11]. Many papers which are related to the application of operational matrix of derivative can be found in the literature [12], [15], [18], [19], [30], [33], [34], [48], [49], [50], [56].

Wavelet theory is very significant in science, engineering and technology and in recent years, wavelets have achieved to attract an enormous attention in many fields of investigation, such as spectroscopy, signal analysis, feature detection in earth science, time-frequency analysis, and image manipulation, among others. Many scholars have contributed to the development of wavelets. Especially, Daubechies, Belkin, Meyer and Mallat are some of them. Thanks to their contribition, there has been a substantial increment in the number of studies on wavelets. There are a wide variety of wavelet functions such as Daubechies, Haar, Laguerre, Legendre, Shannon, Lagrange, Hermitian and Chebyshev wavelets available. Among them, we choose Legendre wavelet in this thesis because of their orthonormality and explicitity. Many applications of Legendre wavelets can be viewed from [12-20].

This thesis focuses on the applications of high order and fractional order differential equations and systems of such equations by utilizing the LWOMM. The most important advantage of the proposed method is that it presents a comprehensible algorithm in reducing high order and fractional differential equations and the system of such equations to a system of algebraic equations. This thesis consists of seven chapters and the third, fourth, fifth and sixth chapters of this thesis have originality. First, we begin with presenting some basic definitions and fundamental relations relevant to the fractional calculus theory, orthogonal polynomials (especially shifted Legendre polynomials), wavelets (especially Legendre wavelets), approximations of these functions and operational matrix of derivative. The operational matrix of fractional derivative is then natively derivated in some special conditions in Chapter 2. Chapter 3 and Chapter 4 generalizes these operational matrices to high order differential equations and the system of such equations of the linear and non-linear forms. Similarly, Chapter 5 and Chapter 6 generalizes these operational matrices to fractional order differential equations and the system of such equations of the linear and non-linear forms. At the end of each chapter, several illustrative examples are tested on the introduced method. Finally, last chapter includes the conclusion and suggestions.

1.2 Objective of the Thesis

This thesis aims to improve an effective and comprehensive technique to solve high order differential equations and the system of such equations together with fractional order differential equations and the system of such equations. The most advantageous characteristic of this method is that it gives an understandable procedure in reducing these equations and the system of such equations to a system of algebraic equations by utilizing operational matrix of derivative and fractional derivative. So, we can easily obtain the desired solution.

1.3 Hypothesis

In this thesis, the operational matrix of fractional derivative is natively derivated in some special conditions by taking advantage of some notable features of Legendre wavelets and shifted Legendre polynomials and these operational matrices are generalized to equations mentioned above and the system of such equations of the linear and non-linear forms for the first time. Numerical solutions of these equations and the system of such equations obtained by using introduced method have originality.

CHAPTER 2

BASIC CONCEPTS

2.1 Fractional Calculus

Fractional calculus is the study of any real-order or complex-order derivative and integral composed of combining and extending the consepts of multiple integral and integer order derivative. The origin of fractional calculus arose in a message from Leibniz to L'Hospital in 1695. For three centuries, analysis of fractional calculus has been restraint to the discipline of pure theoretical mathematics. But, this topic has received attention in recent years, because of its suitability for the explanation of numerous physical, engineering and economics phenomena, such as the fluid-dynamic traffic model, damping laws, continuum and statistical mechanics, diffusion process, solid mechanics, control theory, colored noise, viscoelasticitiy, electrochemistry and electromagnetic, among others.

Let $D = \dfrac{d}{dt}$ be a differential operator and n be a positive integer. It is well known that, the meaning of the $D^n u(t)$ is the n^{th} derivative of the function $u(t)$. But if n is not a positive integer, it is difficult to comment the meaning of the $D^{-\alpha}$ or D^{α} for $\mathrm{Re}(\alpha) > 0$. The meaning of these symbols will be explained in this section.

A variety of definitions that satisfy the idea of fractional derivative have been found by several great mathematician. But Riemann-Liouville and Caputo fractional derivatives are most commonly utilised definements in the world of fractional calculus. The differential and integral operators with fractional analysis operators are denoted as $(D_a^{\alpha} u)(t)$ and $(I_a^{\alpha} u)(t)$ respectively for $\mathrm{Re}(\alpha) > 0$, where a is the boundary value of the fractional differentiation and integration operations [21, [22], [57].

It is necessary to know some mathematical definitions to understand the definitions and applications of the fractional analysis required for this thesis. Some of these definitions and theorems are presented below.

2.1.1 The Gamma Function

It is said that the Gamma function is obviously the generalization of the factorial for all real numbers. This function is defined by [57-58]

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, \quad n \in \mathbb{R}^+ \tag{2.1}$$

Using the following equation related to exponential function of the factorial function

$$n! = \int_0^\infty e^{-t} t^n dt = \int_0^\infty e^{-t} t^{(n+1)-1} dt = \Gamma(n+1) \tag{2.2}$$

The relation between Gamma function and factorial function is appropriated. Using integration by parts we can obtain for $n > 0$

$$\begin{aligned} \Gamma(n+1) &= \int_0^\infty e^{-t} t^n dt = -e^{-t} t^n \Big|_0^\infty - \int_0^\infty \left(-e^{-t}\right) n t^{n-1} dt \\ &= n \int_0^\infty e^{-t} t^{n-1} dt = n\Gamma(n) \end{aligned} \tag{2.3}$$

The Gamma function is directly related to the fractional derivative and integral. These relations can be found by using following properties of Gamma function [57-58].

i) For all $n > 0$, the integral $\int_0^\infty e^{-t} t^{n-1} dt$ is convergent.

ii) The Gamma function Γ is positive for all $n > 0$,

iii) $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$.

iv) For $0 < n < 1$, $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$ and for $n = \dfrac{1}{2}$, $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$.

v) The Gamma function Γ is continous for all $n > 0$.

vi) The Gamma function Γ is differentiable for all $n > 0$ and we get

$$\Gamma'(n) = \int_0^\infty e^{-t} \ln(t) t^{n-1} dt \tag{2.4}$$

Table 2.1 Some numerical values of the Gamma function

$\Gamma\left(-\frac{3}{2}\right)$	$\frac{4\pi}{3}$	$\Gamma(1.0)$	1.0000
$\Gamma(2)$	1	$\Gamma(1.1)$	0.9514
$\Gamma\left(-\frac{1}{2}\right)$	$-2\sqrt{\pi}$	$\Gamma(1.2)$	0.9182
$\Gamma\left(\frac{5}{2}\right)$	$\frac{3\sqrt{\pi}}{4}$	$\Gamma(1.3)$	0.8975
$\Gamma(0)$	undefined	$\Gamma(1.4)$	0.8873
$\Gamma(3)$	2	$\Gamma(1.5)$	0.8862
$\Gamma\left(\frac{1}{2}\right)$	$\sqrt{\pi}$	$\Gamma(1.6)$	0.8935
$\Gamma\left(\frac{7}{2}\right)$	$\frac{15\sqrt{\pi}}{8}$	$\Gamma(1.7)$	0.9086
$\Gamma(1)$	1	$\Gamma(1.8)$	0.9314
$\Gamma(4)$	6	$\Gamma(1.9)$	0.9618
$\Gamma(\infty)$	∞	$\Gamma(2.0)$	1.0000

2.1.2 The Beta Function

The Beta function is defined by a definite integral. Its definition is presented by [58]

$$\beta(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, dt, \quad m,n \in \mathbb{R}^+ \tag{2.5}$$

We can also express the Beta function in terms of the Gamma function:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m,n \in \mathbb{R}^{+} \tag{2.6}$$

2.1.3 L^p Spaces

We consider L^p -spaces of functions whose pth powers are integrable.

Definition 2.1 Let (X, A, μ) be a measure space and $1 \le p < \infty$. The space $L^p(X)$ be composed of equivalence classes of measurable functions $f : X \to \mathbb{R}$ such that

$$\int |f|^p \, d\mu < \infty \tag{2.7}$$

where two measurable functions are equivalent if they are equal $\mu - a.e.$ The L^p -norm of $f \in L^p(X)$ is defined by

$$\|f\|_{L^p} = \left(\int |f|^p \, d\mu\right)^{1/p} \tag{2.8}$$

The notation $L^p(X)$ presumes that the measure μ on X is understood. We say that $f_n \to f$ in L^p if $\|f_n - f\|_{L^p} \to 0$. The reason to regard functions that are equal $a.e$ as equivalent is so that $\|f\|_{L^p} = 0$ implies that $f = 0$ [57].

2.1.4 The Riemann-Liouville Fractional Integral

Let α be a real nonnegative number. For $t \in [a,b]$ in $L^1[a,b]$, the definition of the Riemann-Liouville fractional integral is given as [57-58]

$$\left(I_a^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} u(\xi) d\xi, \;\; t \ge a, \;\; \alpha > 0 \tag{2.9}$$

where $\Gamma(\alpha)$ is the Gamma function.

An important feature of Riemann-Liouville fractional integral is that $I_a^0 = I$ is an identity operator for $a = 0$.

Some properties of the Riemann-Liouville fractional integral are as follows.

Suppose that $u(t)=(t-a)^{\beta}$ where $\beta>-1$, then the Riemann-Liouville fractional integral of $u(t)$ of order α is

$$\left(I_a^{\alpha}u\right)(t)=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta} \tag{2.10}$$

2.1.5 The Riemann-Liouville Fractional Derivative

Using the definition of the Riemann-Liouville fractional integral, then we can define the fractional derivative. Assume that $v=n-\alpha$, where $0<v<1$ and n is the smallest integer greater than α. Then, the definition of the Riemann-Liouville fractional derivative can be expressed as [57-58]

$$\begin{aligned}\left(D_a^{\alpha}u\right)(t)&=\left(\frac{d}{dt}\right)^{n}\left(I_a^{v}u\right)(t)\\ &=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_a^t(t-\xi)^{v-1}f(\xi)d\xi\end{aligned} \tag{2.11}$$

where $D_a^0=I$ is an identity operator for $a=0$.

Some properties of the Riemann-Liouville fractional derivative are as follows.

Suppose that $u(t)=(t-a)^{\beta}$ where $\beta>-1$, then the Riemann-Liouville fractional derivative of $u(t)$ of order α is

$$\left(D_a^{\alpha}u\right)(t)=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha} \tag{2.12}$$

2.1.6 The Caputo Fractional Derivative

Over the years, a variety of definitions that satisfy the idea of fractional derivative have been found by several great mathematician. But Riemann-Liouville and Caputo fractional derivatives are most commonly utilised definements in the world of fractional calculus. Altough the theory about Riemann-Liouville definition was constituted very well, this consept has some troubles with using to real-world problems. To make a success of these troubles, Caputo derivative was established.

Definition 2.2 The fractional-order derivative in the Caputo sense is defined as [18-19]

$$ {}^{C}D^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{u^{(n)}(\xi)}{(t-\xi)^{\alpha+1-n}}d\xi, \quad n-1<\alpha\le n,\ n\in \mathrm{N} \tag{2.13}$$

Some properties of the Caputo derivative are as follows.

$$ {}^{C}D^{\alpha}C=0 \tag{2.14}$$

where C is a constant.

$$ {}^{C}D^{\alpha}t^{\beta}=\begin{cases} 0, & \beta\in \mathrm{N}_0 \ \ and \ \ \beta<\lceil\alpha\rceil \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}, & \beta\in \mathrm{N}_0 \ \ and \ \ \beta\ge\lceil\alpha\rceil \ \ or \ \ \beta\notin \mathrm{N} \ \ and \ \ \beta>\lfloor\alpha\rfloor \end{cases} \tag{2.15}$$

Here, $\lfloor\alpha\rfloor$ is the largest integer less than or equal to α and $\lceil\alpha\rceil$ indicates the smallest integer greater than or equal to α .

The Caputo fractional order derivative is a linear operation as the integer-order derivative:

$$ {}^{C}D^{\alpha}\left(\eta u(t)+\zeta v(t)\right)=\eta\ {}^{C}D^{\alpha}u(t)+\zeta\ {}^{C}D^{\alpha}v(t) \tag{2.16}$$

where η and ζ are constant.

2.2 Orthogonal Polynomials

The orthogonal functions and polynomial series are very important field in science and engineering. They are basis of several numerical methods developed for the solution of differential equations and integro-differential equations. The reason is that the use of orthogonal polynomials is easy. Because they have good convergence properties and they properly represent the weight distribution of a function on a definite network. Mentioned equations are solved by truncating series of orthogonal basis functions. Block-pulse fuctions, sine-cosine functions, Legendre, Hermite, Jacobi, Laguerre and Chebyshev polynomials are the most commonly utilized among these functions. What makes these functions important is that they permit the undertaking problem to be reduced to system of algebraic equations and the approximation of analytic functions.

2.2.1 Legendre Polynomials

One of the kind of particular orthogonal polynomials used in the solution of real-world problems is the class of functions called Legendre polynomials. They are the everywhere regular solutions of a very significant differential equation, the Legendre Equation.

$$(1-x^2)\frac{d^2u}{dx^2}-2x\frac{du}{dx}+m(m+1)u=0 \tag{2.17}$$

Since the Legendre differential equation is a second-order ordinary differential equation, it has two linearly independent solutions. A solution $L_m(x)$ which is regular at finite points is called a Legendre function of the first kind, while a solution $Q_m(x)$ which is singular at ± 1 is called a Legendre function of the second kind. If m is an integer, the function of the first kind reduces to a polynomial known as the Legendre polynomial. We write the solution for a particular value of m as $L_m\left(x\right)$. It is a polynomial of degree m. If m is even/odd then the polynomial is even/odd. They are normalised such that $L_m\left(1\right)=1$.

The equation takes its name from Adrien Marie Legendre (1752-1833), a French mathematician who became a professor in Paris in 1775. He made important contributions to special functions, elliptic integrals, number theory, and the calculus of variations [59].

The well-known Legendre polynomials are defined on the interval $[-1,1]$ and can be designated by the help of the following formulae.

$$(m+1)L_{m+1}(x)=(2m+1)xL_m(x)-mL_{m-1}(x), \quad m=1,2,3,... \tag{2.18}$$

where $L_0\left(x\right)=1$ and $L_1(x)=x$. Defining the so-called shifted Legendre polynomials by presenting the change of variable $x=2t-1$, we can use Legendre polynomials on the interval $[0,1]$. Let the shifted Legendre polynomials $L_m(2t-1)$ be symbolized by $P_m(t)$. Then we can express $P_m(t)$ as follows:

$$(m+1)P_{m+1}(t)=(2m+1)(2t-1)P_m(t)-mP_{m-1}(t), \quad m=1,2,3,... \tag{2.19}$$

The shifted Legendre polynomial $P_m(t)$ has the following analytic form [12]

$$P_m\left(t\right)=\sum_{k=0}^{m}\left(-1\right)^{m+k}\frac{\left(m+k\right)!}{\left(m-k\right)!}\frac{t^k}{\left(k!\right)^2} \tag{2.20}$$

and the orthogonality condition is

$$\int_0^1 P_m(t)P_n(t)\,dt = \begin{cases} \dfrac{1}{2m+1}, & for\ m=n \\ 0, & for\ m \neq n \end{cases} \tag{2.21}$$

The Legendre polynomials are a special case of the Gegenbauer polynomials with $\alpha = \dfrac{1}{2}$, a special case of the Jacobi polynomials $P_m^{(\alpha,\beta)}$ with $\alpha = \beta = 0$ [59].

2.3 Wavelets

Wavelet theory is very significant in science, engineering and technology and in recent years, wavelets have achieved to attract an enormous attention in many fields of investigation, such as spectroscopy, signal analysis, feature detection in earth science, time-frequency analysis, and image manipulation, among others. Many scholars have contributed to the development of wavelets. Especially, Daubechies, Belkin, Meyer and Mallat are some of them. Thanks to their contribition, there has been a substantial increment in the number of studies on wavelets. Many applications of wavelets can be viewed from [12-17], [23-26], [33-34], [48-50], [56].

Wavelets establish a family of functions formulated from dilation parameter a and the translation parameter b change continuously, we have the following family of continuous wavelets [12]:

$$\psi_{ab}(t) = |a|^{-1/2}\,\psi\left(\frac{t-b}{a}\right), \quad a,b \in R, \quad a \neq 0 \tag{2.22}$$

If these parameters a and b are restraint to discrete values as $a = 2^{-k}$, $b = n2^{-k}$, then

$$\psi_{kn}(t) = 2^{k/2}\psi\left(2^k t - n\right) \tag{2.23}$$

forms an orthogonal basis. We use multiresolution of analysis (MRA) for structure wavelets.

Definition 2.3 Let $\{V_j\}_{j\in Z}$ of subset of $L^2(R)$ be the increasing sequence and ϕ be the scaling function. If it satisfies the following conditions, we call $\{V_j\}_{j\in Z}$ with scaling function ϕ MRA [14].

i) $\cup_j V_j$ is dense in $L^2(R)$

ii) $\cap_j V_j = \{0\}$

iii) $f(t) \in V_j \Leftrightarrow f(2^{-j}t) \in V_0$

iv) $\{\varphi(t-n)\}_{n\in Z}$ is an orthogonal basis for V_0.

2.3.1 Legendre Wavelets

There are a wide variety of wavelet functions such as Daubechies, Haar, Laguerre, Legendre, Shannon, Lagrange, Hermitian and Chebyshev wavelets available. Among them, we choose Legendre wavelet in this thesis because of their orthonormality and explicitity.

Legendre wavelet basis is constructed using a linear combination of Legendre polynomial functions. Legendre wavelets $\psi_{nm}(t) = \psi(k,n,m,t)$ have four parameters: where n parameter, k can be presumed any positive integer, m is the order of Shifted Legendre polynomials and t is the normalized time. They are defined on the interval $[0,1]$ by

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k+1}{2}} \sqrt{m+\frac{1}{2}} P_m\left(2^k t - n\right), & \frac{n}{2^k} \le t \le \frac{n+1}{2^k} \\ 0, & otherwise \end{cases} \tag{2.24}$$

where $m = 0,1,...,M;\ \ n = 0,1,...,(2^k - 1)$. The coefficient $\sqrt{\frac{m+1}{2}}$ is for orthonormality [12].

2.3.1.1 Function Approximations

Suppose that a function $u(t)$ is defined over $[0,1]$. Then $u(t)$ may be expanded in the terms of Legendre wavelet as

$$u(t) \cong \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t) \tag{2.25}$$

where $c_{nm} = (u(t), \psi_{nm}(t))$ in which $(.,.)$ denotes the inner product. Let the infinite series in (2.25) be truncated, then it can be written as

$$u(t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} c_{nm}\psi_{nm}(t) = C^T\psi(t) \tag{2.26}$$

where C and $\psi(t)$ are matrices given by [12]

$$\begin{aligned} C &= \left[c_{0,0}, c_{0,1}, \ldots, c_{0,M}, \ldots, c_{2,M}, \ldots, c_{2^k-1,0}, c_{2^k-1,1}, \ldots, c_{2^k-1,M} \right]^T \\ \psi &= \left[\psi_{0,0}, \psi_{0,1}, \ldots, \psi_{0,M}, \ldots, \psi_{2,M}, \ldots, \psi_{2^k-1,0}, \psi_{2^k-1,1}, \ldots, \psi_{2^k-1,M} \right]^T \end{aligned} \tag{2.27}$$

2.4 The Operational Matrix of Derivative

The operational matrix of derivative D is given by

$$\frac{d\psi(t)}{dt} \cong D\psi(t)$$

where $\psi(t) = [\psi_1, \psi_2, \ldots, \psi_N]$ and ψ_i $(i = 1, 2, \ldots, N)$ are orthogonal basis functions, orthogonal on a certain interval $[a, b]$. The matrix D can be uniquely identified on the basis of the specific orthogonal functions. Many papers which are related to the application of operational matrix of derivative can be found in the literature.

F. Mohammadi derived Legendre wavelet operational matrix of derivative in his paper [12]. In this section, the theorem and corollary are just mentioned as follows.

Theorem 2.1 Let $\psi(t)$ be the Legendre wavelets vector given in (2.24), then we get

$$\frac{d\psi(t)}{dt} \cong D\psi(t) \tag{2.28}$$

where D is the $2^k(M+1)$ operational matrix of derivative defined as follows:

$$D = \begin{bmatrix} U & O & \cdots & O \\ O & U & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & U \end{bmatrix} \tag{2.29}$$

in which U is an $(M+1)(M+1)$ matrix and its $(r,s)th$ element is defined as follows:

$$U_{r,s} = \begin{cases} 2^{k+1}\sqrt{(2r-1)(2s-1)}, & r = 2, \ldots, (M+1),\ s = 1, \ldots, r-1 \ and\ (r+s)\ odd \\ 0, & otherwise \end{cases} \tag{2.30}$$

Corollary 2.1 If we use (2.28) then we have operational matrix for *nth* derivative as

$$\frac{d^n\psi(t)}{dt^n} \cong D^n\psi(t) \tag{2.31}$$

where D^n is the *nth* power of matrix D.

By using the property of the product of two Legendre wavelets vector functions, we have

$$e^T\psi\psi^T = \psi^T E \tag{2.32}$$

where e is a given vector and E is a $(2^k M+1)x(2^k M+1)$ matrix dependent on vector e [12].

2.5 The Operational Matrix of Fractional Derivative

A. Saadatmandi and M. Dehghan derived the operational matrix of fractional derivative by using shifted Legendre polynomials in [18]. In this section, the Legendre wavelet operational matrix of fractional derivative is derived in some special conditions by taking advantage of theorem given in [18].

Lemma 2.1. Let $\psi(t)$ be the Legendre wavelets vector presented in Equation (2.24) and assume that $k=0$ then

$$D^\alpha\psi_r(t) = 0, \quad r = 0,1,\ldots,\lceil\alpha\rceil-1, \quad \alpha > 0 \tag{2.33}$$

Proof. The Lemma can be proved by using Equations (2.14) and (2.16) in Equation (2.24).

Theorem 2.2. Let $\psi(t)$ be the Legendre wavelets vector presented in Equation (2.24). Assume that $k=0$ and $\alpha > 0$, then

$$D^\alpha\psi(t) \cong D^{(\alpha)}\psi(t) \tag{2.34}$$

where $D^{(\alpha)}$ is the $(\mathrm{M}+1)x(\mathrm{M}+1)$ operational matrix of the fractional derivative of the order $\alpha > 0, \; N-1 < \alpha \le N$ in the Caputo sense and is expressed as follows

$$D^{(\alpha)}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{h=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\xi_{\lceil\alpha\rceil,0,h} & \sum_{h=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\xi_{\lceil\alpha\rceil,1,h} & \cdots & \sum_{h=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\xi_{\lceil\alpha\rceil,m,h} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{h=\lceil\alpha\rceil}^{r}\xi_{\mathrm{r},0,h} & \sum_{h=\lceil\alpha\rceil}^{r}\xi_{\mathrm{r},1,h} & \cdots & \sum_{h=\lceil\alpha\rceil}^{r}\xi_{\mathrm{r},\mathrm{m},h} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{h=\lceil\alpha\rceil}^{m}\xi_{\mathrm{m},0,h} & \sum_{h=\lceil\alpha\rceil}^{m}\xi_{\mathrm{m},1,h} & \cdots & \sum_{h=\lceil\alpha\rceil}^{m}\xi_{\mathrm{m},\mathrm{m},h} \end{pmatrix} \tag{2.35}$$

where $\xi_{r,s,h}$ is presented by

$$\xi_{r,s,h}=\sqrt{2r+1}\sqrt{2s+1}\sum_{l=0}^{s}\frac{(-1)^{r+s+h+l}(r+h)!(s+l)!}{(r-h)!h!\Gamma(h-\alpha+1)(s-l)!(l!)^2(h+l-\alpha+1)} \tag{2.36}$$

Take in consideration in $D^{(\alpha)}$, the first $\lceil\alpha\rceil$ rows are all zero.

Proof. Assume that $\psi_r(\mathrm{t})$ be the r^{th} element of the vector $\psi(t)$ presented in Equation (2.24), where $r=nM+(m+1)$, $m=0,1,...,M$, $n=0,1,...,(2^k-1)$. Then $\psi_r(t)$ can be expressed as

$$\psi_r(\mathrm{t})=2^{\frac{k+1}{2}}\sqrt{r+\frac{1}{2}}P_r(2^k t-n)\chi_{\left[\frac{n}{2^k},\frac{n+1}{2^k}\right]} \tag{2.37}$$

Suppose that $k=0$ and by utilising the shifted Legendre polynomial, we get

$$\psi_r(t)=\sqrt{2}\sqrt{r+\frac{1}{2}}\sum_{h=0}^{r}\frac{(-1)^{r+h}(r+h)!}{(r-h)!(h!)^2}t^h\chi_{[0,1]} \tag{2.38}$$

If we utilise Equations (2.15), (2.16) and (2.38), then we get

$$\begin{aligned} D^{(\alpha)}\psi_r(t)&=\sqrt{2}\sqrt{r+\frac{1}{2}}\sum_{h=0}^{r}\frac{(-1)^{r+h}(r+h)!}{(r-h)!(h!)^2}D^{\alpha}(t^h)\chi_{[0,1]} \\ &=\sqrt{2r+1}\sum_{h=\lceil\alpha\rceil}^{r}\frac{(-1)^{r+h}(r+h)!}{(r-h)!(h!)\Gamma(h-\alpha+1)}t^{h-\alpha}\chi_{[0,1]},\quad r=\lceil\alpha\rceil,...,m \end{aligned} \tag{2.39}$$

Approximating $t^{h-\alpha}$ by $(m+1)$ terms of Legendre wavelets, then we get

$$t^{h-\alpha} \cong \sum_{s=0}^{m} b_{h,s}\psi_s\left(t\right) \tag{2.40}$$

here

$$\begin{aligned} b_{h,s} &= \int_0^1 t^{h-\alpha}\psi_s\left(t\right)dt = \sqrt{2}\sqrt{s+\frac{1}{2}}\sum_{l=0}^{s}\frac{(-1)^{s+l}(s+l)!}{\left(s-l\right)!(l!)^2}\int_0^1 t^{h+l-\alpha}dt \\ &= \sqrt{2s+1}\sum_{l=0}^{s}\frac{(-1)^{s+l}(s+l)!}{\left(s-l\right)!(l!)^2(h+l-\alpha+1)} \end{aligned} \tag{2.41}$$

If we utilise Equations (2.39) and (2.41), then we have

$$\begin{aligned} D^{\alpha}\psi_r\left(t\right) &\cong \sqrt{2r+1}\sum_{h=\lceil\alpha\rceil}^{r}\sum_{s=0}^{m}\frac{\left(-1\right)^{r+h}(r+h)!}{(r-h)!(h!)\Gamma\left(h-\alpha+1\right)}b_{h,s}\psi_s\left(t\right)\chi_{[0,1]} \\ &= \sum_{s=0}^{m}\left(\sum_{h=\lceil\alpha\rceil}^{r}\xi_{r,s,h}\right)\psi_s(t)\chi_{[0,1]}, \quad r=\lceil\alpha\rceil,\dots,m \end{aligned} \tag{2.42}$$

in which $\xi_{r,s,h}$ is presented in Equations (2.36). Also if we use Lemma 2.1, then we can write

$$D^{\alpha}\psi_r(t) = 0, \quad r = 0,1,\dots,\lceil\alpha\rceil-1, \quad \alpha>0 \tag{2.43}$$

Combining Equations (2.42) and (2.43), then we obtain the result.

2.6 Differential Equations

2.6.1 Ordinary Differential Equations

The general nth order linear differential equation for the function $u = u\left(t\right)$ is written as [57]

$$h_0\left(t\right)\frac{d^n u}{dt^n} + h_1\left(t\right)\frac{d^{n-1}u}{dt^{n-1}} + \dots + h_{n-1}\left(t\right)\frac{du}{dt} + h_n\left(t\right)u = g(t), \qquad \mathrm{t}_0 < t < t_1$$

For example,

$$4\frac{d^4u}{dt^4}(t) - \cos t\frac{d^3u}{dt^3} + 2\sin t\frac{d^2u}{dt^2} + 2\cos t\frac{du}{dt} + 16u = 4e^t\sin 2t, \quad 0 < t < 1$$

The above differential equation is fourth order linear differential equation.

The general nth order non-linear differential equation for the function $u = u\left(t\right)$ is written as

$$\frac{d^n u}{dt^n}(t) = H\left(t, u(t), \frac{du}{dt}(t), ..., \frac{d^{n-1}u}{dt^{n-1}}(t)\right), \qquad \mathrm{t}_0 < t < t_1$$

For instance,

$$\frac{d^5 u}{dt^5} + 24e^{-5u} = \frac{48}{(1+t)^5}, \qquad 0 < t < 1$$

The previous differential equation is fifth order non-linear differential equation.

2.6.2 Fractional Order Differential Equations

The general linear FDE for the function $u = u(t)$ is written as [57]

$$D^\alpha u(t) = h_0(t) D^{\eta_0} u(t) + ... + h_{k-1}(t) D^{\eta_{k-1}} u(t) + h_k(t) u(t) + g(t)$$

Linearity of the this equation arises from linearity of the fractional differential operator. For instance,

$$4(\mathrm{t}+1)\,\mathrm{D}^{\frac{5}{2}}\, u(t) + 4\,\mathrm{D}^{\frac{3}{2}}\, u(t) + \frac{1}{\sqrt{t+1}} u(t) = \sqrt{t} + \sqrt{\pi}$$

and

$$D^2 u(t) + D^{\frac{3}{2}} u(t) + u(t) = 1 + t$$

are linear FDEs.

The general non- linear fractional order differential equation for the function $u = u(t)$ is written as

$$D^\alpha u(t) = H\left(t, u(t), D^{\eta_1} u(t), ..., D^{\eta_k} u(t)\right)$$

For instance,

$$D^3 u(t) + D^{\frac{5}{2}} u(t) + u^2(t) = t^4$$

and

$$D^{1.3} u(t) + u^2(t) = \frac{20}{7} \frac{t^{0.7}}{\Gamma(0.7)} + t^4$$

are non-linear FDEs.

2.7 Systems of Differential Equations

2.7.1 Systems of Ordinary Differential Equations

The general system of linear differential equations for functions $u_j = u_j(t)$ $(j = 1, 2, ..., m)$ is written as [57]

$$\begin{aligned}
\frac{d^n u_1}{dt^n} &= h_{11}(t)u_1 + ... + h_{1n}(t)u_m + k_1(t) \\
\frac{d^n u_2}{dt^n} &= h_{21}(t)u_1 + ... + h_{2n}(t)u_m + k_2(t) \\
&\vdots \qquad \vdots \quad \vdots \qquad \vdots \\
\frac{d^n u_m}{dt^n} &= h_{m1}(t)u_1 + ... + h_{mn}(t)u_m + k_m(t)
\end{aligned}$$

where $h_{ij}(t)$ and $k_i(t)$ $i = 1, 2, ..., m$ $j = 1, 2, ..., n$ are known functions on some interval $a < t < b$. The unknowns are the functions $u_1(t), ..., u_m(t)$.

If all $k_i = 0$ then the system is called homogeneous, otherwise it is called non-homogeneous. Linearity of the system arises from linearity of the diferential equations. That is to say, if all diferential equations is linear then the system is a linear system.

For instance,

$$\begin{aligned}
\frac{du(t)}{dt} &= w(t) - \cos t \\
\frac{dv(t)}{dt} &= w(t) - e^t \\
\frac{dw(t)}{dt} &= u(t) - v(t)
\end{aligned}$$

The above system is first order, non-homogeneous and linear system of differential equations.

The general system of non-linear differential equations for functions $u_j = u_j(t)$ $(j = 1, 2, ..., m)$ is written as

$$
\begin{aligned}
\frac{d^n u_1}{dt^n}(t) &= H_1\left(t, u_1(t), \ldots, \frac{d^{n-1}u_1}{dt^{n-1}}(t), u_2(t), \ldots, \frac{d^{n-1}u_2}{dt^{n-1}}(t), \ldots, u_m(t), \ldots, \frac{d^{n-1}u_m}{dt^{n-1}}(t)\right) \\
\frac{d^n u_2}{dt^n}(t) &= H_2\left(t, u_1(t), \ldots, \frac{d^{n-1}u_1}{dt^{n-1}}(t), u_2(t), \ldots, \frac{d^{n-1}u_2}{dt^{n-1}}(t), \ldots, u_m(t), \ldots, \frac{d^{n-1}u_m}{dt^{n-1}}(t)\right) \\
&\vdots \\
\frac{d^n u_m}{dt^n}(t) &= H_m\left(t, u_1(t), \ldots, \frac{d^{n-1}u_1}{dt^{n-1}}(t), u_2(t), \ldots, \frac{d^{n-1}u_2}{dt^{n-1}}(t), \ldots, u_m(t), \ldots, \frac{d^{n-1}u_m}{dt^{n-1}}(t)\right)
\end{aligned}
$$

For example,

$$
\begin{aligned}
\frac{du(t)}{dt} &= -2u(t) + u^2(t)\,\mathrm{v(t)} \\
\frac{dv(t)}{dt} &= u(t) - u^2(t)\,\mathrm{v(t)}
\end{aligned}
$$

The above Brusselator system is first order non-linear system of differential equations.

2.7.2 Systems of Fractional Order Differential Equations

The general system of FDEs for functions $u_j = u_j(t)$ $\left(j = 1, 2, \ldots, m\right)$ is written as [57]

$$
\begin{aligned}
D^{\eta_1} u_1(t) &= U_1(t, u_1, u_2, \ldots, u_m), \\
D^{\eta_2} u_2(t) &= U_2(t, u_1, u_2, \ldots, u_m), \\
&\vdots \\
D^{\eta_n} u_m(t) &= U_m(t, u_1, u_2, \ldots, u_m),
\end{aligned}
$$

If U_i's are linear functions of $t, u_1, u_2, \ldots, u_m$, then the system is a linear system of FDEs.

For example,

$$
\begin{aligned}
D^{\alpha} u(t) &= u(t) + v(t) \\
D^{\alpha} v(t) &= -u(t) + v(t)
\end{aligned}
$$

The above system is a system of linear FDEs.

If U_i's are non-linear functions of $t, u_1, u_2, \ldots, u_m$, then the system is a non-linear system of FDEs.

For instance,

$$D^{\frac{3}{2}}u(t) = -8u(t) + v^2(t) - 4t^6 + 4t^3 + \frac{8t^{\frac{3}{2}}}{\sqrt{\pi}} - 1$$

$$D^{\frac{1}{2}}v(t) = t^2 Du(t) + v(t) - 3t^4 - 2t^3 + \frac{32t^{\frac{5}{2}}}{5\sqrt{\pi}} - 1$$

The previous system is a system of non-linear FDEs.

CHAPTER 3

THE APPLICATION OF THE OPERATIONAL MATRIX OF DERIVATIVE TO HIGH ORDER DIFFERENTIAL EQUATIONS

High-order differential equations have substantial attention, because of their fascinating mathematical structures and properties and they play an important role in the thermal science and mechanical engineering. Fluid-flow, heat transfer and other related physical phenomena of interest are gained by principles of conservation and are symbolized in terms of differential equations denoting these principles.

For example, fourth-order DEs are used in the numerical analysis of viscoelastic and inelastic flows, the free vibration analysis of beam structures, deformation of beams and plate deflection theory [1]. A fourth order analogue of it is the Orr-Sommerfield equation explain to great correctness the cross-stream behavior of channel fluid-flow. Moreover, sixth-order differential equations arise in the free vibration analysis of ring structures and astrophysics [2]. Some related applications of high-order DEs can be found in [1-8]. In [1], Noor and Mohyud-Din presented the variation iteration method for solving fourth-order BVPs and in [4], [7] they illustrated the homotopy perturbation method for solving fifth-order and sixth-order BVPs, respectively. Also, in [8], a Legendre Petrov-Galerkin method was demonstrated for the solution of the fourth-order BVPs. In [5-6], the numerical solution of fifth-order BVPs was presented by using a new cubic B-spline method and a sixth-degree B-spline approximation, respectively. In [2], El-Gamel *et al*. applied sinc-Galerkin method for solving sixth-order BVPs. Secer and writer of this thesis submitted a paper which deals with the numerical solution of high order differential equations by using Legendre wavelet operational matrix method in [56].

In this chapter, the operational matrix of Legendre wavelet is generalized in order to solve high-order linear and non-linear multi-point: initial and boundary value problems.

3.1 Solving High Order Linear Differential Equations

This section introduces an alternative solution technique called LWOMM to obtain the numerical solution of high order linear DEs. Consider the following equation

$$h_0(t)\frac{d^n u}{dt^n}+h_1(t)\frac{d^{n-1}u}{dt^{n-1}}+\ldots+h_{n-1}(t)\frac{du}{dt}+h_n(t)u=g(t), \qquad \mathrm{t}_0<t<t_1 \tag{3.1}$$

with these initial conditions

$$u(t_0)=u_0,\ \frac{du}{dt}(t_0)=u_1,\ \frac{d^2u}{dt^2}(t_0)=u_2,\ldots,\frac{d^{n-1}u}{dt^{n-1}}(t_0)=u_{n-1} \tag{3.2}$$

or with these boundary conditions

$$\begin{aligned}
&u(t_0)=u_0,\ \frac{du}{dt}(t_0)=u_1,\ \frac{d^2u}{dt^2}(t_0)=u_2,\ldots,\frac{d^iu}{dt^i}(t_0)=u_i\\
&u(t_1)=u_0',\ \frac{du}{dt}(t_1)=u_1',\ \frac{d^2u}{dt^2}(t_1)=u_2',\ldots,\frac{d^iu}{dt^i}(t_1)=u_i' \qquad i=0,1,\ldots,n/2\ \ if\ \ n\ even\\
&u(t_0)=u_0,\ \frac{du}{dt}(t_0)=u_1,\ \frac{d^2u}{dt^2}(t_0)=u_2,\ldots,\frac{d^iu}{dt^i}(t_0)=u_i\\
&u(t_1)=u_0',\ \frac{du}{dt}(t_1)=u_1',\ \frac{d^2u}{dt^2}(t_1)=u_2',\ldots,\frac{d^{i-1}u}{dt^{i-1}}(t_1)=u_{i-1}' \qquad i=0,1,\ldots,(n+1)/2\ \ if\ \ n\ \text{odd}
\end{aligned} \tag{3.3}$$

First of all, approximating $u(t)$ by the Legendre wavelets, then we obtain

$$u(t)\cong C^T\psi(t) \tag{3.4}$$

where C is an unknown vector and $\psi(t)$ is the vector defined in Equation (2.27). If we utilise Equation (2.31) then we have

$$\frac{du}{dt}(t)\cong C^TD\psi(t),\quad \frac{d^2u}{dt^2}(t)\cong C^TD^2\psi(t),\ldots,\frac{d^nu}{dt^n}(t)\cong C^TD^n\psi(t) \tag{3.5}$$

Also approximating $h_0(t),h_1(t),\ldots,h_n(t)\ and\ g(t)$, then we get

$$h_0(t)\cong H_0{}^T\psi(t),\ \ h_1(t)\cong H_1{}^T\psi(t),\ldots,h_n(t)\cong H_n{}^T\psi(t),\ \text{ and }\ g(t)\cong G^T\psi(t) \tag{3.6}$$

where vectors $H_0,H_1,\ldots,H_n\ and\ G$ are given by Equation (2.26). Substituting Equations (3.5)-(3.6) in Equation (3.1) we obtain

$$
\begin{aligned}
R(t) = &\left(H_0{}^T\psi(t)\right)\left(C^T D^n\psi(t)\right) + \left(H_1{}^T\psi(t)\right)\left(C^T D^{n-1}\psi(t)\right) + \ldots \\
&+ \left(H_{n-1}{}^T\psi(t)\right)\left(C^T D\psi(t)\right) + \left(H_n{}^T\psi(t)\right)\left(C^T\psi(t)\right) - G^T\psi(t) = 0
\end{aligned}
\tag{3.7}
$$

If we use the product operation matrix of Legendre wavelets, then we obtain

$$
\begin{aligned}
R(t) = &\left(H_0{}^T\psi(t)\right)\left(\psi^T(t)\left(D^n\right)^T C\right) + \left(H_1{}^T\psi(t)\right)\left(\psi^T(t)\left(D^{n-1}\right)^T C\right) + \ldots \\
&+ \left(H_{n-1}{}^T\psi(t)\right)\left(\psi^T(t)(D)^T C\right) + \left(H_n{}^T\psi(t)\right)\left(\psi^T(t)C\right) - \psi^T(t)G \\
= &\,\psi^T(t)\tilde{H}_0\left(D^n\right)^T C + \psi^T(t)\tilde{H}_1\left(D^{n-1}\right)^T C + \ldots + \psi^T(t)\tilde{H}_{n-1}(D)^T C + \psi^T(t)\tilde{H}_n C - \psi^T(t)G
\end{aligned}
\tag{3.8}
$$

where $\tilde{H}_0, \tilde{H}_1, \ldots, \tilde{H}_n$ are the product operation matrices and can be calculated by using Equation (2.32).

We obtain $2^k(M+1)-n$ linear equations by computing

$$
\int_0^1 \psi_j(t)R(t)\,dt = 0, \quad j = 1, \ldots, 2^k(M+1)-n \tag{3.9}
$$

Also, if we substitute these initial conditions (3.2) in Equations (3.4)-(3.5) we obtain

$$
\begin{aligned}
&u(t_0) \cong C^T\psi(t_0) = u_0, \quad \frac{du}{dt}(t_0) \cong C^T D\psi(t_0) = u_1 \\
&\frac{d^2u}{dt^2}(t_0) \cong C^T D^2\psi(t_0) = u_2, \ldots, \frac{d^{n-1}u}{dt^{n-1}}(t_0) \cong C^T D^{n-1}\psi(t_0) = u_{n-1}
\end{aligned}
\tag{3.10}
$$

We obtain $2^k(M+1)$ set of linear equations by using Equations (3.9) and (3.10). These linear equations can be solved for unknown coefficients of the vector C. Accordingly, $u(t)$ which is given in Equation (3.1) can be computed.

3.2 Solving High Order Non-Linear Differential Equations

In this section, the LWOMM is implemented for solving $n-th$ order non-linear DEs. Consider the following equation

$$
\frac{d^n u}{dt^n}(t) = H\left(t, u(t), \frac{du}{dt}(t), \ldots, \frac{d^{n-1}u}{dt^{n-1}}(t)\right), \qquad \mathrm{t}_0 < t < t_1 \tag{3.11}
$$

with the initial conditions

$$u(t_0)=u_0,\ \frac{du}{dt}(t_0)=u_1,\ \frac{d^2u}{dt^2}(t_0)=u_2,\ldots,\frac{d^{n-1}u}{dt^{n-1}}(t_0)=u_{n-1} \tag{3.12}$$

or boundary conditions

$$\begin{array}{ll}
u(t_0)=u_0,\ \frac{du}{dt}(t_0)=u_1,\ \frac{d^2u}{dt^2}(t_0)=u_2,\ldots,\frac{d^iu}{dt^i}(t_0)=u_i & \\
 & i=0,1,\ldots,n/2\ \ if\ \ n\ even \\
u(t_1)=u_0',\ \frac{du}{dt}(t_1)=u_1',\ \frac{d^2u}{dt^2}(t_1)=u_2',\ldots,\frac{d^iu}{dt^i}(t_1)=u_i' & \\
u(t_0)=u_0,\ \frac{du}{dt}(t_0)=u_1,\ \frac{d^2u}{dt^2}(t_0)=u_2,\ldots,\frac{d^iu}{dt^i}(t_0)=u_i & \\
 & i=0,1,\ldots,(n+1)/2\ \ if\ \ n\ \text{odd} \\
u(t_1)=u_0',\ \frac{du}{dt}(t_1)=u_1',\ \frac{d^2u}{dt^2}(t_1)=u_2',\ldots,\frac{d^{i-1}u}{dt^{i-1}}(t_1)=u_{i-1}' &
\end{array} \tag{3.13}$$

First we presume that the unknown function $u(t)$ is approximated and given by

$$u(t)\cong C^T\psi(t) \tag{3.14}$$

where C is an unknown vector and $\psi(t)$ is the vector which given in Equation (2.27). By utilising Equation (3.11) then we obtain

$$C^TD^n\psi(t)\cong H\left(t,C^T\psi(t),C^TD\psi(t),\ldots,C^TD^{n-1}\psi(t)\right) \tag{3.15}$$

Also by substituting initial conditions (3.12) in Equations (3.4) and (3.5) we obtain

$$\begin{array}{l}
u(t_0)\cong C^T\psi(t_0)=u_0,\ \ \frac{du}{dt}(t_0)\cong C^TD\psi(t_0)=u_1 \\
\frac{d^2u}{dt^2}(t_0)\cong C^TD^2\psi(t_0)=u_2,\ldots,\frac{d^{n-1}u}{dt^{n-1}}(t_0)\cong C^TD^{n-1}\psi(t_0)=u_{n-1}
\end{array} \tag{3.16}$$

To obtain the solution $u(t)$, we first compute Equation (3.15) at $2^k(M+1)-n$ points. For a better result, we utilise the first $2^k(M+1)-n$ roots of shifted Legendre polynomials $P_{2^k(M+1)}(t)$. If we use these equations collectively with Equation (3.16), then we obtain $2^k(M+1)$ non-linear equations. These non-linear equations can be solved for unknown coefficients of the vector C. Accordingly $u(t)$ given in Equation (3.11) can be computed.

3.3 Applications

In this section, we present some examples to demonstrate the performance of the introduced tecnique for solving high order linear and non-linear DEs. It is shown that the LWOMM yield better results.

Example 3.1 Consider the following second-order non-linear BVP

$$\frac{d^2u}{dt}+2\left(\frac{du}{dt}\right)^2+8u(t)=0, \quad 0<t<1 \tag{3.17}$$

with these boundary conditions

$$u(0)=0, \quad u(1)=0 \tag{3.18}$$

The exact solution of the previous system is $u(t)=t-t^2$

To solve the above problem, we implemented the method presented in Section 3.2 with $M=2, \; k=0$. Approximating solution following as

$$u(t)\cong C^T\psi(t), \quad \frac{du}{dt}\cong C^T D\psi(t), \quad \frac{d^2u}{dt^2}\cong C^T D^2\psi(t)$$

We get

$$D=\begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}, \quad D^2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12\sqrt{5} & 0 & 0 \end{pmatrix}$$

If we consider (3.17) with (3.18), we have

$$C^T D^2\psi(t)+2\left(C^T D\psi(t)\right)^2+8\left(C^T D\psi(t)\right)=0 \tag{3.19}$$

Calculating Equation (3.19) at the first root of $P_3(t)$, i.e. $t_0=\frac{1}{2}-\frac{\sqrt{15}}{10}$

We get

$$\begin{aligned} &4c_{0,2}\sqrt{15}\sqrt{3}+2\left(2c_{0,1}\sqrt{3}+2c_{0,2}\sqrt{15}\sqrt{3}(-1+2t)\right)^2 \\ &\quad +8c_{0,0}+8c_{0,1}\sqrt{3}(-1+2t)+8c_{0,2}\sqrt{5}\left(6t^2-6t+1\right)=0 \end{aligned}$$

and by utilising boundary conditions we have

$$c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}=0$$
$$c_{0,0}+\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}=0$$

If we solve this system of nonlinear algebraic equations, we get

$$C^T=\left[c_{0,0},c_{0,1},c_{0,2}\right]=\left[0.1666666666,\text{-}0.4309487636\ 10^{-22},\text{-}0.07453559926\right]$$

Consequently,

$$u(t)=C^T\psi(t)=\left[0.1666666666,\text{-}0.4309487636\ 10^{-22},\text{-}0.07453559926\right]\begin{bmatrix}1\\ \sqrt{3}(-1+2t)\\ \sqrt{5}\left(6t^2-6t+1\right)\end{bmatrix}$$

The approximate solution with the exact solution are displayed in Table. 3.1.

Table 3.1 Comparison between the exact solution and our numerical solution for Example 3.1

t	Exact Solution $u(t)$	$M=2,\ \ k=0$ Approximate Solution	Absolute Error
0.0	0.00	-0.1 10^{-9}	0.1 10^{-9}
0.1	0.09	0.08999999994	0.6 10^{-10}
0.2	0.16	0.15999999999	0.1 10^{-9}
0.3	0.21	0.20999999999	0.1 10^{-9}
0.4	0.24	0.23999999999	0.1 10^{-9}
0.5	0.25	0.24999999999	0.1 10^{-9}
0.6	0.24	0.23999999999	0.1 10^{-9}
0.7	0.21	0.20999999999	0.1 10^{-9}
0.8	0.16	0.15999999999	0.1 10^{-9}
0.9	0.09	0.08999999999	0.6 10^{-10}
1.0	0.00	-0.1 10^{-9}	0.1 10^{-9}

Example 3.2. We consider the following fourth-order linear IVP

$$4\frac{d^4u}{dt^4}(t)-\cos t\frac{d^3u}{dt^3}+2\sin t\frac{d^2u}{dt^2}+2\cos t\frac{du}{dt}+16u=4e^t\sin 2t,\qquad 0<t<1 \tag{3.20}$$

with these initial conditions

$$u(0)=0,\quad \frac{du}{dt}(0)=1,\quad \frac{d^2u}{dt^2}(0)=2,\quad \frac{d^3u}{dt^3}(0)=2 \tag{3.21}$$

The exact solution of the previous system is known as $u(t)=e^t\sin t$

To solve Equation (3.20), we applied the method presented in Section 3.1 with $M=5,\ k=0$. Approximating solution following as

$$u(t)\cong C^T\psi(t),\quad \frac{du(t)}{dt}\cong C^T D\psi(t)$$

$$\frac{d^2u(t)}{dt^2}\cong C^T D^2\psi(t),\quad \frac{d^3u(t)}{dt^3}\cong C^T D^3\psi(t),\quad \frac{d^4u(t)}{dt^4}\cong C^T D^4\psi(t)$$

Also approximating $h_0(t)=\sin t,\ h_1(t)=\cos t,\ g(t)=e^t\sin 2t$ following as

$$h_0(t)\cong H_0{}^T\psi(t),\ h_1(t)\cong H_1{}^T\psi(t) \text{ and } g(t)\cong G^T\psi(t)$$

where $H_0(t)=\int_0^1 \sin t\psi(t)dt,\ H_1(t)=\int_0^1 \cos t\psi(t)dt$ and $G(t)=\int_0^1 e^t\sin 2t\psi(t)\,dt$

We get

$$D=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 2\sqrt{3} & 0 & 0 & 0 & 0 & 0\\ 0 & 2\sqrt{15} & 0 & 0 & 0 & 0\\ 0 & 0 & 2\sqrt{35} & 0 & 0 & 0\\ 0 & 0 & 0 & 2\sqrt{63} & 0 & 0\\ 0 & 0 & 0 & 0 & 2\sqrt{99} & 0 \end{pmatrix},$$

$$D^2=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 4\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0\\ 0 & 4\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0\\ 0 & 0 & 4\sqrt{63}\sqrt{35} & 0 & 0 & 0\\ 0 & 0 & 0 & 4\sqrt{99}\sqrt{63} & 0 & 0 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 8\sqrt{35}\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 8\sqrt{63}\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 8\sqrt{99}\sqrt{63}\sqrt{35} & 0 & 0 & 0 \end{pmatrix}$$

$$D^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 16\sqrt{63}\sqrt{35}\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 16\sqrt{99}\sqrt{63}\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 \end{pmatrix}$$

If we consider (3.20) with (3.21), we have

$$R(t) = 4C^T D^4\psi(t) - H_1{}^T D^3\psi(t) + 2H_0{}^T D^2\psi(t) + 2H_1{}^T D\psi(t) + 16C^T\psi(t) - 4G^T\psi(t) \tag{3.22}$$

By computing

$$\int_0^1 \psi_j(t)R(t)\,dt = 0, \quad j = 1,2$$

We obtain two linear equations following as

$$\begin{aligned} &-5.387308351 + 16c_{0,0} + 5.829881996c_{0,1} + 22.57903587c_{0,2} \\ &\quad - 222.6322963c_{0,3} + 20349.82385c_{0,4} + 118.3624723c_{0,5} = 0 \\ &-3.173505657 + 15.06490716c_{0,1} + 25.84945221c_{0,2} \\ &\quad + 121.4121245c_{0,3} - 1101.248082c_{0,4} + 116247.0813c_{0,5} = 0 \end{aligned}$$

and by utilising initial conditions we have

$$\begin{aligned} &c_{0,0} - \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} - \sqrt{7}c_{0,3} + 3c_{0,4} - \sqrt{11}c_{0,5} = 0 \\ &2\sqrt{3}c_{0,1} - 6\sqrt{5}c_{0,2} + 10\sqrt{7}c_{0,3} - 42c_{0,4} + 18\sqrt{11}c_{0,5} = 1 \\ &12\sqrt{5}c_{0,2} - 60\sqrt{7}c_{0,3} + 420c_{0,4} - 252\sqrt{11}c_{0,5} = 2 \\ &120\sqrt{7}c_{0,3} - 2520c_{0,4} - 1260\sqrt{11}c_{0,5} = 2 \end{aligned}$$

If we solve this system of linear algebraic equations, we get

$C^T = \left[c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4}, c_{0,5}\right]$

$= \left[0.9077018418, 0.6482676542, 0.1005609239, 0.00298064330, -.00071311177, -.0000889418471\right]$

Consequently,

$$u(t) = C^T\psi(t) = \begin{bmatrix} 0.9077018418 \\ 0.6482676542 \\ 0.1005609239 \\ 0.00298064330 \\ -.00071311177 \\ -.0000889418471 \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}\left(-1+2t\right) \\ \sqrt{5}\left(6t^2-6t+1\right) \\ \sqrt{7}\left(20t^3-30t^2+12t-1\right) \\ 3\left(70t^4-140t^3+90t^2-20t+1\right) \\ \sqrt{11}\left(252t^5-630t^4+560t^3-210t^2+30t-1\right) \end{bmatrix}$$

The exact solution and our approximate solution are displayed in Table. 3.2.

Table 3.2 Comparison between the exact solution and the approximate solution for Example 3.2

t	Exact Solution $u(t)$	$M=5,\ k=0$ Approximate Solution	Absolute Error
0.0	0.0000000000	0.060 10^{-11}	0.060 10^{-11}
0.1	0.1103329887	0.1126216474	0.0022886587
0.2	0.2426552686	0.2466635725	0.0040083039
0.3	0.3989105540	0.4038963982	0.0049858442
0.4	0.5809439009	0.5859989519	0.0050550509
0.5	0.7904390834	0.7944690603	0.0040299768
0.6	1.0288456660	1.0305343470	0.0016886798
0.7	1.2972951110	1.2950630240	0.0022320878
0.8	1.5965053400	1.5884746930	0.0080306463
0.9	1.9266733040	1.9106511420	0.0160221600
1.0	2.2873552870	2.2608471370	0.0265081500

Example 3.3 Consider the following fifth order linear IVP [5]

$$\begin{aligned}&\frac{d^5u}{dt^5}+(t-2)\frac{d^4u}{dt^4}+2\frac{d^3u}{dt^3}-\left(t^2+2t-1\right)\frac{d^2u}{dt^2}+\left(2t^2+4t\right)\frac{du}{dt}\\&-2t^2u=4e^t\cos t-2t^4+4t^3+6t^2-4t+2,\quad 0<t<1\end{aligned} \tag{3.23}$$

subject to these initial conditions

$$u(0)=0,\ \frac{du}{dt}(0)=2,\ \frac{d^2u}{dt^2}(0)=6,\ \frac{d^3u}{dt^3}(0)=4,\ \frac{d^4u}{dt^4}(0)=0 \tag{3.24}$$

The exact solution of the above system is $u(t)=2e^t\sin t+t^2$

To solve the above problem, we applied the method presented in Section 3.1 with $M=7,\ k=0$. Approximating solution following as

$$u(t)\cong C^T\psi(t),\quad \frac{du(t)}{dt}\cong C^TD\psi(t),\quad \frac{d^2u(t)}{dt^2}\cong C^TD^2\psi(t),$$

$$\frac{d^3u(t)}{dt^3}\cong C^TD^3\psi(t),\quad \frac{d^4u(t)}{dt^4}\cong C^TD^4\psi(t),\quad \frac{d^5u(t)}{dt^5}\cong C^TD^5\psi(t)$$

Also, approximating $h_0(t)=t-2,\ h_1(t)=t^2+2t-1,\ h_2(t)=2t^2+4t,\ h_3(t)=2t^2,$ and $g(t)=4e^t\cos t-2t^4+4t^3+6t^2-4t+2$ following as

$$h_0(t)\cong H_0{}^T\psi(t),\ h_1(t)\cong H_1{}^T\psi(t),\ h_2(t)\cong H_2{}^T\psi(t),\ h_3(t)\cong H_3{}^T\psi(t)\ \text{and}\ g(t)\cong G^T\psi(t)$$

where

$$H_0(t)=\int_0^1(t-2)\psi(t)dt,\ H_1(t)=\int_0^1\left(t^2+2t-1\right)\psi(t)dt,\ H_2(t)=\int_0^1\left(2t^2+4t\right)\psi(t)dt,$$

$$H_3(t)=\int_0^1\left(2t^2\right)\psi(t)dt\ \text{ and }\ G(t)=\int_0^1\left(4e^t\cos t-2t^4+4t^3+6t^2-4t+2\right)\psi(t)\,dt$$

We get

$$D=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{63} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sqrt{143} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{195} & 0 \end{pmatrix},$$

$$D^2=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\sqrt{63}\sqrt{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\sqrt{99}\sqrt{63} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\sqrt{143}\sqrt{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\sqrt{195}\sqrt{143} & 0 & 0 \end{pmatrix}$$

$$D^3=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8\sqrt{35}\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8\sqrt{63}\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8\sqrt{99}\sqrt{63}\sqrt{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8\sqrt{143}\sqrt{99}\sqrt{63} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8\sqrt{195}\sqrt{143}\sqrt{99} & 0 & 0 & 0 \end{pmatrix}$$

$$D^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16\sqrt{63}\sqrt{35}\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16\sqrt{99}\sqrt{63}\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16\sqrt{143}\sqrt{99}\sqrt{63}\sqrt{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16\sqrt{195}\sqrt{143}\sqrt{99}\sqrt{63} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32\sqrt{99}\sqrt{63}\sqrt{35}\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 32\sqrt{143}\sqrt{99}\sqrt{63}\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 32\sqrt{195}\sqrt{143}\sqrt{99}\sqrt{63}\sqrt{35} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If we consider (3.23) with (3.24), we have

$$\begin{aligned} R(t) = {} & C^T D^5 \psi(t) + \left(H_0{}^T \psi(t)\right)\left(C^T D^4 \psi(t)\right) + 2\left(C^T D^3 \psi(t)\right) - \left(H_1{}^T \psi(t)\right)\left(C^T D^2 \psi(t)\right) \\ & + \left(H_2{}^T \psi(t)\right)\left(C^T D \psi(t)\right) - \left(H_3{}^T \psi(t)\right)\left(C^T \psi(t)\right) - G^T \psi(t) \end{aligned} \tag{3.25}$$

By computing

$$\int_0^1 \psi_j(t) R(t)\, dt = 0, \quad j = 1, 2, 3$$

We obtain three linear equations following as

$$\begin{aligned} & -0.6666666667 c_{0,0} + 8.660254039 c_{0,1} + 4.323064761 c_{0,2} + 557.3716095 c_{0,3} \\ & \quad - 7574.000000 c_{0,4} + 108652.6281 c_{0,5} - 0.3\ 10^{-7} c_{0,6} - 0.15\ 10^{-4} c_{0,7} - 8.112098454 = 0 \end{aligned}$$

$$\begin{aligned} & -1.782876059 - 0.5773502693 c_{0,0} + 5.200000001 c_{0,1} - 2.065591121 c_{0,2} - 18.46123351 c_{0,3} \\ & \quad + 4221.354229 c_{0,4} - 43449.57403 c_{0,5} + 715526.8907 c_{0,6} - 0.1\ 10^{-4} c_{0,7} = 0 \end{aligned}$$

$$\begin{aligned} & -0.1490711985 c_{0,0} - 0.986\ 10^{-10} c_{0,1} + 9.238095242 c_{0,2} - 38.82075211 c_{0,3} - 47.53241642 c_{0,4} \\ & \quad + 14713.31402 c_{0,5} - 134121.8771 c_{0,6} + 2550146.901 c_{0,7} - 0.3879252900 = 0 \end{aligned}$$

and by utilising the initial conditions we have

$$u(0) \cong C^T \psi\left(0\right) = 0, \quad \frac{du(0)}{dt} \cong C^T D\psi\left(0\right) = 2, \quad \frac{d^2u(0)}{dt^2} \cong C^T D^2\psi\left(0\right) = 6$$

$$\frac{d^3u(0)}{dt^3} \cong C^T D^3\psi\left(0\right) = 4, \quad \frac{d^4u(0)}{dt^4} \cong C^T D^4\psi\left(0\right) = 0$$

If we solve this system of linear algebraic equations, we get

$$C^T = \left[c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4}, c_{0,5}, c_{0,6}, c_{0,7}\right] = \begin{bmatrix} 2.142158342 \\ 1.578792178 \\ 0.2730371723 \\ 0.0054607618 \\ -0.001414419958 \\ -0.000175507821 \\ -0.86373431\ 10^{-5} \\ -0.96647696\ 10^{-7} \end{bmatrix}^T$$

Consequently,

$$u\left(t\right) = C^T\psi\left(t\right) = \begin{bmatrix} 2.142158342 \\ 1.578792178 \\ 0.2730371723 \\ 0.0054607618 \\ -0.001414419958 \\ -0.000175507821 \\ -0.86373431\ 10^{-5} \\ -0.96647696\ 10^{-7} \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}\left(-1+2t\right) \\ \sqrt{5}\left(6t^2-6t+1\right) \\ \sqrt{7}\left(20t^3-30t^2+12t-1\right) \\ 3\left(70t^4-140t^3+90t^2-20t+1\right) \\ \sqrt{11}\left(252t^5-630t^4+560t^3-210t^2+30t-1\right) \\ \sqrt{13}\left(924t^6-2772t^5+3150t^4-1680t^3+420t^2-42t+1\right) \\ \sqrt{15}\left(3432t^7-12012t^6+16632t^5-11550t^4+4200t^3-756t^2+56t-1\right) \end{bmatrix}$$

The approximate solution with the exact solution are displayed in Table. 3.3.

Table 3.3 Comparison between the exact solution and the approximate solution for Example 3.3

t	Exact Solution $u(t)$	$M=7,\ k=0$ Approximate Solution	Absolute Error
0.0	0.0000000000	$-0.5425130 10^{-9}$	$0.5425130 10^{-9}$
0.1	0.2306659774	0.2349757869	0.0043098115
0.2	0.5253105372	0.5326889428	0.0073784060
0.3	0.8878211080	0.8965862650	0.0087651579
0.4	1.3218878020	1.3298579860	0.0079701859
0.5	1.8308781670	1.8353120110	0.0044338456
0.6	2.4176913320	2.4152287240	0.0024626060
0.7	3.0845902220	3.0711957340	0.0133944863
0.8	3.8330106800	3.8039211894	0.0290887837
0.9	4.6633466080	4.6130299590	0.0503166471
1.0	5.5747105740	5.4968272320	0.0778833389

Example 3.4 Consider the following sixth order linear BVP [2]

$$\frac{d^6u}{dt^6}+\frac{d^3u}{dt^3}+\frac{d^2u}{dt^2}-u=e^{-t}\left(-15t^2+78t-114\right), \quad 0<t<1 \tag{3.26}$$

subject to these boundary conditions

$$u(0)=0, \quad \frac{du}{dt}(0)=0, \quad \frac{d^2u}{dt^2}(0)=0, u(1)=1/e, \quad \frac{du}{dt}(1)=2/e, \quad \frac{d^2u}{dt^2}(1)=1/\mathrm{e} \tag{3.27}$$

The exact solution of the above system is $u(t)=t^3e^{-t}$

To solve the above problem, we applied the method presented in Section 3.1 with $M=7,\ k=0$. Approximating solution following as

$$u(t)\cong C^T\psi(t), \qquad \frac{du(t)}{dt}\cong C^TD\psi(t)$$

$$\frac{d^2u(t)}{dt^2} \cong C^T D^2 \psi(t), \quad \frac{d^3u(t)}{dt^3} \cong C^T D^3 \psi(t), \quad \frac{d^6u(t)}{dt^6} \cong C^T D^6 \psi(t)$$

Also approximating $g(t) = e^{-t}\left(-15t^2 + 78t - 114\right)$ following as $g(t) \cong G^T \psi(t)$

We get

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{63} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sqrt{143} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{195} & 0 \end{pmatrix},$$

$$D^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\sqrt{63}\sqrt{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\sqrt{99}\sqrt{63} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\sqrt{143}\sqrt{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\sqrt{195}\sqrt{143} & 0 & 0 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8\sqrt{35}\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8\sqrt{63}\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8\sqrt{99}\sqrt{63}\sqrt{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8\sqrt{143}\sqrt{99}\sqrt{63} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8\sqrt{195}\sqrt{143}\sqrt{99} & 0 & 0 & 0 \end{pmatrix}$$

$$D^6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 64\sqrt{143}\sqrt{99}\sqrt{63}\sqrt{35}\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 64\sqrt{195}\sqrt{143}\sqrt{99}\sqrt{63}\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If we consider (3.26) with (3.27), we have

$$R(t) = C^T D^6 \psi(t) + C^T D^3 \psi(t) + C^T D^2 \psi(t) - C^T \psi(t) - G^T \psi(t)$$

By computing

$$\int_0^1 \psi_j(t) R(t)\, dt = 0, \quad j = 1, 2$$

We obtain two linear equations following as

$$53.85997844 + 26.83281573 c_{0,2} - c_{0,0} + 317.4901573 c_{0,3} + 2398701.153 c_{0,6} = 0$$

$$-26.41233933 - c_{0,1} + 91.65151390 c_{0,3} + 1454.922678 c_{0,4} - 0.5\ 10^{-8} c_{0,5} - 0.49\ 10^{-7} c_{0,6} + 19338946.95 c_{0,7} = 0$$

and by utilising the boundary conditions we have

$$u(0) \cong C^T \psi(0) = 0, \quad u(1) \cong C^T \psi(1) = \frac{1}{e}$$
$$\frac{du}{dt}(0) \cong C^T D \psi(0) = 0, \quad \frac{du}{dt}(1) \cong C^T D \psi(1) = \frac{2}{e}$$
$$\frac{d^2u}{dt^2}(0) \cong C^T D^2 \psi(0) = 0, \quad \frac{d^2u}{dt^2}(1) \cong C^T D^2 \psi(1) = \frac{1}{e}$$

If we solve this system of linear algebraic equations, we get

$$u(t)=C^T\psi(t)=\begin{bmatrix} 0.1161815354 \\ 0.1069989365 \\ 0.03242759289 \\ -0.001046948131 \\ -0.001556841602 \\ 0.0004149720161 \\ -0.0000226295487 \\ 0.1493379251\ 10^{-5} \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}(6t^2-6t+1) \\ \sqrt{7}(20t^3-30t^2+12t-1) \\ 3(70t^4-140t^3+90t^2-20t+1) \\ \sqrt{11}(252t^5-630t^4+560t^3-210t^2+30t-1) \\ \sqrt{13}(924t^6-2772t^5+3150t^4-1680t^3+420t^2-42t+1) \\ \sqrt{15}(3432t^7-12012t^6+16632t^5-11550t^4+4200t^3-756t^2+56t-1) \end{bmatrix}$$

The exact solution and our approximate solution that has been obtained by using proposed method with $M=7,\ k=0$ are displayed in Table. 3.4.

Table 3.4 Comparison between the exact solution and the approximate solution for Example 3.4

t	Exact Solution $u(t)$	$M=7,\ k=0$ Approximate Solution	Absolute Error
0.0	0.0000000000	$0.306542 10^{-9}$	$0.306542 10^{-9}$
0.1	0.0009048374	0.0031672437	0.0022624063
0.2	0.0065498460	0.0089879497	0.0024381036
0.3	0.0200020919	0.0221105281	0.0021084361
0.4	0.0429004829	0.0449373620	0.0020368791
0.5	0.0758163324	0.0782004350	0.0023841026
0.6	0.1185433134	0.1214626682	0.0029193547
0.7	0.1703287592	0.1735549706	0.0032262115
0.8	0.2300564296	0.2329590100	0.0029025806
0.9	0.2963892819	0.2981457037	0.0017564224
1.0	0.3678794412	0.3678794407	$0.255522 10^{-9}$

Example 3.5 Consider the following fifth order non-linear BVP [6]

$$\frac{d^5u}{dt^5} + 24e^{-5u} = \frac{48}{(1+t)^5}, \quad 0 < t < 1 \tag{3.28}$$

with these boundary conditions

$$u(0) = 0, \;\; \mathrm{u}(1) = \ln 2, \;\; \frac{du}{dt}(0) = 1, \;\; \frac{du}{dt}(1) = 0.5, \;\; \frac{d^2u}{dt^2}(0) = -1 \tag{3.29}$$

The exact solution of the above system is $u(t) = \ln(1+t)$

To solve the above problem, we applied the LWOMM introduced in Section 3.2 with $M = 8, \;\; k = 0$. Approximating solution following as

$$u(t) \cong C^T\psi(t), \quad \frac{du(t)}{dt} \cong C^T D\psi(t)$$

$$\frac{d^2u(t)}{dt^2} \cong C^T D^2\psi(t), \quad \frac{d^5u(t)}{dt^5} \cong C^T D^5\psi(t)$$

Also approximating $g(t) = \dfrac{48}{(1+t)^5}$ following as $g(t) \cong G^T\psi(t)$

where $G(t) = \int_0^1 \left(\dfrac{48}{(1+t)^5}\right)\psi(t)\,dt$

We get

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{63} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{99} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sqrt{143} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{195} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{255} & 0 \end{pmatrix},$$

$$D^2=\begin{pmatrix}0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\4\sqrt{15}\sqrt{3}&0&0&0&0&0&0&0&0\\0&4\sqrt{35}\sqrt{15}&0&0&0&0&0&0&0\\0&0&4\sqrt{63}\sqrt{35}&0&0&0&0&0&0\\0&0&0&4\sqrt{99}\sqrt{63}&0&0&0&0&0\\0&0&0&0&4\sqrt{143}\sqrt{99}&0&0&0&0\\0&0&0&0&0&4\sqrt{195}\sqrt{143}&0&0&0\\0&0&0&0&0&0&4\sqrt{255}\sqrt{195}&0&0\end{pmatrix},$$

$$D^5=\begin{pmatrix}0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\32\sqrt{99}\sqrt{63}\sqrt{35}\sqrt{15}\sqrt{3}&0&0&0&0&0&0&0&0\\0&32\sqrt{143}\sqrt{99}\sqrt{63}\sqrt{35}\sqrt{15}&0&0&0&0&0&0&0\\0&0&32\sqrt{195}\sqrt{143}\sqrt{99}\sqrt{63}\sqrt{35}&0&0&0&0&0&0\\0&0&0&32\sqrt{255}\sqrt{195}\sqrt{143}\sqrt{99}\sqrt{63}&0&0&0&0&0\end{pmatrix}$$

If we consider (3.28) with (3.29), we have

$$R(t)=C^T D^5\psi(t)+24e^{-5\left(C^T\psi(t)\right)}-G^T\psi\left(t\right)=0 \tag{3.30}$$

Calculating Equation (3.30) at the first four roots of $P_9\left(t\right)$, i.e.

$$t_0=\frac{1}{2}$$

$$t_1=\frac{1}{2}+\frac{1}{2}\sqrt{1+4RootOf\,(24310\_\mathrm{Z}^4+11440\_\mathrm{Z}^3+1716\_\mathrm{Z}^2+88\_\mathrm{Z}+1,\mathrm{index}=1)}$$

$$t_2=\frac{1}{2}-\frac{1}{2}\sqrt{1+4RootOf\,(24310\_\mathrm{Z}^4+11440\_\mathrm{Z}^3+1716\_\mathrm{Z}^2+88\_\mathrm{Z}+1,\mathrm{index}=1)}$$

$$t_3=\frac{1}{2}+\frac{1}{2}\sqrt{1+4RootOf\,(24310\_\mathrm{Z}^4+11440\_\mathrm{Z}^3+1716\_\mathrm{Z}^2+88\_\mathrm{Z}+1,\mathrm{index}=2)}$$

We obtain four non-linear equations and by utilising boundary conditions we have a system of non-linear algebraic equations

If we solve this system of non-linear algebraic equations, we get

$$C^T = \left[c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4}, c_{0,5}, c_{0,6}, c_{0,7}, c_{0,8}\right] = \begin{bmatrix} 0.3877430836 \\ 0.1961116976 \\ -0.01825787356 \\ 0.002530037833 \\ -0.0001049565766 \\ 0.6036072799\ 10^{-4} \\ -0.7854936709\ 10^{-5} \\ 0.1064101460\ 10^{-5} \\ -0.1105550721\ 10^{-6} \end{bmatrix}^T$$

Consequently,

$$u(t) = \begin{bmatrix} 0.3877430836 \\ 0.1961116976 \\ -0.01825787356 \\ 0.002530037833 \\ -0.0001049565766 \\ 0.6036072799\ 10^{-4} \\ -0.7854936709\ 10^{-5} \\ 0.1064101460\ 10^{-5} \\ -0.1105550721\ 10^{-6} \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}\left(-1+2t\right) \\ \sqrt{5}\left(6t^2-6t+1\right) \\ \sqrt{7}\left(20t^3-30t^2+12t-1\right) \\ 3\left(70t^4-140t^3+90t^2-20t+1\right) \\ \sqrt{11}\left(252t^5-630t^4+560t^3-210t^2+30t-1\right) \\ \sqrt{13}\left(924t^6-2772t^5+3150t^4-1680t^3+420t^2-42t+1\right) \\ \sqrt{15}\left(3432t^7-12012t^6+16632t^5-11550t^4+4200t^3-756t^2+56t-1\right) \\ \sqrt{17}\left(12870t^8-51480t^7+84084t^6-72072t^5+34650t^4-9240t^3+1260t^2-72t+1\right) \end{bmatrix}$$

The approximate solution with the exact solution are given in Table 3.5.

Table 3.5 Comparison between the exact solution and the approximate solution for Example 3.5

t	Exact Solution $u(t)$	$M=8,\ k=0$ Approximate Solution	Absolute Error
0.0	0.0000000000	-0.851428 10^{-10}	0.851428 10^{-10}
0.1	0.0953101798	0.09685278434	0.00154260454
0.2	0.1823215568	0.1848673057	0.0025457489
0.3	0.2623642645	0.2654062222	0.0030419577
0.4	0.3364722366	0.3395145459	0.0030423093
0.5	0.4054651081	0.4080466567	0.0025815486
0.6	0.4700036292	0.4717568670	0.0017532378
0.7	0.5306282511	0.5313642136	0.0007359625
0.8	0.5877866649	0.5875997863	0.0001868786
0.9	0.6418538862	0.6412425416	0.0006113446
1.0	0.6931471806	0.6931471809	0.3 10^{-9}

Example 3.6 Consider the following sixth order non-linear BVP [7]

$$\frac{d^6u}{dt^6}(t)=e^t u^2(t), \quad 0<t<1 \tag{3.31}$$

subject to these boundary conditions

$$\begin{aligned} &u(0)=1, \quad \frac{du}{dt}(0)=-1, \quad \frac{d^2u}{dt^2}(0)=1, \\ &u(1)=1/e, \quad \frac{du}{dt}(1)=-1/e, \quad \frac{d^2u}{dt^2}(1)=1/\mathrm{e} \end{aligned} \tag{3.32}$$

The exact solution of the above system is $u(t)=e^{-t}$

To solve the above problem, we applied the method presented in Section 3.2 with $M=7,\ k=0$. Approximating solution following as

$$u(t)\cong C^T\psi(t), \quad \frac{du(t)}{dt}\cong C^T D\psi(t), \quad \frac{d^2u(t)}{dt^2}\cong C^T D^2\psi(t), \quad \frac{d^6u(t)}{dt^6}\cong C^T D^6\psi(t)$$

Also approximating $g(t)=e^t$ following as $g(t)\cong G^T\psi\left(t\right)$

where $G(t)=\int_0^1 e^t\psi\left(t\right)dt$

We get

$$D=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{63} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sqrt{143} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{195} & 0 \end{pmatrix},$$

$$D^2=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\sqrt{63}\sqrt{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\sqrt{99}\sqrt{63} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\sqrt{143}\sqrt{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\sqrt{195}\sqrt{143} & 0 & 0 \end{pmatrix}$$

$$D^6=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 64\sqrt{143}\sqrt{99}\sqrt{63}\sqrt{35}\sqrt{15}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 64\sqrt{195}\sqrt{143}\sqrt{99}\sqrt{63}\sqrt{35}\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If we consider (3.31) with (3.32), we have

$$R(t) = C^T D^6 \psi(t) - \left(G^T \psi(t)\right)\left(C^T \psi(t)\right)^2 \tag{3.33}$$

Calculating Equation (3.30) at the first two roots of $P_8(t)$, i.e.

$$t_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4RootOf(12870\_Z^4 + 6864\_Z^3 + 1188\_Z^2 + 72\_Z + 1, \text{index} = 1)}$$

$$t_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4RootOf(12870\_Z^4 + 6864\_Z^3 + 1188\_Z^2 + 72\_Z + 1, \text{index} = 1)}$$

We obtain two non-linear equations and by utilising boundary conditions we have

$$u(0) \cong C^T \psi(0) = 1, \quad u(1) \cong C^T \psi(1) = \frac{1}{e}$$
$$\frac{du}{dt}(0) \cong C^T D\psi(0) = -1, \quad \frac{du}{dt}(1) \cong C^T D\psi(1) = -\frac{1}{e}$$
$$\frac{d^2u}{dt^2}(0) \cong C^T D^2\psi(0) = 1, \quad \frac{d^2u}{dt^2}(1) \cong C^T D^2\psi(1) = \frac{1}{e}$$

If we solve this system of non-linear algebraic equations, we get

$$C^T = \left[c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4}, c_{0,5}, c_{0,6}, c_{0,7}\right] = \begin{bmatrix} 0.6318783499 \\ -0.1784202709 \\ 0.2307661619 \\ -0.002863931751 \\ 0.0001531565124 \\ 0.0001658259796 \\ 0.2825296276\ 10^{-6} \\ -0.9394504404\ 10^{-8} \end{bmatrix}^T$$

Consequently,

$$u(t) = C^T \psi(t) = \begin{bmatrix} 0.6318783499 \\ -0.1784202709 \\ 0.2307661619 \\ -0.002863931751 \\ 0.0001531565124 \\ 0.0001658259796 \\ 0.2825296276\ 10^{-6} \\ -0.9394504404\ 10^{-8} \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}(6t^2 - 6t + 1) \\ \sqrt{7}(20t^3 - 30t^2 + 12t - 1) \\ 3(70t^4 - 140t^3 + 90t^2 - 20t + 1) \\ \sqrt{11}(252t^5 - 630t^4 + 560t^3 - 210t^2 + 30t - 1) \\ \sqrt{13}(924t^6 - 2772t^5 + 3150t^4 - 1680t^3 + 420t^2 - 42t + 1) \\ \sqrt{15}(3432t^7 - 12012t^6 + 16632t^5 - 11550t^4 + 4200t^3 - 756t^2 + 56t - 1) \end{bmatrix}$$

The approximate solution with the exact solution are displayed in Table. 3.6.

Table 3.6 Comparison between the exact solution and the approximate solution for Example 3.6

t	Exact Solution $u(t)$	$M=7,\ k=0$ Approximate Solution	Absolute Error
0.0	1.0000000000	1.0000000000	0.0000000000
0.1	0.9048374180	0.9035595802	0.0012778377
0.2	0.8187307531	0.8165310322	0.0021997209
0.3	0.7408182207	0.7385408499	0.0022773707
0.4	0.6703200460	0.6687962996	0.0015237462
0.5	0.6065306597	0.6062498912	0.0002807683
0.6	0.5488116361	0.5497646530	0.0009530170
0.7	0.4965853038	0.4982801468	0.0016948430
0.8	0.4493289641	0.4509791614	0.0016501970
0.9	0.4065696597	0.4074550200	0.0008853600
1.0	0.3678794412	0.3678794407	0.227001010^{-10}

CHAPTER 4

THE APPLICATION OF THE OPERATIONAL MATRIX OF DERIVATIVE TO SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

There are a lot of tecniques that have been studied by many researchers to solve systems of ordinary differential equations numerically. Some related applications of such systems can be found in [51-54]. In [51], Patil and Khambayat presented the differential transform method for solving systems of linear differential equations and in [52], Adio illustrated the same method for solving the system of second order differential equations. Also, in [53], the differential transform method and Laplace transform method were demonstrated for the solution of the such systems. In [54], the numerical solution of the system of differential equations was presented by using the Adomian decomposition method.

In this chapter, the Legendre wavelet operational matrix of derivative is generalized in order to solve systems of linear and non-linear differential equations.

4.1 Solving Systems of Linear Differential Equations

In this section, the LWOMM is implemented to solve systems of linear differential equations. Consider the following system

$$
\begin{aligned}
\frac{d^n u_1}{dt^n} &= h_{11}(t)u_1 + \ldots + h_{1n}(t)u_m + k_1(t) \\
\frac{d^n u_2}{dt^n} &= h_{21}(t)u_1 + \ldots + h_{2n}(t)u_m + k_2(t) \\
\vdots \quad & \quad \vdots \quad \vdots \qquad \vdots \\
\frac{d^n u_m}{dt^n} &= h_{m1}(t)u_1 + \ldots + h_{mn}(t)u_m + k_m(t)
\end{aligned}
\tag{4.1}
$$

with these initial conditions

$$
\begin{aligned}
&u_1(t_0)=u_{10},\ \frac{du_1}{dt}(t_0)=u_{11},\ \frac{d^2u_1}{dt^2}(t_0)=u_{12},\dots,\frac{d^{n-1}u_1}{dt^{n-1}}(t_0)=u_{1(n-1)}\\
&u_2(t_0)=u_{20},\ \frac{du_2}{dt}(t_0)=u_{21},\ \frac{d^2u_2}{dt^2}(t_0)=u_{22},\dots,\frac{d^{n-1}u_2}{dt^{n-1}}(t_0)=u_{2(n-1)}\\
&\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
&u_m(t_0)=u_{m0},\ \frac{du_m}{dt}(t_0)=u_{m1},\ \frac{d^2u_m}{dt^2}(t_0)=u_{m2},\dots,\frac{d^{n-1}u_m}{dt^{n-1}}(t_0)=u_{\mathrm{m}(n-1)}
\end{aligned}
\tag{4.2}
$$

or with these boundary conditions

$$
\begin{aligned}
&u_1(t_0)=u_{10},\ \frac{du_1}{dt}(t_0)=u_{11},\ \frac{d^2u_1}{dt^2}(t_0)=u_{12},\dots,\frac{d^iu_1}{dt^i}(t_0)=u_{1i}\\
&u_2(t_0)=u_{20},\ \frac{du_2}{dt}(t_0)=u_{21},\ \frac{d^2u_2}{dt^2}(t_0)=u_{22},\dots,\frac{d^iu_2}{dt^i}(t_0)=u_{2i}\\
&\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
&u_j(t_0)=u_{j0},\ \frac{du_j}{dt}(t_0)=u_{j1},\ \frac{d^2u_j}{dt^2}(t_0)=u_{j2},\dots,\frac{d^iu_j}{dt^i}(t_0)=u_{ji}\\
&u_1(t_1)=u_{10}',\ \frac{du_1}{dt}(t_1)=u_{11}',\ \frac{d^2u_1}{dt^2}(t_1)=u_{12}',\dots,\frac{d^iu_1}{dt^i}(t_1)=u_{1i}'\\
&u_2(t_1)=u_{20}',\ \frac{du_2}{dt}(t_1)=u_{21}',\ \frac{d^2u_2}{dt^2}(t_1)=u_{22}',\dots,\frac{d^iu_2}{dt^i}(t_1)=u_{2i}'\\
&\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
&u_j(t_1)=u_{j0}',\ \frac{du_j}{dt}(t_1)=u_{j1}',\ \frac{d^2u_j}{dt^2}(t_1)=u_{j2}',\dots,\frac{d^iu_j}{dt^i}(t_1)=u_{ji}'\\
&i=0,1,\dots,n/2\ \ if\ \ n\ even,\ \ \mathrm{j}=0,1,\dots,j/2\ \ if\ \ m\ even\\
&u_1(t_0)=u_{10},\ \frac{du_1}{dt}(t_0)=u_{11},\ \frac{d^2u_1}{dt^2}(t_0)=u_{12},\dots,\frac{d^iu_1}{dt^i}(t_0)=u_{1i}\\
&u_2(t_0)=u_{20},\ \frac{du_2}{dt}(t_0)=u_{21},\ \frac{d^2u_2}{dt^2}(t_0)=u_{22},\dots,\frac{d^iu_2}{dt^i}(t_0)=u_{2i}\\
&\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
&u_j(t_0)=u_{j0},\ \frac{du_j}{dt}(t_0)=u_{j1},\ \frac{d^2u_j}{dt^2}(t_0)=u_{j2},\dots,\frac{d^iu_j}{dt^i}(t_0)=u_{ji}\\
&u_1(t_1)=u_{10}',\ \frac{du_1}{dt}(t_1)=u_{11}',\ \frac{d^2u_1}{dt^2}(t_1)=u_{12}',\dots,\frac{d^{i-1}u_1}{dt^{i-1}}(t_1)=u_{1(i-1)}'\\
&u_2(t_1)=u_{20}',\ \frac{du_2}{dt}(t_1)=u_{21}',\ \frac{d^2u_2}{dt^2}(t_1)=u_{22}',\dots,\frac{d^{i-1}u_2}{dt^{i-1}}(t_1)=u_{2(i-1)}'\\
&\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
&u_{j-1}(t_1)=u_{(j-1)0}',\ \frac{du_{j-1}}{dt}(t_1)=u_{(j-1)1}',\ \frac{d^2u_{j-1}}{dt^2}(t_1)=u_{(j-1)2}',\dots,\frac{d^{i-1}u_{j-1}}{dt^{i-1}}(t_1)=u_{(j-1)(i-1)}'\\
&i=0,1,\dots,(n+1)/2\ \ if\ \ n\ odd,\ \ \mathrm{j}=0,1,\dots,(m+1)/2\ \ if\ \ m\ \mathrm{odd}
\end{aligned}
\tag{4.3}
$$

First we presume that unknown functions $u_1(t), u_2(t), \ldots, u_m(t)$ are approximated and given by

$$u_1(t) \cong C_1^T \psi(t),\ u_2(t) \cong C_2^T \psi(t), \ldots, u_m(t) \cong C_m^T \psi(t) \tag{4.4}$$

where $C_1, C_2, \ldots, C_m$ are unknown vectors and $h_{11}(t),\ldots,h_{1n}(t),h_{21}(t),\ldots,h_{2n}(t),\ldots,h_{m1}(t),\ldots,h_{mn}(t)$ can be any function of the independent variable t and dependent variable h_{ij} $(i=1,2,\ldots,m\ \ j=1,2,\ldots,n)$ and $\psi(t)$ is the vector which given in Equation (2.27). By utilising Equation (2.31) we obtain

$$\begin{gathered}
\frac{du_1}{dt}(t) \cong C_1^T D\psi(t),\ \ \frac{du_2}{dt}(t) \cong C_2^T D\psi(t), \ldots, \frac{du_m}{dt}(t) \cong C_m^T D\psi(t) \\
\frac{d^2u_1}{dt^2}(t) \cong C_1^T D^2\psi(t),\ \ \frac{d^2u_2}{dt^2}(t) \cong C_2^T D^2\psi(t), \ldots, \frac{d^2u_m}{dt^2}(t) \cong C_m^T D^2\psi(t) \\
\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
\frac{d^nu_1}{dt^n}(t) \cong C_1^T D^n\psi(t),\ \ \frac{d^nu_2}{dt^n}(t) \cong C_2^T D^n\psi(t), \ldots, \frac{d^nu_m}{dt^n}(t) \cong C_m^T D^n\psi(t)
\end{gathered} \tag{4.5}$$

Approximating $h_{11}(t),\ldots,h_{1n}(t),h_{21}(t),\ldots,h_{2n}(t),\ldots,h_{m1}(t),\ldots,h_{mn}(t)$, we get

$$\begin{gathered}
h_{11}(t) \cong H_{11}^T \psi(t),\ \ h_{12}(t) \cong H_{12}^T \psi(t), \ldots, h_{1n}(t) \cong H_{1n}^T \psi(t) \\
h_{21}(t) \cong H_{21}^T \psi(t),\ \ h_{22}(t) \cong H_{22}^T \psi(t), \ldots, h_{2n}(t) \cong H_{2n}^T \psi(t) \\
\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
h_{m1}(t) \cong H_{m1}^T \psi(t),\ \ h_{m2}(t) \cong H_{m2}^T \psi(t), \ldots, h_{mn}(t) \cong H_{mn}^T \psi(t)
\end{gathered} \tag{4.6}$$

We can also approximate $k_1(t), k_2(t), \ldots, k_m(t)$ as

$$k_1(t) \cong K_1^T \psi(t),\ \ k_2(t) \cong K_2^T \psi(t), \ldots, k_m(t) \cong K_m^T \psi(t) \tag{4.7}$$

where vectors $K_1, K_2, \ldots, K_m$ are given by (2.26). Substituting Equations (4.5)-(4.6) and (4.7) in Equation (4.1) we obtain

$$\begin{gathered}
R_1(t) = \left(C_1^T D^n \psi(t)\right) - \left(H_{11}^T \psi(t)\right)\left(C_1^T \psi(t)\right) - \ldots - \left(H_{1n}^T \psi(t)\right)\left(C_m^T \psi(t)\right) - K_1^T \psi(t) = 0 \\
R_2(t) = \left(C_2^T D^n \psi(t)\right) - \left(H_{21}^T \psi(t)\right)\left(C_1^T \psi(t)\right) - \ldots - \left(H_{2n}^T \psi(t)\right)\left(C_m^T \psi(t)\right) - K_2^T \psi(t) = 0 \\
\vdots \\
R_m(t) = \left(C_m^T D^n \psi(t)\right) - \left(H_{m1}^T \psi(t)\right)\left(C_1^T \psi(t)\right) - \ldots - \left(H_{mn}^T \psi(t)\right)\left(C_m^T \psi(t)\right) - K_m^T \psi(t) = 0
\end{gathered} \tag{4.8}$$

We obtain $2^k(M+1)-mn$ linear equations by computing

$$\int_0^1 \psi_j(t)R_i(t)\,dt=0,\;\; j=1,\ldots,2^k(M+1),\;\; i=1,2,\ldots,m \tag{4.9}$$

Also by substituting initial conditions (4.2) in Equations (4.4)-(4.5) we obtain

$$\begin{array}{llll}
u_1(t_0)=C_1^T\psi(t_0)=u_{10}, & \dfrac{du_1}{dt}(t_0)=C_1^TD\psi(t_0)=u_{11},\ldots, & \dfrac{d^{n-1}u_1}{dt^{n-1}}(t_0)=C_1^TD^{n-1}\psi(t_0)=u_{1(n-1)} \\
u_2(t_0)=C_2^T\psi(t_0)=u_{20}, & \dfrac{du_2}{dt}(t_0)=C_2^TD\psi(t_0)=u_{21},\ldots, & \dfrac{d^{n-1}u_2}{dt^{n-1}}(t_0)=C_2^TD^{n-1}\psi(t_0)=u_{2(n-1)} \\
\vdots & \vdots & \vdots \\
u_m(t_0)=C_m^T\psi(t_0)=u_{m0}, & \dfrac{du_m}{dt}(t_0)=C_m^TD\psi(t_0)=u_{m1},\ldots, & \dfrac{d^{n-1}u_m}{dt^{n-1}}(t_0)=C_m^TD^{n-1}\psi(t_0)=u_{\mathrm{m}(n-1)}
\end{array} \tag{4.10}$$

We obtain $2^k(M+1)$ set of linear equations by using Equations (4.9)-(4.10). These linear equations can be solved for unknown coefficients of the vector C. Accordingly $u(t)$ which is given in Equation (4.1) can be computed.

4.2 Solving Systems of Non-Linear Differential Equations

In this section, the LWOMM is implemented for solving systems of non-linear differential equations. Consider the following system with the initial conditions presented in (4.2) or boundary conditions presented in (4.3).

$$\begin{array}{l}
\dfrac{d^nu_1}{dt^n}(t)=H_1\left(t,u_1(t),\ldots,\dfrac{d^{n-1}u_1}{dt^{n-1}}(t),u_2(t),\ldots,\dfrac{d^{n-1}u_2}{dt^{n-1}}(t),\ldots,u_m(t),\ldots,\dfrac{d^{n-1}u_m}{dt^{n-1}}(t)\right) \\
\dfrac{d^nu_2}{dt^n}(t)=H_2\left(t,u_1(t),\ldots,\dfrac{d^{n-1}u_1}{dt^{n-1}}(t),u_2(t),\ldots,\dfrac{d^{n-1}u_2}{dt^{n-1}}(t),\ldots,u_m(t),\ldots,\dfrac{d^{n-1}u_m}{dt^{n-1}}(t)\right) \\
\qquad\vdots \\
\dfrac{d^nu_m}{dt^n}(t)=H_m\left(t,u_1(t),\ldots,\dfrac{d^{n-1}u_1}{dt^{n-1}}(t),u_2(t),\ldots,\dfrac{d^{n-1}u_2}{dt^{n-1}}(t),\ldots,u_m(t),\ldots,\dfrac{d^{n-1}u_m}{dt^{n-1}}(t)\right)
\end{array} \tag{4.11}$$

First we presume that unknown functions $u_1(t),u_2(t),\ldots,u_m(t)$ are approximated and given by

$$u_1(t)\cong C_1^T\psi(t),\;\; u_2(t)\cong C_2^T\psi(t),\ldots,u_m(t)\cong C_m^T\psi(t) \tag{4.12}$$

where $C_1, C_2, ..., C_m$ are unknown vectors and $\psi(t)$ is the vector which is given in Equation (2.27). If we use Equation (2.31), we obtain

$$\begin{aligned}
&\frac{du_1}{dt}(t) \cong C_1^T D\psi(t),\ \frac{du_2}{dt}(t) \cong C_2^T D\psi(t),..., \frac{du_m}{dt}(t) \cong C_m^T D\psi(t) \\
&\frac{d^2u_1}{dt^2}(t) \cong C_1^T D^2\psi(t),\ \frac{d^2u_2}{dt^2}(t) \cong C_2^T D^2\psi(t),..., \frac{d^2u_m}{dt^2}(t) \cong C_m^T D^2\psi(t) \\
&\qquad\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&\frac{d^nu_1}{dt^n}(t) \cong C_1^T D^n\psi(t),\ \frac{d^nu_2}{dt^n}(t) \cong C_2^T D^n\psi(t),..., \frac{d^nu_m}{dt^n}(t) \cong C_m^T D^n\psi(t)
\end{aligned} \tag{4.13}$$

By utilising Equation (4.11) we obtain

$$\begin{aligned}
&C_1^T D^n\psi(t) \cong H_1\left(t, C_1^T\psi(t),...,C_1D^{n-1}\psi(t), C_2^T\psi(t),...,C_2D^{n-1}\psi(t),....,C_m^T\psi(t),...,C_mD^{n-1}\psi(t)\right) \\
&C_2^T D^n\psi(t) \cong H_2\left(t, C_1^T\psi(t),...,C_1D^{n-1}\psi(t), C_2^T\psi(t),...,C_2D^{n-1}\psi(t),....,C_m^T\psi(t),...,C_mD^{n-1}\psi(t)\right) \\
&\qquad\vdots \\
&C_m^T D^n\psi(t) \cong H_m\left(t, C_1^T\psi(t),...,C_1D^{n-1}\psi(t), C_2^T\psi(t),...,C_2D^{n-1}\psi(t),....,C_m^T\psi(t),...,C_mD^{n-1}\psi(t)\right)
\end{aligned}$$

(4.14)

Also by substituting initial conditions (4.2) in Equations (4.4)-(4.5) we obtain

$$\begin{aligned}
&u_1(t_0) \cong C_1^T\psi(t_0) = u_{10},\ \frac{du_1}{dt}(t_0) \cong C_1^T D\psi(t_0) = u_{11},..., \frac{d^{n-1}u_1}{dt^{n-1}}(t_0) \cong C_1^T D^{n-1}\psi(t_0) = u_{1(n-1)} \\
&u_2(t_0) \cong C_2^T\psi(t_0) = u_{20},\ \frac{du_2}{dt}(t_0) \cong C_2^T D\psi(t_0) = u_{21},..., \frac{d^{n-1}u_2}{dt^{n-1}}(t_0) \cong C_2^T D^{n-1}\psi(t_0) = u_{2(n-1)} \\
&\quad\vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \qquad\qquad\quad \vdots \\
&u_m(t_0) \cong C_m^T\psi(t_0) = u_{m0},\ \frac{du_m}{dt}(t_0) \cong C_m^T D\psi(t_0) = u_{m1},..., \frac{d^{n-1}u_m}{dt^{n-1}}(t_0) \cong C_m^T D^{n-1}\psi(t_0) = u_{\mathrm{m}(n-1)}
\end{aligned}$$

(4.15)

To obtain the solution $u_1(t), u_2(t),...,u_m(t)$, we first compute Equation (4.14) at $2^k(M+1)-mn$ points. For a better result, we utilise the first $2^k(M+1)-mn$ roots of shifted Legendre polynomials $P_{2^k(M+1)}(t)$. If we use these equations collectively with Equation (4.15), then we obtain $2^k(M+1)$ non-linear equations. These non-linear equations can be solved for unknown coefficients of the vector C. Accordingly $u_1(t), u_2(t),...,u_m(t)$ given in Equation (4.11) can be computed.

4.3 Applications

In this section, we present some examples we have wanted to demonstrate the performance of the proposed tecnique solving systems of linear and non-linear differential equations. It is shown that the LWOMM yield better results.

Example 4.1 We first consider the following system of non-homogeneous differential equations of the linear form [51]

$$\begin{aligned} \frac{du(t)}{dt} &= w(t) - \cos t \\ \frac{dv(t)}{dt} &= w(t) - e^t \\ \frac{dw(t)}{dt} &= u(t) - v(t) \end{aligned} \tag{4.16}$$

subject to these initial conditions

$$u(0) = 1, \quad v(0) = 0, \quad w(0) = 2 \tag{4.17}$$

The exact solution of the above system is

$$u(t) = e^t, \quad v(t) = \sin t, \quad w(t) = e^t + \cos t$$

To solve the above problem, we applied the method presented in Section 4.1 with $M = 2,\ \ k = 0$. Approximating solution following as

$$u(t) \cong C^T \psi(t), \quad v(t) \cong S^T \psi(t), \quad w(t) \cong W^T \psi(t)$$
$$\frac{du(t)}{dt} \cong C^T D\psi(t), \quad \frac{dv(t)}{dt} \cong S^T D\psi(t), \quad \frac{dw(t)}{dt} \cong W^T D\psi(t)$$

Also approximating $h_0(t) = \cos t$ and $h_1(t) = e^t$ following as

$$h_0(t) \cong H_0{}^T \psi(t) \quad \text{and} \quad h_1(t) \cong H_1{}^T \psi(t)$$

where

$$H_0(t) = \int_0^1 \cos t\, \psi(t) dt \quad \text{and} \quad H_1(t) = \int_0^1 e^t \psi(t) dt$$

We get

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}$$

If we consider (4.15) with (4.16), we have

$$\begin{aligned} R_1(t) &= C^T D\psi(t) - W^T\psi(t) + H_0{}^T\psi(t) \\ R_2(t) &= S^T D\psi(t) - W^T\psi(t) + H_1{}^T\psi(t) \\ R_3(t) &= W^T D\psi(t) - C^T\psi(t) + S^T\psi(t) \end{aligned} \tag{4.18}$$

By computing

$$\int_0^1 \psi_i(t) R_j(t)\, dt = 0, \quad i = 1,2 \;\; j = 1,2,3$$

We obtain six linear equations following as

$$\begin{aligned} &3.464101615 c_{0,1} - w_{0,0} + 0.8414709848 = 0 \\ &3.464101615 s_{0,1} - w_{0,0} + 1.718281828 = 0 \\ &3.464101616 w_{0,1} - c_{0,0} + s_{0,0} = 0 \\ &-0.1349690260 - w_{0,1} + 7.745966692 c_{0,2} = 0 \\ &0.4879501870 - w_{0,1} + 7.745966692 s_{0,2} = 0 \\ &7.745966698 w_{0,2} - c_{0,1} + s_{0,1} = 0 \end{aligned}$$

and by utilising initial conditions we have

$$\begin{aligned} &c_{0,0} - \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} = 1 \\ &s_{0,0} - \sqrt{3}s_{0,1} + \sqrt{5}s_{0,2} = 0 \\ &w_{0,0} - \sqrt{3}w_{0,1} + \sqrt{5}w_{0,2} = 2 \end{aligned}$$

If we solve this system of linear algebraic equations, we get

$$C^T = \left[c_{0,0}, c_{0,1}, c_{0,2}\right] = [1.713532543, 0.4950067009, 0.06432908766]$$

$$S^T = \left[s_{0,0}, s_{0,1}, \mathrm{s}_{0,2}\right] = [0.4549484087, 0.2418932127, -0.01608943696]$$

$$W^T = \left[w_{0,0}, w_{0,1}, w_{0,2}\right] = [2.556224497, 0.3633219442, 0.03267681083]$$

Consequently,

$$u(t)=C^T\psi(t)=[1.713532543, 0.4950067009, 0.06432908766]\begin{bmatrix}1\\ \sqrt{3}(-1+2t)\\ \sqrt{5}(6t^2-6t+1)\end{bmatrix}$$

$$v(t)=S^T\psi(t)=[0.4549484087, 0.2418932127, -0.01608943696]\begin{bmatrix}1\\ \sqrt{3}(-1+2t)\\ \sqrt{5}(6t^2-6t+1)\end{bmatrix}$$

$$w(t)=W^T\psi(t)=[2.556224497, 0.3633219442, 0.03267681083]\begin{bmatrix}1\\ \sqrt{3}(-1+2t)\\ \sqrt{5}(6t^2-6t+1)\end{bmatrix}$$

The approximate solution and the exact solution are displayed in Table. 4.1, Table 4.2 and Table 4.3.

Table 4.1 Comparison between the exact solution and the approximate solution $u(t)$ for Example 4.1

t	Exact Solution $u(t)$	$M=2,\ k=0$ Approximate Solution	Absolute Error
0.0	1.0000000000	0.9999999996	0.4 10^{-9}
0.1	1.105170918	1.093799476	0.011371442
0.2	1.221402758	1.204860258	0.016542500
0.3	1.349858808	1.333182345	0.016676463
0.4	1.491824698	1.478765738	0.013058960
0.5	1.648721271	1.641610436	0.007110835
0.6	1.822118800	1.821716441	0.000402359
0.7	2.013752707	2.019083751	0.005331044
0.8	2.225540928	2.233712366	0.008171438
0.9	2.459603111	2.465602287	0.005999176
1.0	2.718281828	2.714753512	0.003528316

Table 4.2 Comparison between the exact solution and the approximate solution $v(t)$ for Example 4.1

t	Exact Solution $v(t)$	$M=2,\ k=0$ Approximate Solution	Absolute Error
0.0	0.00000000000	-0.55 10^{-9}	0.55 10^{-9}
0.1	0.09983341665	0.1032218867	0.00338847005
0.2	0.1986693308	0.2021265251	0.0034571943
0.3	0.2955202067	0.2967139144	0.0011937077
0.4	0.3894183423	0.3869840548	0.0024342875
0.5	0.4794255386	0.4729369463	0.0064885923
0.6	0.5646424734	0.5545725887	0.0100698847
0.7	0.6442176872	0.6318909822	0.0123267050
0.8	0.7173560909	0.7048921268	0.0124639641
0.9	0.7833269096	0.7735760223	0.0097508873
1.0	0.8414709848	0.8379426685	0.0035283163

Table 4.3 Comparison between the exact solution and the approximate solution $w(t)$ for Example 4.1

t	Exact Solution $w(t)$	$M=2,\ k=0$ Approximate Solution	Absolute Error
0.0	2.000000000	2.000000000	0.000000000
0.1	2.100175083	2.086401925	0.013773158
0.2	2.201469336	2.181571959	0.019897377
0.3	2.305195297	2.285510102	0.019685195
0.4	2.412885692	2.398216352	0.014669340
0.5	2.526303833	2.519690711	0.006613122
0.6	2.647454415	2.649933179	0.002478764
0.7	2.778594894	2.788943756	0.010348862
0.8	2.922247637	2.936722440	0.014474803
0.9	3.081213079	3.093269233	0.012056154
1.0	3.258584134	3.258584134	0.000000000

Example 4.2 Consider the following system of differential equations [53]

$$\begin{aligned}&\frac{du(t)}{dt}+\frac{dv(t)}{dt}+u(t)+v(t)=1\\&\frac{dv(t)}{dt}=2u(t)+v(t)\end{aligned}\tag{4.19}$$

subject to these initial conditions

$$u(0)=0,\ v(0)=1\tag{4.20}$$

The exact solution of the above system is known as

$$u(t)=e^{-t}-1,\ \ v(t)=2-e^{-t}$$

To solve the above system, we used the method presented in Section 4.2 with $M=4,\ \ k=0$. Approximating solution following as

$$u(t)=C^T\psi(t)\ \ \text{and}\ \ v(t)=S^T\psi(t)$$

$\frac{du(t)}{dt} = C^T D\psi(t)$ and $\frac{dv(t)}{dt} = S^T D\psi(t)$

We get

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{63} & 0 \end{pmatrix}$$

If we consider (4.18) with (4.19), we have

$$R_1(t) = C^T D\psi(t) + S^T D\psi(t) + C^T\psi(t) + S^T\psi(t) - 1 = 0 \tag{4.21}$$

$$R_2(t) = S^T D\psi(t) - 2C^T\psi(t) - S^T\psi(t) = 0 \tag{4.22}$$

Calculating Equations (4.21) and (4.22) at the first four roots of $P_5(t)$, i.e.

$$t_0 = \frac{1}{2},\ \ t_1 = \frac{1}{2} - \frac{\sqrt{245 - 14\sqrt{70}}}{42},\ \ t_2 = \frac{1}{2} - \frac{\sqrt{245 + 14\sqrt{70}}}{42},\ \ t_3 = \frac{1}{2} + \frac{\sqrt{245 - 14\sqrt{70}}}{42}$$

and by utilising initial conditions we have

$$c_{0,0} - \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} - \sqrt{7}c_{0,3} + 3c_{0,4} = 0$$
$$s_{0,0} - \sqrt{3}s_{0,1} + \sqrt{5}s_{0,2} - \sqrt{7}s_{0,3} + 3s_{0,4} = 1$$

If we solve this system of nonlinear algebraic equations, we get

$$C^T = [\text{-}0.3719399342, \text{-}0.1812731838, 0.02340226770, \text{-}0.00198837331, 0.000125255710]$$

$$S^T = [1.371939934, 0.1812731837, \text{-}0.02340226756, 0.001988373268, \text{-}0.0001252558]$$

Consequently,

$$u(t) = C^T\psi(t) = \begin{bmatrix} -0.3719399342 \\ -0.1812731838 \\ 0.02340226770 \\ -0.00198837331 \\ 0.00012525571 \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2 - 6t + 1\right) \\ \sqrt{7}\left(20t^3 - 30t^2 + 12t - 1\right) \\ 3\left(70t^4 - 140t^3 + 90t^2 - 20t + 1\right) \end{bmatrix}$$

$$v(t)=S^T\psi(t)=\begin{bmatrix}1.371939934\\0.1812731837\\-0.02340226756\\0.001988373268\\-0.0001252558\end{bmatrix}^T\begin{bmatrix}1\\\sqrt{3}(-1+2t)\\\sqrt{5}\left(6t^2-6t+1\right)\\\sqrt{7}\left(20t^3-30t^2+12t-1\right)\\3\left(70t^4-140t^3+90t^2-20t+1\right)\end{bmatrix}$$

The approximate solution with the exact solution are displayed in Table 4.4 and Table 4.5.

Table 4.4 Comparison between the exact solution and the approximate solution $u(t)$ for Example 4.2

t	Exact Solution $u(t)$	$M=2,\ k=0$ Approximate Solution	$M=4,\ k=0$ Approximate Solution	Absolute Error
0.0	0.0000000000	-0.14 10^{-9}	0.1120 10^{-9}	0.1120 10^{-9}
0.1	-0.0951625820	-0.0941420851	-0.09635576876	0.00119318676
0.2	-0.1812692469	-0.1807908742	-0.1835093329	0.00224008600
0.3	-0.2591817793	-0.2599463674	-0.2623129320	0.0031311527
0.4	-0.3296799540	-0.3316085648	-0.3335556777	0.0038757237
0.5	-0.3934693403	-0.3957774661	-0.3979635520	0.0044942117
0.6	-0.4511883639	-0.4524530716	-0.4561994083	0.0050110444
0.7	-0.5034146962	-0.5016353813	-0.5088629711	0.0054482749
0.8	-0.5506710359	-0.5433243950	-0.5564908362	0.0058198003
0.9	-0.5934303403	-0.5775201129	-0.5995564702	0.0061261299
1.0	-0.6321205588	-0.6042225348	-0.6384702111	0.0063496523

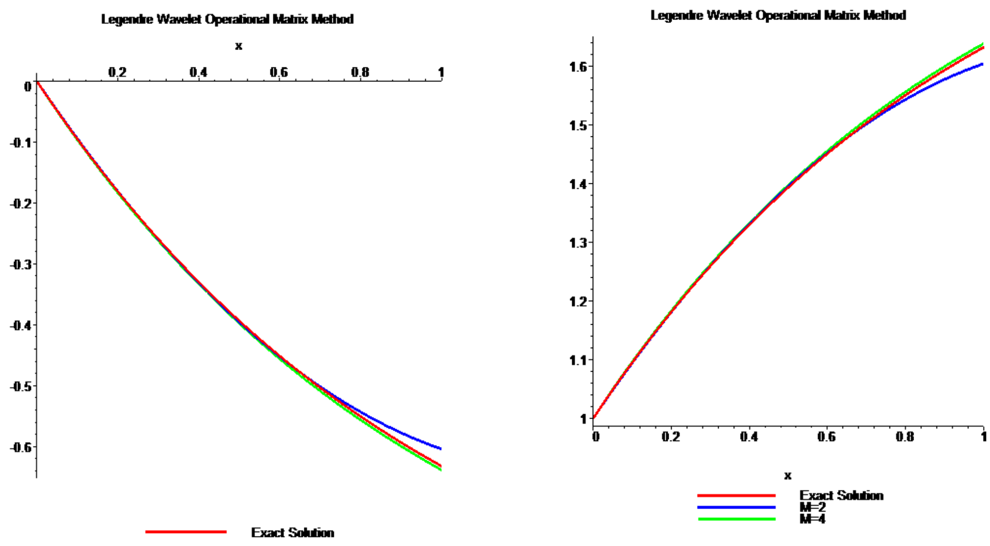

Figure 4.1 Comparison of our solutions $u(t), v(t)$ and the exact solution when $M=2$ *and* $M=4$ for Example 4.2

Table 4.5 Comparison between the exact solution and the approximate solution $v(t)$ for Example 4.2

t	Exact Solution $v(t)$	$M=2,\ k=0$ Approximate Solution	$M=4,\ k=0$ Approximate Solution	Absolute Error
0.0	1.000000000	1.000000001	1.000000001	$0.1\ 10^{-8}$
0.1	1.095162582	1.094142086	1.096355770	0.001193188
0.2	1.181269247	1.180790875	1.183509333	0.002240086
0.3	1.259181779	1.259946368	1.262312933	0.003131154
0.4	1.329679954	1.331608566	1.333555679	0.003875725
0.5	1.393469340	1.395777467	1.397963553	0.004494213
0.6	1.451188364	1.452453073	1.456199409	0.005011045
0.7	1.503414696	1.501635383	1.508862972	0.005448276
0.8	1.550671036	1.543324395	1.556490838	0.005819802
0.9	1.593430340	1.577520114	1.599556471	0.006126131
1.0	1.632120559	1.604222535	1.638470212	0.006349653

Example 4.3 Consider the following system of linear differential equations [53]

$$\begin{aligned} &\frac{d^2u(t)}{dt^2}+v(t)=1 \\ &\frac{d^2v(t)}{dt^2}+u(t)=0 \end{aligned} \tag{4.23}$$

subject to these initial conditions

$$u(0)=0, \quad \frac{du}{dt}(0)=0, \; v(0)=0, \quad \frac{dv}{dt}(0)=0, \tag{4.24}$$

The exact solution of the above system is

$$u(t)=\frac{1}{4}\left(e^{t}+e^{-t}-2\cos t\right), \; v(t)=\frac{1}{4}\left(4-e^{t}-e^{-t}-2\cos t\right)$$

To solve the above linear system, we applied the method presented in Section 4.1 with $M=4, \; k=0$. Approximating solution following as

$$u(t)=C^T\psi(t), \; v(t)=S^T\psi(t)$$

$$\frac{du(t)}{dt}=C^T D\psi(t), \quad \frac{dv(t)}{dt}=S^T D\psi(t)$$

$$\frac{d^2u(t)}{dt^2}=C^T D^2\psi(t), \quad \frac{d^2v(t)}{dt^2}=S^T D^2\psi(t)$$

We get

$$D=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{63} & 0 \end{pmatrix}, \; D^2=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4\sqrt{3}\sqrt{15} & 0 & 0 & 0 & 0 \\ 0 & 4\sqrt{15}\sqrt{35} & 0 & 0 & 0 \\ 0 & 0 & 4\sqrt{35}\sqrt{63} & 0 & 0 \end{pmatrix}$$

If we consider (4.23) with (4.24), we have

$$\begin{aligned} R_1(t)&=C^T D^2\psi(t)+S^T\psi(t)-1 \\ R_2(t)&=S^T D^2\psi(t)+C^T\psi(t) \end{aligned} \tag{4.25}$$

By computing

$$\int_0^1 \psi_i(t) R_j(t)\, dt = 0, \quad i = 1,2,3 \;\; j = 1,2$$

We obtain six linear equations following as

$$26.83281574 c_{0,2} + s_{0,0} - 1 = 0$$

$$26.83281574 s_{0,2} + c_{0,0} = 0$$

$$-0.1\,10^{-7} s_{0,3} + 91.65151397 c_{0,3} + 1.000000001 s_{0,1} = 0$$

$$-0.1\,10^{-7} c_{0,3} + 91.65151397 s_{0,3} + 1.000000001 c_{0,1} = 0$$

$$-0.1\,10^{-7} s_{0,4} + 0.1\,10^{-8} s_{0,3} + 187.8297100 c_{0,4} + 0.9999999992 s_{0,2} = 0$$

$$-0.1\,10^{-7} c_{0,4} + 0.1\,10^{-8} c_{0,3} + 187.8297100 s_{0,4} + 0.9999999992 c_{0,2} = 0$$

and by utilising initial conditions we have

$$\begin{aligned}
&c_{0,0} - \sqrt{3} c_{0,1} + \sqrt{5} c_{0,2} - \sqrt{7} c_{0,3} + 3 c_{0,4} = 0 \\
&s_{0,0} - \sqrt{3} s_{0,1} + \sqrt{5} s_{0,2} - \sqrt{7} s_{0,3} + 3 s_{0,4} = 0 \\
&2\sqrt{3} c_{0,1} - 6\sqrt{5} c_{0,2} + 10\sqrt{7} c_{0,3} - 42 c_{0,4} = 0 \\
&2\sqrt{3} s_{0,1} - 6\sqrt{5} s_{0,2} + 10\sqrt{7} s_{0,3} - 42 s_{0,4} = 0
\end{aligned}$$

If we solve this system of linear algebraic equations, we get

$$C^T = [0.1680546197, 0.1456667117, 0.03781985846, 0.0001588513170, 0.00003334417056]$$

$$S^T = [-0.01481329344, -0.01455896371, -0.006263025891, -0.001589354125, -0.000201351845]$$

Consequently,

$$u(t) = C^T \psi(t) = \begin{bmatrix} 0.1680546197 \\ 0.1456667117 \\ 0.03781985846 \\ 0.000158851317 \\ 0.000033344170 \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}(6t^2 - 6t + 1) \\ \sqrt{7}(20t^3 - 30t^2 + 12t - 1) \\ 3(70t^4 - 140t^3 + 90t^2 - 20t + 1) \end{bmatrix}$$

$$
v(t) = S^T \psi(t) = \begin{bmatrix} -0.01481329344 \\ -0.01455896371 \\ -0.006263025891 \\ -0.001589354125 \\ -0.000201351845 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2 - 6t + 1\right) \\ \sqrt{7}\left(20t^3 - 30t^2 + 12t - 1\right) \\ 3\left(70t^4 - 140t^3 + 90t^2 - 20t + 1\right) \end{bmatrix}
$$

The approximate solution with the exact solution are displayed in Table. 4.6 and Table 4.7.

Table 4.6 Comparison between the exact solution and the approximate solution $u(t)$ for Example 4.3

t	Exact Solution $u(t)$	$M = 2,\ k = 0$ Approximate Solution	$M = 4,\ k = 0$ Approximate Solution	Absolute Error
0.0	0.0000000000	0.00000000000	-0.1685 10^{-9}	0.1685 10^{-9}
0.1	0.0050000014	0.00514285717	0.00505714932	0.000057147920
0.2	0.0200000889	0.02057142862	0.02016653121	0.00016644231
0.3	0.0450010126	0.04628571436	0.04531976012	0.00031874752
0.4	0.0800056890	0.08228571438	0.08052525613	0.00051956713
0.5	0.1250217017	0.1285714286	0.1258082447	0.0007865430
0.6	0.1800648016	0.1851428573	0.1812107570	0.0011459554
0.7	0.2451634092	0.2520000001	0.2467916294	0.0016282202
0.8	0.3203641184	0.3291428572	0.3226265038	0.0022623854
0.9	0.4057382085	0.4165714287	0.4088078275	0.0030696190
1.0	0.5013891643	0.5142857144	0.5054448535	0.0040556892

Table 4.7 Comparison between the exact solution and the approximate solution $v(t)$ for Example 4.3

t	Exact Solution $v(t)$	$M=2,\ k=0$ Approximate Solution	$M=4,\ k=0$ Approximate Solution	Absolute Error
0.0	0.0000000000	$0.1\ 10^{-10}$	$0.150\ 10^{-10}$	$0.150\ 10^{-10}$
0.1	$-0.41666\ 10^{-5}$	-0.000857142850	-0.000604747500	0.0006005809000
0.2	-0.0000666667	-0.003428571425	-0.001510714801	0.001444048101
0.3	-0.0003375018	-0.007714285713	-0.002867321525	0.002529819725
0.4	-0.0010666830	-0.01371428572	-0.004925468639	0.003858785639
0.5	-0.0026042637	-0.02142857143	-0.008037538438	0.005433274738
0.6	-0.0054004164	-0.03085714285	-0.01265739455	0.00725697815
0.7	-0.0100055964	-0.04200000000	-0.01934038193	0.00933478553
0.8	-0.0170708276	-0.05485714286	-0.02874332686	0.01167249926
0.9	-0.0273481769	-0.06942857143	-0.04162453697	0.01427636007
1.0	-0.0416914703	-0.08571428571	-0.05884380119	0.01715233089

Example 4.4 Consider the following system of non-linear differential equations [49]

$$\begin{aligned}\frac{du(t)}{dt} &= -1002u(t) + 1000v^2(t)\\ \frac{dv(t)}{dt} &= u(t) - v(t) - v^2(t)\end{aligned} \tag{4.26}$$

subject to these initial conditions

$$u(0)=1,\ \ v(0)=1 \tag{4.27}$$

The exact solution of the above system is

$$u(t)=e^{-2t},\ \ v(t)=e^{-t}$$

To solve the above system, we implemented the method presented in Section 4.2 with $M=4,\ \ k=0$. Approximating solution following as

$u(t) = C^T\psi(t)$ and $v(t) = S^T\psi(t)$ $\dfrac{du(t)}{dt} = C^T D\psi(t)$ and $\dfrac{dv(t)}{dt} = S^T D\psi(t)$

We get

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{63} & 0 \end{pmatrix}$$

If we consider (4.26) with (4.27), we have

$$\begin{aligned} R_1(t) &= C^T D\psi(t) + 1002\left(C^T\psi(t)\right) - 1000\left(S^T\psi(t)\right)^2 \\ R_2(t) &= S^T D\psi(t) - C^T\psi(t) + S^T\psi(t) + \left(S^T\psi(t)\right)^2 \end{aligned} \tag{4.28}$$

Calculating Equation (4.28) at the first four roots of $P_5(t)$, i.e.

$$t_0 = \frac{1}{2},\ \ t_1 = \frac{1}{2} - \frac{\sqrt{245 - 14\sqrt{70}}}{42},\ \ t_2 = \frac{1}{2} - \frac{\sqrt{245 + 14\sqrt{70}}}{42},\ \ t_3 = \frac{1}{2} + \frac{\sqrt{245 - 14\sqrt{70}}}{42}$$

and by utilising initial conditions we have

$$\begin{aligned} c_{0,0} - \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} - \sqrt{7}c_{0,3} + 3c_{0,4} &= 1 \\ s_{0,0} - \sqrt{3}s_{0,1} + \sqrt{5}s_{0,2} - \sqrt{7}s_{0,3} + 3s_{0,4} &= 1 \end{aligned}$$

If we solve this system of nonlinear algebraic equations, we get

$$C^T = [0.4286121774,-0.2341012934,0.06111143031,-0.008961478897,0.001851110478]$$

$$S^T = [0.6280482694,-0.1812595585,0.02343895161,-0.001955630037,0.0001385886922]$$

Consequently,

$$u(t) = C^T\psi(t) = \begin{bmatrix} 0.4286121774 \\ -0.2341012934 \\ 0.06111143031 \\ -0.008961478897 \\ 0.001851110478 \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2 - 6t + 1\right) \\ \sqrt{7}\left(20t^3 - 30t^2 + 12t - 1\right) \\ 3\left(70t^4 - 140t^3 + 90t^2 - 20t + 1\right) \end{bmatrix}$$

$$v(t) = S^T\psi(t) = \begin{bmatrix} 0.6280482694 \\ -0.1812595585 \\ 0.02343895161 \\ -0.001955630037 \\ 0.0001385886922 \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}(6t^2-6t+1) \\ \sqrt{7}(20t^3-30t^2+12t-1) \\ 3(70t^4-140t^3+90t^2-20t+1) \end{bmatrix}$$

Comparison of our results and the exact solution supports that our results approaches the exact solution as the value of M increases. Finally, we also present the numerical computations for u(t) and v(t) with the exact solution in Table 4.8 and Table 4.9.

Table 4.8 Comparison between the exact solution and the approximate solution $u(t)$ for Example 4.4

t	Exact Solution $u(t)$	$M=3,\ k=0$ Approximate Solution	$M=4,\ k=0$ Approximate Solution	Absolute Error
0.0	1.0000000000	1.000000000	1.000000000	0.0000000000
0.1	0.8187307531	0.8165316851	0.8164539898	0.0022767633
0.2	0.6703200460	0.6660452088	0.6665620472	0.0037579988
0.3	0.5488116361	0.5437106133	0.5442136317	0.0045980044
0.4	0.4493289641	0.4446979406	0.4442311630	0.0050978011
0.5	0.3678794412	0.3641772334	0.3623700203	0.0055094209
0.6	0.3011942119	0.2973185321	0.2953185417	0.0058756702
0.7	0.2465969639	0.2392918810	0.2406980273	0.0058989366
0.8	0.2018965180	0.1852673211	0.1970627334	0.0048337846
0.9	0.1652988882	0.1304148956	0.1638998799	0.0013990083
1.0	0.1353352832	0.0699046455	0.1416296414	-0.0062943582

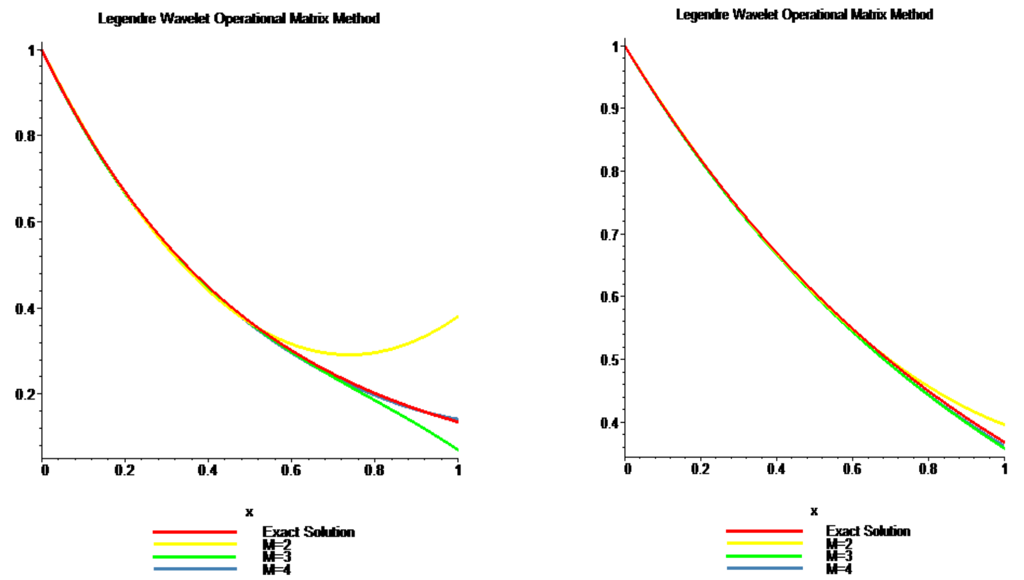

Figure 4.2 Comparison of our solutions $u(t), v(t)$ and the exact solution when $M=2,\ M=3\ and\ M=4$ for Example 4.4

Table 4.9 Comparison between the exact solution and the approximate solution v(t) for Example 4.4

t	Exact Solution $v(t)$	$M=3,\ k=0$ Approximate Solution	$M=4,\ k=0$ Approximate Solution	Absolute Error
0.0	1.0000000000	1.000000000	1.000000000	0.0000000000
0.1	0.9048374180	0.9037386313	0.9036350375	0.0012023805
0.2	0.8187307531	0.8165106052	0.8164828594	0.0022478937
0.3	0.7408182207	0.7375975458	0.7376781019	0.0031401188
0.4	0.6703200460	0.6662810770	0.6664252500	0.0038947960
0.5	0.6065306597	0.6018428229	0.6019986373	0.0045320224
0.6	0.5488116361	0.5435644074	0.5437424462	0.0050691899
0.7	0.4965853038	0.4907274546	0.4910707077	0.0055145961
0.8	0.4493289641	0.4426135883	0.4434673016	0.0058616625
0.9	0.4065696597	0.3985044327	0.4004859565	0.0060837032
1.0	0.3678794412	0.3576816116	0.3617502495	0.0061291917

Example 4.5 Consider the following Brusselator system presented in [47] and [48]

$$\begin{aligned} \frac{du(t)}{dt} &= -2u(t) + u^2(t)\,\mathrm{v(t)} \\ \frac{dv(t)}{dt} &= u(t) - u^2(t)\,\mathrm{v(t)} \end{aligned} \tag{4.29}$$

subject to these initial conditions

$$u(0) = 1, \quad v(0) = 1 \tag{4.30}$$

The approximate solution of this system when $\alpha = 1$ was presented by Chang and Isah using Legendre wavelet operational matrix of fractional derivative through wavelet-polynomial transformation (LWPT) in [48] and by Bota and Caruntu using the polynomial least squares method (PLSM) in [47]. These numerical solutions of this system are given by

$$\begin{aligned} u_{LWPT}(t) &= 1 - 1.0120t + 0.1211t^2 + 0.1517t^3, \\ v_{LWPT}(t) &= 1 + 0.0096t + 0.4069t^2 - 0.2461t^3 \end{aligned}$$

$$\begin{aligned} u_{PLSM}(t) &= 1 - 1.02827t + 0.201028t^2 + 0.0750974t^3, \\ v(t)_{PLSM} &= 1 + 0.0271107t + 0.334087t^2 - 180088t^3 \end{aligned}$$

To solve the above Brusselator system, we implemented the method presented in Section 4.2 with $M = 4, \; k = 0$. Approximating solution following as

$$u(t) \cong C^T\psi(t), \; v(t) \cong S^T\psi(t) \text{ and } \frac{du(t)}{dt} \cong C^T D\psi(t), \; \frac{dv(t)}{dt} \cong S^T D\psi(t)$$

We get

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{63} & 0 \end{pmatrix}$$

If we consider (4.29) with (4.30), we have

$$C^T D\psi(t) + 2C^T\psi(t) - \left(C^T\psi(t)\right)^2 \left(S^T\psi(t)\right) \tag{4.31}$$

$$S^T D\psi(t) - C^T\psi(t) + \left(C^T\psi(t)\right)^2 \left(S^T\psi(t)\right) \tag{4.32}$$

Calculating Equations (4.31) and (4.32) at the first four roots of $P_5(t)$, i.e.

$$t_0=\frac{1}{2},\ \ t_1=\frac{1}{2}-\frac{\sqrt{245-14\sqrt{70}}}{42},\ \ t_2=\frac{1}{2}-\frac{\sqrt{245+14\sqrt{70}}}{42},\ \ t_3=\frac{1}{2}+\frac{\sqrt{245-14\sqrt{70}}}{42}$$

and by utilising initial conditions we have

$$c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}-\sqrt{7}c_{0,3}+3c_{0,4}=1$$
$$s_{0,0}-\sqrt{3}s_{0,1}+\sqrt{5}s_{0,2}-\sqrt{7}s_{0,3}+3s_{0,4}=1$$

If we solve this system of nonlinear algebraic equations, we get

$$C^T=[0.5745339305,-0.2192448444,0.02298952682,0.001262676590,-0.0007808523319]$$

$$S^T=[1.069834705,0.05308541875,0.005100236422,-0.003352119819,0.0006128621024]$$

Consequently,

$$u(t)=C^T\psi(t)=\begin{bmatrix}0.5745339305\\-0.2192448444\\0.02298952682\\0.001262676590\\-0.0007808523319\end{bmatrix}^T\begin{bmatrix}1\\\sqrt{3}(-1+2t)\\\sqrt{5}(6t^2-6t+1)\\\sqrt{7}(20t^3-30t^2+12t-1)\\3(70t^4-140t^3+90t^2-20t+1)\end{bmatrix}$$

$$v(t)=S^T\psi(t)=\begin{bmatrix}1.069834705\\0.05308541875\\0.005100236422\\-0.003352119819\\0.0006128621024\end{bmatrix}^T\begin{bmatrix}1\\\sqrt{3}(-1+2t)\\\sqrt{5}(6t^2-6t+1)\\\sqrt{7}(20t^3-30t^2+12t-1)\\3(70t^4-140t^3+90t^2-20t+1)\end{bmatrix}$$

Comparison of our results and these approximate solutions introduced in [47] and [48] are also displayed in Figure 4.3. The figures support that our results approaches the approximate solutions presented in [47] and [48]. Finally, we also present the numerical computations for $u(t)$ and $v(t)$ in Table 4.10 and Table 4.11.

Table 4.10 Comparison between our approximate solution u_{LOWMM} , u_{LWPT} and u_{PLSM} for Example 4.5.

t	u_{LWOMM}	u_{LWPT}	u_{PLSM}
0.0	1.000000000	1.0000000	1.0000000000
0.1	0.9022538826	0.9001627	0.8992583774
0.2	0.8065945278	0.8036576	0.8029878992
0.3	0.7148002463	0.7113949	0.7116391498
0.4	0.6282557996	0.6242848	0.6256627136
0.5	0.5479523993	0.5432375	0.5455091750
0.6	0.4744877078	0.4691632	0.4716291184
0.7	0.4080658376	0.4029721	0.4044731282
0.8	0.3484973516	0.3455744	0.3444917888
0.9	0.2951992633	0.2978803	0.2921356846
1.0	0.2471950367	0.2608000	0.2478554000

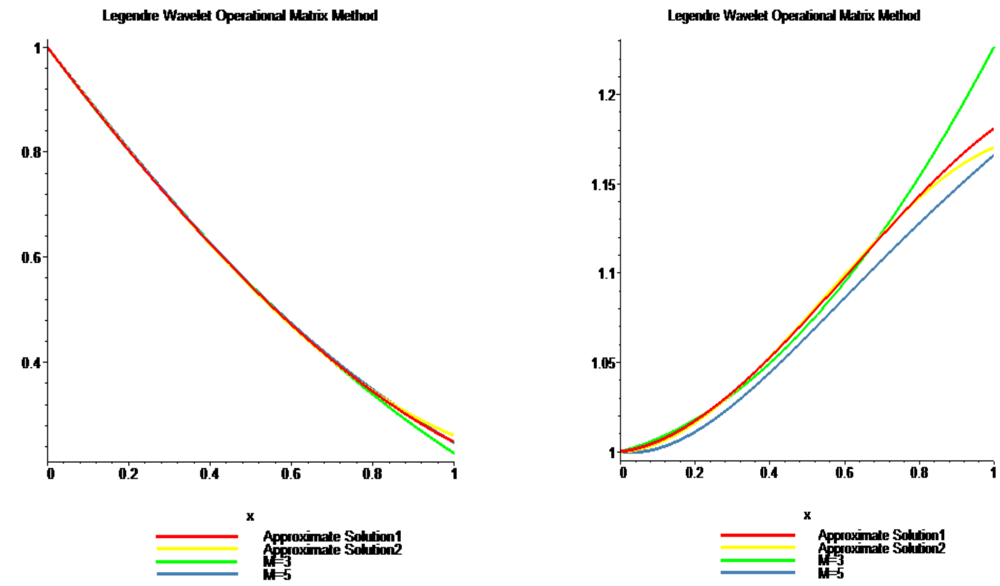

Figure 4.3 Comparison of our solutions u_{LOWMM} , v_{LOWMM} with the approximate solution u_{LPST} , v_{LPST} and the approximate solution u_{PLSM} , v_{PLSM} for Example 4.5.

Table 4.11 Comparison between our approximate solution v_{LOWMM} , v_{LWPT} and v_{PLSM} for Example 4.5.

t	v_{LWOMM}	v_{LWPT}	v_{PLSM}
0.0	0.9999999996	1.0000000	1.000000000
0.1	1.001804569	1.0047829	1.005871852
0.2	1.011179960	1.0162272	1.017344916
0.3	1.025980819	1.0328563	1.033338664
0.4	1.044370674	1.0531936	1.052772568
0.5	1.064821937	1.0757625	1.074566100
0.6	1.086115901	1.0990864	1.097638732
0.7	1.107342743	1.1216887	1.120909936
0.8	1.127901522	1.1420928	1.143299184
0.9	1.147500177	1.1588221	1.163725948
1.0	1.166155534	1.1704000	1.181109700

CHAPTER 5

THE APPLICATION OF THE OPERATIONAL MATRIX OF FRACTIONAL DERIVATIVE TO FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

There are a lot of tecniques that have been studied by many researchers to solve FDEs numerically. Some related applications of FDEs can be found in [18-37].

5.1 Solving Linear Fractional Differential Equations

In this section, we apply the Legendre wavelet operational matrix of fractional derivative for solving linear FDEs. Consider the following equation

$$D^{\alpha}u\left(t\right)=h_0(t)D^{\eta_0}u\left(t\right)+\ldots+h_{k-1}(t)D^{\eta_{k-1}}u\left(t\right)+h_k(t)u(t)+g(t) \tag{5.1}$$

with these initial conditions

$$u\left(t_0\right)=u_0,\ \frac{du}{dt}\left(t_0\right)=u_1,\ \frac{d^2u}{dt^2}\left(t_0\right)=u_2,\ldots,\frac{d^{n-1}u}{dt^{n-1}}\left(t_0\right)=u_{n-1} \tag{5.2}$$

where $h_0(t),h_1(t),\ldots,h_k(t)$ can be any function of the independent variable t and dependent variable $h_i\ (i=0,1,\ldots,k)$ and $n<\alpha\le n+1,\quad 0<\eta_0<\eta_1<\ldots<\eta_{k-1}<\alpha$ and D^{α} indicates the Caputo fractional derivative of order α .

First approximating $u\left(t\right), g\left(t\right)$ and $h_0(t),h_1(t),\ldots,h_k(t)$ by the Legendre wavelets, then we obtain

$$u\left(t\right)\cong\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}c_{n,m}\psi_{n,m}=C^T\psi\left(t\right) \tag{5.3}$$

$$g(t)\cong\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}s_{n,m}\psi_{n,m}=S^T\psi\left(t\right) \tag{5.4}$$

$$h_0(t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} h_{0n,m}\psi_{n,m} = H_0^{\,T}\psi(t)$$
$$\vdots \tag{5.5}$$
$$h_k(t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} h_{kn,m}\psi_{n,m} = H_k^{\,T}\psi(t)$$

where S and H_i $(i = 0,1,...,k)$ are known vector but C is an unknown vector and $\psi(t)$ is the vector given in Equation (2.27). If we utilise Equations (2.34) and (5.3), then we obtain

$$D^{\alpha}u(t) \cong C^T D^{\alpha}\psi(t) \cong C^T D^{(\alpha)}\psi(t) \tag{5.6}$$

$$D^{\eta_i}u(t) \cong C^T D^{\eta_i}\psi(t) \cong C^T D^{(\eta_i)}\psi(t), \quad i = 0,...,(k-1) \tag{5.7}$$

Substituting Equations (5.3)-(5.4) and (5.5) in Equation (5.1) the residual $R(t)$ can be expressed:

$$\begin{aligned} R(t) \cong\, & C^T D^{(\alpha)}\psi(t) - \left(H_0^{\,T}\psi(t)\right)\left(C^T D^{(\eta_0)}\psi(t)\right) - ... - \left(H_{k-1}^{\,T}\psi(t)\right)\left(C^T D^{(\eta_{k-1})}\psi(t)\right) \\ & - \left(H_k^{\,T}\psi(t)\right)\left(C^T D^{(\eta_k)}\psi(t)\right) - \left(H_{k+1}^{\,T}\psi(t)\right)\left(S^T\psi(t)\right) \end{aligned} \tag{5.8}$$

We get $2^k(M+1)-n$ linear equations by employing

$$\langle R(t), \psi_r(t)\rangle = \int_0^1 \psi_r(t)R(t)\,dt = 0, \quad \mathrm{r} = 0,...,2^k(M+1)-n \tag{5.9}$$

If we substitute Equation (5.3) in Equation (5.2) then we have

$$\begin{aligned} & u(0) \cong C^T\psi(0) = u_0 \\ & \frac{du}{dt}(0) \cong C^T D\psi(0) = u_1 \\ & \frac{d^2u}{dt^2}(0) \cong C^T D^2\psi(0) = u_2 \\ & \vdots \\ & \frac{d^{n-1}u}{dt^{n-1}}(0) \cong C^T D^{(n-1)}\psi(0) = u_{n-1} \end{aligned} \tag{5.10}$$

$2^k(M+1)$ set of linear equations are obtained Equations (5.9) and (5.10). We can solve these linear equations for unknown coefficients of the vector C. Consequently, $u(t)$ presented in Equation (5.1) can be computed.

5.2 Solving Non-Linear Fractional Differential Equations

In this section, we apply Legendre wavelet operational matrix of fractional derivative for solving non-linear FDEs. Consider the following equation

$$D^{\alpha}u(t) = H\left(t, u(t), D^{\eta_1}u(t), \dots, D^{\eta_k}u(t)\right) \tag{5.11}$$

with these initial conditions

$$u(t_0) = u_0,\ \frac{du}{dt}(t_0) = u_1,\ \frac{d^2u}{dt^2}(t_0) = u_2, \dots, \frac{d^{n-1}u}{dt^{n-1}}(t_0) = u_{n-1} \tag{5.12}$$

where $n < \alpha \le n+1,\ \ 0 < \eta_1 < \eta_2 < \dots < \eta_k < \alpha$ and D^{α} indicates the Caputo fractional derivative of order α .

First approximating $u(t),\ D^{\alpha}u(t)$ and for $i = 1, 2, \dots, k$ $D^{\eta_i}u(t)$ by the Legendre wavelets as Equations (5.3), (5.6) and (5.7) respectively and substituting these equations in Equation (5.11), then we obtain

$$C^T D^{(\alpha)}\psi(t) \cong H\left(t, C^T\psi(t), C^T D^{(\eta_1)}\psi(t), \dots, C^T D^{(\eta_k)}\psi(t)\right) \tag{5.13}$$

Also, if we substitute Equation (5.3) in Equation (5.122) then we have

$$\begin{aligned}
&u(0) \cong C^T\psi(0) = u_0 \\
&\frac{du}{dt}(0) \cong C^T D\psi(0) = u_1 \\
&\frac{d^2u}{dt^2}(0) \cong C^T D^2\psi(0) = u_2 \\
&\vdots \\
&\frac{d^{n-1}u}{dt^{n-1}}(0) \cong C^T D^{(n-1)}\psi(0) = u_{n-1}
\end{aligned} \tag{5.14}$$

First collocating Equation (5.13) at $2^k(M+1)-n$ points, then we can obtain the solution $u(t)$. We should use the first $2^k(M+1)-n$ roots of shifted Legendre polynomials $P_{2^k(M+1)}(t)$ to get a better result. Utilising these equations together with Equation (5.14), then we have $2^k(M+1)$ non-linear equations. These non-linear equations can be solved for unknown coefficients of the vector C. Consequently, $u(t)$ presented in Equation (5.11) can be computed.

5.3 Applications

In this section, we solve five linear and non-linear fractional differential equations by using LWOMM.

Example 5.1 We first consider the following FDE of the linear form [32]

$$4(\mathrm{t}+1)\,\mathrm{D}^{\frac{5}{2}}u(t)+4\,\mathrm{D}^{\frac{3}{2}}u(t)+\frac{1}{\sqrt{t+1}}u(t)=\sqrt{t}+\sqrt{\pi} \tag{5.15}$$

subject to

$$u(0)=\sqrt{\pi},\quad \frac{du}{dt}(0)=\frac{\sqrt{\pi}}{2},\quad u(1)=\sqrt{2\pi} \tag{5.16}$$

The exact solution of above system is $u(t)=\sqrt{\pi(t+1)}$

We implemented the method illustrated in Section 5.1 with $M=3,\ k=0$. Approximating solution following as

$$u(t)\cong C^T\psi(t),\quad Du(t)\cong C^TD\psi(t),\quad D^{\frac{3}{2}}u(t)\cong C^TD^{\left(\frac{3}{2}\right)}\psi(t),\quad D^{\frac{5}{2}}u(t)\cong C^TD^{\left(\frac{5}{2}\right)}\psi(t)$$

Also, approximating $h_0(t)=t+1,\quad h_1(t)=\dfrac{1}{\sqrt{t+1}}$ and $g(t)=\sqrt{t}+\sqrt{\pi}$ following as

$$h_0(t)\cong H_0{}^T\psi(t),\quad h_1(t)\cong H_1{}^T\psi(t)\quad \text{and}\quad g(t)\cong G^T\psi(t)$$

where $H_0(t)=\int_0^1(t+1)\psi(t)dt,\ H_1(t)=\int_0^1\frac{1}{\sqrt{t+1}}\psi(t)dt$ and $G(t)=\int_0^1\left(\sqrt{t}+\sqrt{\pi}\right)\psi(t)\,dt$

We have

$$D=\begin{pmatrix}0&0&0&0\\2\sqrt{3}&0&0&0\\0&2\sqrt{15}&0&0\\0&0&2\sqrt{35}&0\end{pmatrix},\quad D^{\left(\frac{3}{2}\right)}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\\frac{16\sqrt{5}}{\sqrt{\pi}}&\frac{16\sqrt{5}\sqrt{3}}{5\sqrt{\pi}}&-\frac{16}{7\sqrt{\pi}}&\frac{16\sqrt{7}\sqrt{5}}{105\sqrt{\pi}}\\-\frac{16\sqrt{7}}{\sqrt{\pi}}&\frac{80\sqrt{7}\sqrt{3}}{7\sqrt{\pi}}&\frac{16\sqrt{7}\sqrt{5}}{3\sqrt{\pi}}&\frac{-80}{11\sqrt{\pi}}\end{pmatrix},$$

$$D^{\left(\frac{5}{2}\right)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{160\sqrt{7}}{\sqrt{\pi}} & \frac{32\sqrt{7}\sqrt{3}}{\sqrt{\pi}} & -\frac{32\sqrt{7}\sqrt{5}}{7\sqrt{\pi}} & \frac{32}{3\sqrt{\pi}} \end{pmatrix}$$

If we consider (5.15) with (5.16), we have

$$R(t) = 4\left(H_0{}^T\psi(t)\right)\left(C^T D^{\left(\frac{5}{2}\right)}\psi(t)\right) + 4C^T D^{\left(\frac{3}{2}\right)}\psi(t) + \left(H_1{}^T\psi(t)\right)\left(C^T\psi(t)\right) - G^T\psi(t)$$

By computing

$$\int_0^1 \psi_1(t)R(t)\,dt = 0$$

We have

$$-2.439120518 + 0.8284271247c_{0,0} - 0.08206204062c_{0,1} + 80.75114673c_{0,2} + 1432.995536c_{0,3} = 0$$

and by utilising initial conditions we have

$$c_{0,0} - \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} - \sqrt{7}c_{0,3} = \sqrt{\pi}$$
$$2\sqrt{3}c_{0,1} - 6\sqrt{5}c_{0,2} + 10\sqrt{7}c_{0,3} = \frac{\sqrt{\pi}}{2}$$
$$c_{0,0} + \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} + \sqrt{7}c_{0,3} = \sqrt{2\pi}$$

If we solve this system of linear algebraic equations, we get

$$\begin{aligned} C^T &= \left[c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}\right] \\ &= [2.161308708, 0.2103901234, -0.00973478719, 0.00101325788] \end{aligned}$$

Consequently,

$$u(t) = C^T\psi(t) = \begin{bmatrix} 2.161308708 \\ 0.2103901234 \\ -0.00973478719 \\ 0.00101325788 \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2 - 6t + 1\right) \\ \sqrt{7}\left(20t^3 - 30t^2 + 12t - 1\right) \end{bmatrix}$$

The approximate solution with the exact solution are displayed in Table 5.1.

Table 5.1 Comparison between the exact solution and the approximate solution for Example 5.1

t	Exact Solution $u(t)$	$M=3,\ k=0$ Approximate Solution	Absolute Error
0.0	1.772455923	1.772453851	0.2072 10^{-5}
0.1	1.858967455	1.859556019	0.000588564
0.2	1.941628183	1.942759272	0.001131089
0.3	2.020910686	2.022385308	0.001474622
0.4	2.097198131	2.098755828	0.001557697
0.5	2.170806302	2.172192531	0.001386229
0.6	2.241999108	2.243017118	0.001018010
0.7	2.310999784	2.311551286	0.000551502
0.8	2.377999159	2.378116734	0.000117575
0.9	2.443161886	2.443035165	0.000126721
1.0	2.506631205	2.506628274	0.2931 10^{-5}

Example 5.2 Consider the following fractional Bagley-Torvik differential equation of the linear form with the initial conditions [18]

$$\frac{d^2u(t)}{dt^2}+D^{\frac{3}{2}}u(t)+u(t)=1+t \tag{5.17}$$

subject to

$$u(0)=0,\ \frac{du}{dt}(0)=1 \tag{5.18}$$

The exact solution of above system is $u(t)=1+t$.

We implemented the method illustrated in Section 5.1 to the above problem with $M=2$, $k=0$. Approximating solution following as

$$u(t)\cong C^T\psi(t),\ \frac{du(t)}{dt}\cong C^T D\psi(t),\ \frac{d^2u(t)}{dt^2}\cong C^T D^2\psi(t),\ D^{\frac{3}{2}}u(t)\cong C^T D^{\left(\frac{3}{2}\right)}\psi(t)$$

Also, approximating $g(t)=1+t$ following as $g(t)\cong G^T\psi(t)$

We have

$$D=\begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix},\quad D^2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4\sqrt{3}\sqrt{15} & 0 & 0 \end{pmatrix},\quad D^{\left(\frac{3}{2}\right)}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{16\sqrt{5}}{\sqrt{\pi}} & \frac{16\sqrt{3}\sqrt{5}}{5\sqrt{\pi}} & -\frac{16}{7\sqrt{\pi}} \end{pmatrix}$$

If we consider (5.17) with (5.18), we have

$$R(t)=C^T D^2\psi(t)+C^T D^{\left(\frac{3}{2}\right)}\psi(t)+C^T\psi(t)-G^T\psi(t)$$

By computing

$$\int_0^1 \psi_1(t)R(t)\,dt=0$$

we have

$$47.01787591c_{0,2}+c_{0,0}\text{-}1.500000000\ =\ 0$$

and by utilising initial conditions we have

$$c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}=1$$
$$2\sqrt{3}c_{0,1}-6\sqrt{5}c_{0,2}=1$$

If we solve this system of linear algebraic equations, we get

$$C^T=\left[c_{0,0},c_{0,1},c_{0,2}\right]=\left[1.499999999,\ 0.2886751345,\ 0.2\ 10^{10}\right]$$

Consequently,

$$u(t)=C^T\psi(t)=\left[1.499999999,\ 0.2886751345,\ 0.2\ 10^{10}\right]\begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2-6t+1\right) \end{bmatrix}$$

The approximate solution with the exact solution are displayed in Table 5.2.

Table 5.2 Comparison between the exact solution and the approximate solution for Example 5.2

t	Exact Solution $u(t)$	$M=2,\ k=0$ Approximate Solution	Absolute Error
0.0	1.0000000000	0.999999999	0.110^{-8}
0.1	1.1000000000	1.099999999	0.110^{-8}
0.2	1.2000000000	1.199999999	0.110^{-8}
0.3	1.3000000000	1.299999999	0.110^{-8}
0.4	1.4000000000	1.399999999	0.110^{-8}
0.5	1.5000000000	1.499999999	0.110^{-8}
0.6	1.6000000000	1.599999999	0.110^{-8}
0.7	1.7000000000	1.699999999	0.110^{-8}
0.8	1.8000000000	1.799999999	0.110^{-8}
0.9	1.9000000000	1.899999999	0.110^{-8}
1.0	2.0000000000	1.999999999	0.110^{-8}

Example 5.3 Consider the following FDE of the non-linear form [33]

$$D^3u(t)+D^{\frac{5}{2}}u(t)+u^2(t)=t^4 \tag{5.19}$$

subject to these initial conditions

$$u(0)=0,\quad \frac{du}{dt}(0)=0,\quad \frac{d^2u}{dt^2}(0)=2, \tag{5.20}$$

The exact solution of above system is $u(t)=t^2$

We implemented the method illustrated in Section 5.2 with $M=3,\ k=0$. Approximating solution following as

$$u(t)=C^T\psi(t),\quad \frac{du(t)}{dt}=C^TD\psi(t)$$

$$\frac{d^2u(t)}{dt^2}=C^TD^2\psi(t),\quad \frac{d^3u(t)}{dt^3}=C^TD^3\psi(t)\quad D^{\frac{5}{2}}u(t)=C^TD^{\left(\frac{5}{2}\right)}\psi(t)$$

Also, approximating $g(t)=t^4$ following as $g(t)=G^T\psi(t)$

We have

$$D=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 \end{pmatrix}, \quad D^2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4\sqrt{3}\sqrt{15} & 0 & 0 & 0 \\ 0 & 4\sqrt{15}\sqrt{35} & 0 & 0 \end{pmatrix}$$

$$D^3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8\sqrt{3}\sqrt{15}\sqrt{35} & 0 & 0 & 0 \end{pmatrix}, \quad D^{\left(\frac{5}{2}\right)}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{160\sqrt{7}}{\sqrt{\pi}} & \frac{32\sqrt{7}\sqrt{3}}{\sqrt{\pi}} & -\frac{32\sqrt{7}\sqrt{5}}{7\sqrt{\pi}} & \frac{32}{3\sqrt{\pi}} \end{pmatrix}$$

If we consider (5.19) with (5.20), we have

$$R(t)=C^T D^3\psi(t)+C^T D^{\left(\frac{5}{2}\right)}\psi(t)+\left(C^T\psi(t)\right)^2-G^T\psi(t) \tag{5.21}$$

Calculating Equation (5.21) at the first root of $P_4(t)$, i.e. $t_0=\frac{1}{2}-\frac{\sqrt{525-70\sqrt{30}}}{70}$

and by utilising initial conditions we have

$$c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}-\sqrt{7}c_{0,3}=0$$
$$2\sqrt{3}c_{0,1}-6\sqrt{5}c_{0,2}+10\sqrt{7}c_{0,3}=0$$
$$12\sqrt{5}c_{0,2}-60\sqrt{7}c_{0,3}=2$$

If we solve this system of non-linear algebraic equations, we get

$$\begin{aligned} C^T &= \left[c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}\right] \\ &= \left[0.3333333334,\ 0.2886751345,\ 0.07453559922,\ \text{-}0.3178931093\ 10^{-12}\right] \end{aligned}$$

Consequently,

$$u(t)=C^T\psi(t)=\begin{bmatrix} 0.3333333334 \\ 0.2886751345 \\ 0.07453559922 \\ \text{-}0.3178931093\ 10^{-12} \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2-6t+1\right) \\ \sqrt{7}\left(20t^3-30t^2+12t-1\right) \end{bmatrix}$$

The approximate solution with the exact solution are displayed in Table 5.3.

Table 5.3 Comparison between the exact solution and the approximate solution for Example 5.3

t	Exact Solution $u(t)$	$M=3,\ k=0$ Approximate Solution	Absolute Error
0.0	0.00	$0.8410661107\ 10^{-12}$	$0.8410661107\ 10^{-12}$
0.1	0.01	0.01000000006	$0.6\ 10^{-10}$
0.2	0.04	0.04000000011	$0.11\ 10^{-9}$
0.3	0.09	0.09000000014	$0.14\ 10^{-9}$
0.4	0.16	0.16000000020	$0.2\ 10^{-9}$
0.5	0.25	0.25000000020	$0.2\ 10^{-9}$
0.6	0.36	0.36000000010	$0.1\ 10^{-9}$
0.7	0.49	0.49000000010	$0.1\ 10^{-9}$
0.8	0.64	0.64000000010	$0.1\ 10^{-9}$
0.9	0.81	0.81000000000	0
1.0	1.00	0.99999999999	$0.1\ 10^{-9}$

Example 5.4 Consider the following FDE of the non-linear form with the initial conditions

$$D^{1.3}u(t)+u^2(t)=\frac{20}{7}\frac{t^{0.7}}{\Gamma(0.7)}+t^4 \tag{5.17}$$

subject to

$$u(0)=0,\quad \frac{du}{dt}(0)=0 \tag{5.18}$$

The exact solution of the above system is $u(t)=t^2$

To solve the above problem, we implemented the method presented in Section 5.2 with $M=2,\ k=0$. Approximating solution following as

$$u(t)\cong C^T\psi(t),\ Du(t)\cong C^T D\psi(t),\ D^{1.3}u(t)\cong C^T D^{(1.3)}\psi(t)$$

Also approximating $g(t)=\dfrac{20}{7}\dfrac{t^{0.7}}{\Gamma(0.7)}+t^4$ following as $g(t)\cong G^T\psi(t)$

We get

$$D=\begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix},\quad D^{(1.3)}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 17.37179094 & 7.800806362 & -0.8165511655 \end{pmatrix}$$

If we consider (5.17) with (5.18), we have

$$R(t)=C^T D^{(1.3)}\psi(t)+\left(C^T\psi(t)\right)^2-G^T\psi(t) \tag{5.19}$$

Calculating Equation (5.19) at the first root of $P_3(t)$, i.e. $t_0=\dfrac{1}{2}-\dfrac{\sqrt{15}}{10}$

and by utilising the boundary conditions we have

$$\begin{aligned} &c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}=0 \\ &2\sqrt{3}c_{0,1}-6\sqrt{5}c_{0,2}=0 \end{aligned}$$

If we solve this system of nonlinear algebraic equations, we get

$$C^T=\left[c_{0,0},c_{0,1},c_{0,2}\right]=[0.3364347192,0.2913610134,0.07522909016]$$

Consequently,

$$u(t)=C^T\psi(t)=[0.3364347192,0.2913610134,0.07522909016]\begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2-6t+1\right) \end{bmatrix}$$

The approximate solution and the exact solution are shown in Table 5.4.

Table 5.4 Comparison between the exact solution and the approximate solution for Example 5.4

t	Exact Solution $u(t)$	$M=2,\ k=0$ Approximate Solution	Absolute Error
0.0	0.00	-0.1 10^{-9}	0.1 10^{-9}
0.1	0.01	0.01009304147	0.00009304147
0.2	0.04	0.04037216618	0.00037216618
0.3	0.09	0.09083737403	0.00083737403
0.4	0.16	0.16148866500	0.00148866500
0.5	0.25	0.25232603910	0.00232603910
0.6	0.36	0.36334949640	0.00334949640
0.7	0.49	0.49455903680	0.00455903680
0.8	0.64	0.64595466040	0.00595466040
0.9	0.81	0.81753636710	0.00753636710
1.0	1.00	1.00930415700	0.00930415700

Example 5.5 Consider the following FDE of the linear form [55]

$$D^2u(t)+D^{1/2}u(t)+u(t)=2+t^2+\frac{8}{3\sqrt{\pi}}t^{3/2} \tag{5.20}$$

subject to these boundary conditions

$$u(0)=0,\quad u(1)=1 \tag{5.21}$$

The exact solution of the above system is $u(t)=t^2$

We implemented the method illustrated in Section 5.1 to the above problem with $M=2$, $k=0$. Approximating solution following as

$$u(t)\cong C^T\psi(t),\quad \frac{d^2u(t)}{dt^2}\cong C^TD^2\psi(t),\quad D^{\frac{1}{2}}u(t)\cong C^TD^{\left(\frac{1}{2}\right)}\psi(t)$$

Also, approximating $g(t)=2+t^2+\frac{8}{3\sqrt{\pi}}t^{3/2}$ following as $g(t)=G^T\psi(t)$

where

$$G = \int_0^1 \left(2 + t^2 + \frac{8}{3\sqrt{\pi}} t^{3/2} \right) \psi(t) dt$$

We have

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4\sqrt{3}\sqrt{15} & 0 & 0 \end{pmatrix}$$

$$D^{\left(\frac{1}{2}\right)} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{8\sqrt{3}}{3\sqrt{\pi}} & \frac{8}{5\sqrt{\pi}} & -\frac{8\sqrt{3}\sqrt{5}}{105\sqrt{\pi}} \\ -\frac{8\sqrt{5}}{5\sqrt{\pi}} & \frac{8\sqrt{3}\sqrt{5}}{7\sqrt{\pi}} & \frac{8}{3\sqrt{\pi}} \end{pmatrix}$$

If we consider (5.20) with (5.21), we have

$$R(t) = C^T D^2 \psi(t) + C^T D^{\left(\frac{1}{2}\right)} \psi(t) + C^T \psi(t) - G^T \psi(t)$$

By computing

$$\int_0^1 \psi_1(t) R(t) dt = 0$$

we have

$$24.81430971 c_{0,2} + 2.605880063 c_{0,1} - 2.935135555 + c_{0,0} = 0$$

and by utilising boundary conditions we have

$$c_{0,0} - \sqrt{3} c_{0,1} + \sqrt{5} c_{0,2} = 0$$
$$c_{0,0} + \sqrt{3} c_{0,1} + \sqrt{5} c_{0,2} = 1$$

If we solve this system of linear algebraic equations, we get

$$C^T = \left[c_{0,0}, c_{0,1}, c_{0,2} \right] = [0.3333333332,\ 0.2886751345,\ 0.0745355993]$$

Consequently,

$$u(t)=C^T\psi(t)=[0.3333333332,\ 0.2886751345,\ 0.0745355993]\begin{bmatrix}1\\ \sqrt{3}(-1+2t)\\ \sqrt{5}\left(6t^2-6t+1\right)\end{bmatrix}$$

The approximate solution with the exact solution are displayed in Table 5.5.

Table 5.5 Comparison between the exact solution and the approximate solution for Example 5.5

t	Exact Solution $u(t)$	$M=2,\ k=0$ Approximate Solution	Absolute Error
0.0	0.00	$-0.1\,10^{-9}$	$0.1\,10^{-9}$
0.1	0.01	0.0099999989	$0.11\,10^{-9}$
0.2	0.04	0.0399999988	$0.12\,10^{-9}$
0.3	0.09	0.0899999987	$0.13\,10^{-9}$
0.4	0.16	0.1599999999	$0.1\,10^{-9}$
0.5	0.25	0.2499999998	$0.2\,10^{-9}$
0.6	0.36	0.3599999998	$0.2\,10^{-9}$
0.7	0.49	0.4899999998	$0.2\,10^{-9}$
0.8	0.64	0.6399999998	$0.2\,10^{-9}$
0.9	0.81	0.8099999998	$0.2\,10^{-9}$
1.0	1.00	0.9999999998	$0.2\,10^{-9}$

Example 5.6 Consider the following FDE of the non-linear form with the boundary conditions [31]

$$\frac{d^2u(t)}{dt^2}+\Gamma\left(\frac{4}{5}\right)t^{\frac{6}{5}}D^{\frac{6}{5}}u(t)+\frac{11}{9}\Gamma\left(\frac{5}{6}\right)t^{\frac{1}{6}}D^{\frac{1}{6}}u(\mathrm{t})-\left(\frac{du(t)}{dt}\right)^2=2+\frac{1}{10}t^2 \tag{5.22}$$

subject to

$$u(0)=1,\quad u(1)=2 \tag{5.23}$$

The exact solution of the previous system is $u(t)=1+t^2$

To solve the above problem, we implemented the method presented in Section 5.2 with $M = 2, \; k = 0$. Approximating solution following as

$$u(t) \cong C^T \psi\left(t\right), \quad \frac{du(t)}{dt} \cong C^T D\psi\left(t\right), \quad \frac{d^2u(t)}{dt^2} \cong C^T D^2 \psi\left(t\right)$$

$$D^{\frac{1}{6}} u\left(t\right) \cong C^T D^{\left(\frac{1}{6}\right)} \psi\left(t\right), \quad D^{\frac{6}{5}} u\left(t\right) \cong C^T D^{\left(\frac{6}{5}\right)} \psi\left(t\right)$$

Approximating $f_1(t) = \Gamma\left(\frac{4}{5}\right) t^{\frac{6}{5}} D^{\frac{6}{5}}, \quad f_2(t) = \frac{11}{9}\Gamma\left(\frac{5}{6}\right) t^{\frac{1}{6}}, \quad g(t) = 2 + \frac{1}{10}t^2$ following as

$$f_1(t) \cong F_1^T \psi\left(t\right), \quad f_2(t) \cong F_2^{\,T} \psi\left(t\right), \quad g(t) \cong G^T \psi(t)$$

where

$$F_1(t) = \int_0^1 \Gamma\left(\frac{4}{5}\right) t^{\frac{6}{5}} D^{\frac{6}{5}} \psi\left(t\right) dt, \quad F_1(t) = \int_0^1 \frac{11}{9}\Gamma\left(\frac{5}{6}\right) t^{\frac{1}{6}} \psi\left(t\right) dt, \quad G(t) = \int_0^1 \left(2 + \frac{1}{10}t^2\right) \psi\left(t\right) dt$$

We get

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}$$

$$D^{\left(\frac{1}{6}\right)} = \begin{pmatrix} 0 & 0 & 0 \\ 2.008717540 & 1.023294363 & -0.05743771053 \\ -2.288145774 & 0.5858729181 & 1.235012279 \end{pmatrix}$$

$$D^{\left(\frac{6}{5}\right)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 16.00534636 & 7.920592311 & -0.5381810900 \end{pmatrix}$$

If we consider (5.22) with (5.23), we have

$$R(t) = C^T D^2 \psi\left(t\right) + \left(F_1^T \psi\left(t\right)\right)\left(C^T D^{\left(\frac{6}{5}\right)} \psi\left(t\right)\right) + \left(F_2^{\,T} \psi\left(t\right)\right)\left(C^T D^{\left(\frac{1}{6}\right)} \psi\left(t\right)\right) - \left(C^T D \psi\left(t\right)\right)^2 - G^T \psi\left(t\right) \tag{5.24}$$

Calculating Equation (5.24) at the first root of $P_3\left(t\right)$, i.e. $t_0 = \frac{1}{2} - \frac{\sqrt{15}}{10}$

and by utilising the boundary conditions we have

$$c_{0,0} - \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} = 1$$
$$c_{0,0} + \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} = 2$$

If we solve this system of nonlinear algebraic equations, we get

$$C^T = \left[c_{0,0}, c_{0,1}, c_{0,2}\right] = [1.333325434,\ 0.2886751345,\ 0.07453913186]$$

Consequently,

$$u(t) = C^T\psi(t) = [1.333325434,\ 0.2886751345,\ 0.07453913186]\begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2 - 6t + 1\right) \end{bmatrix}$$

The approximate solution with the exact solution are shown in Table 5.6.

Table 5.6 Comparison between the exact solution and the numerical solution for Example 5.6

t	Exact Solution $u(t)$	$M=2,\ k=0$ Approximate Solution	Absolute Error
0.0	1.00	0.9999999998	0.2 10^{-9}
0.1	1.01	1.009995734	0.4266 10^{-5}
0.2	1.04	1.039992417	0.7583 10^{-5}
0.3	1.09	1.089990047	0.9953 10^{-5}
0.4	1.16	1.159988625	0.000011375
0.5	1.25	1.249988151	0.000011849
0.6	1.36	1.359988625	0.000011375
0.7	1.49	1.489990047	0.9953 10^{-5}
0.8	1.64	1.639992416	0.7584 10^{-5}
0.9	1.81	1.809995734	0.4266 10^{-5}
1.0	2.00	2.000000000	0.000000000

CHAPTER 6

THE APPLICATION OF THE OPERATIONAL MATRIX OF FRACTIONAL DERIVATIVE TO SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS

There are a lot of tecniques that have been studied by many researchers to solve systems of FDEs numerically. Some related applications of such systems can be found in [38-50].

6.1 Solving Systems of Fractional Differential Equations

In this section, the LWOMM is implemented to obtain the numerical solution of the system of FDEs. Consider the following system

$$\begin{aligned} &D^{\eta_1}u_1(t)=U_1(t,u_1,u_2,...,u_m),\\ &D^{\eta_2}u_2(t)=U_2(t,u_1,u_2,...,u_m),\\ &\qquad\qquad\vdots\\ &D^{\eta_n}u_m(t)=U_m(t,u_1,u_2,...,u_m) \end{aligned} \tag{6.1}$$

where U_i's are linear/nonlinear functions of $t,u_1,u_2,...,u_m$, D^{η_i} is the derivative of u_i with order of η_i in the sense of Caputo and $N-1\le\eta_i<N$, subjected to the initial conditions:

$$\begin{aligned} &u_1(t_0)=u_{10},\ \frac{du_1}{dt}(t_0)=u_{11},\ \frac{d^2u_1}{dt^2}(t_0)=u_{12},...,\frac{d^{n-1}u_1}{dt^{n-1}}(t_0)=u_{1(n-1)}\\ &u_2(t_0)=u_{20},\ \frac{du_2}{dt}(t_0)=u_{21},\ \frac{d^2u_2}{dt^2}(t_0)=u_{22},...,\frac{d^{n-1}u_2}{dt^{n-1}}(t_0)=u_{2(n-1)}\\ &\quad\vdots\qquad\qquad\vdots\qquad\qquad\quad\vdots\qquad\qquad\qquad\vdots\\ &u_m(t_0)=u_{m0},\ \frac{du_m}{dt}(t_0)=u_{m1},\ \frac{d^2u_m}{dt^2}(t_0)=u_{m2},...,\frac{d^{n-1}u_m}{dt^{n-1}}(t_0)=u_{\mathrm{m}(n-1)} \end{aligned} \tag{6.2}$$

First of all, approximating $u_1(t), u_2(t), \ldots, u_m(t)$ and $D^{\eta_1}u_1(t), D^{\eta_2}u_2(t), \ldots, D^{\eta_n}u_n(t)$, then we obtain

$$\begin{aligned}
u_1(t) &\cong \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{1n,m}\psi_{n,m} = C_1^{\,T}\psi(t) \\
u_2(t) &\cong \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{2n,m}\psi_{n,m} = C_2^{\,T}\psi(t) \\
&\vdots \\
u_m(t) &\cong \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nn,m}\psi_{n,m} = C_m^{\,T}\psi(t)
\end{aligned} \tag{6.3}$$

where $C_i, \; i = 1, 2, \ldots, m$ are unknown vectors and $\psi(t)$ is the vector introduced in (2.27). If we utilise Equation (2.34) then we have

$$\begin{aligned}
D^{\eta_1}u_1(t) &\cong C_1^{\,T} D^{(\eta_1)}\psi(t) \\
D^{\eta_2}u_2(t) &\cong C_2^{\,T} D^{(\eta_2)}\psi(t) \\
&\vdots \\
D^{\eta_n}u_m(t) &\cong C_m^{\,T} D^{(\eta_n)}\psi(t)
\end{aligned} \tag{6.4}$$

Substituting Equations (6.3)-(6.4) in Equation (6.1), then we obtain

$$\begin{aligned}
R_1(t) &= C_1^{\,T} D^{(\eta_1)}\psi(t) - U_1(t, C_1^{\,T}\psi(t), C_2^{\,T}\psi(t), \ldots, C_m^{\,T}\psi(t)) \\
R_2(t) &= C_2^{\,T} D^{(\eta_2)}\psi(t) - U_2(t, C_1^{\,T}\psi(t), C_2^{\,T}\psi(t), \ldots, C_m^{\,T}\psi(t)) \\
&\vdots \\
R_m(t) &= C_m^{\,T} D^{(\eta_n)}\psi(t) - U_m(t, C_1^{\,T}\psi(t), C_2^{\,T}\psi(t), \ldots, C_m^{\,T}\psi(t))
\end{aligned} \tag{6.5}$$

If U_i's are linear functions of $t, u_1, u_2, \ldots, u_m$, then we produce $2^k(M+1) - mn$ linear equations by implementing

$$\int_0^1 \psi_j(t) R_i(t)\,dt = 0, \quad j = 1, \ldots, 2^k(M+1) - mn, \quad i = 1, 2, \ldots, m \tag{6.6}$$

Also by substituting initial conditions (6.2) in Equation (6.4), then we obtain

$$
\begin{array}{l}
u_1(t_0) \cong C_1^T \psi(t_0) = u_{10},\ \dfrac{du_1}{dt}(t_0) \cong C_1^T D\psi(t_0) = u_{11},\ldots,\dfrac{d^{n-1}u_1}{dt^{n-1}}(t_0) \cong C_1^T D^{n-1}\psi(t_0) = u_{1(n-1)} \\
u_2(t_0) \cong C_2^T \psi(t_0) = u_{20},\ \dfrac{du_2}{dt}(t_0) \cong C_2^T D\psi(t_0) = u_{21},\ldots,\dfrac{d^{n-1}u_2}{dt^{n-1}}(t_0) \cong C_2^T D^{n-1}\psi(t_0) = u_{2(n-1)} \\
\quad\vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\
u_m(t_0) \cong C_m^T \psi(t_0) = u_{m0},\ \dfrac{du_m}{dt}(t_0) \cong C_m^T D\psi(t_0) = u_{m1},\ldots,\dfrac{d^{n-1}u_m}{dt^{n-1}}(t_0) \cong C_m^T D^{n-1}\psi(t_0) = u_{\mathrm{m}(n-1)}
\end{array}
\tag{6.7}
$$

A $2^k(M+1)$ set of linear equations was generated by combining Equations (6.6)-(6.7) . Solution of these linear equations can be obtained for unknown coefficients of the vector C. Consequently $u_1(t), u_2(t), \ldots, u_m(t)$ introduced in Equation (6.1) can be computed.

If U_i's are non-linear functions of $t, u_1, u_2, \ldots, u_m$, first computing $R_1(t), R_2(t), \ldots, R_m(t)$ at $2^k(M+1) - mn$ points and for a better result, using the first $2^k(M+1) - mn$ roots of shifted Legendre polynomials $P_{2^k(M+1)}(t)$, then these equations collectively with Equation (6.7) produce $2^k(M+1)$ non-linear equations. Solution of these non-linear equations can be obtained for unknown coefficients of the vector C. Consequently $u_1(t), u_2(t), \ldots, u_m(t)$ introduced in Equation (6.1) can be computed.

6.2 Applications

In this section, to show applicability and powerfulness of the introduced method, we solve five linear and non-linear system of FDEs.

Example 6.1 We first consider the following linear system of FDEs [41], [43]

$$
\begin{aligned}
D^\alpha u(t) &= u(t) + v(t) \\
D^\alpha v(t) &= -u(t) + v(t)
\end{aligned}
\tag{6.8}
$$

subject to

$$
u(0) = 0, \quad v(0) = 1 \tag{6.9}
$$

The exact solution of this system when $\alpha = 1$ is known to be

$$u(t) = e^t \sin t, \quad v(t) = e^t \cos t$$

To solve the above system when $\alpha = 0.9$, we applied the method presented in Section 6.1 with $M = 2, \ \ k = 0$. Approximating solution following as

$$u(t) = C^T \psi(t), \ \ v(t) = S^T \psi(t), \ \ D^{0.9} u(t) = C^T D^{(0.9)} \psi(t), \quad D^{0.9} v(t) = S^T D^{(0.9)} \psi(t)$$

We get

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}, \ \ D^{(0.9)} = \begin{pmatrix} 0 & 0 & 0 \\ 1.911059300\sqrt{3} & 0.2730084714 & -0.02642017466\sqrt{15} \\ -0.273008472\sqrt{5} & 1.664471004\sqrt{15} & 0.6325806046 \end{pmatrix}$$

If we consider (6.8) with (6.9), we have

$$R_1(t) = C^T D^{(0.9)} \psi(t) - C^T \psi(t) - S^T \psi(t)$$
$$R_2(t) = S^T D^{(0.9)} \psi(t) + C^T \psi(t) - S^T \psi(t)$$

By computing

$$\int_0^1 \psi_i(t) R_j(t)\, dt = 0, \ \ i = 1,2 \ \ j = 1,2$$

We obtain four linear equations following as

$$-0.6104655018 c_{0,2} + 3.310051804 c_{0,1} - c_{0,0} - s_{0,0} = 0$$

$$-0.6104655018 s_{0,2} + 3.310051804 s_{0,1} + c_{0,0} - s_{0,0} = 0$$

$$-0.7269915286 c_{0,1} + 6.446468479 c_{0,2} - s_{0,1} = 0$$

$$-0.7269915286 s_{0,1} + 6.446468479 s_{0,2} + c_{0,1} = 0$$

and by utilising initial conditions we have

$$c_{0,0} - \sqrt{3} c_{0,1} + \sqrt{5} c_{0,2} = 0$$
$$s_{0,0} - \sqrt{3} s_{0,1} + \sqrt{5} s_{0,2} = 1$$

If we solve this system of linear algebraic equations, we get

$$C^T = \left[c_{0,0}, c_{0,1}, c_{0,2} \right] = [1.096167384, 0.7531907067, 0.09319805590]$$

$$S^T = \left[s_{0,0}, s_{0,1}, s_{0,2}\right] = [1.340038676, 0.0532350665, -0.1108342136]$$

$$u(t) = C^T\psi(t) = [1.096167384, 0.7531907067, 0.09319805590]\begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2-6t+1\right) \end{bmatrix}$$

$$v(t) = S^T\psi(t) = [1.340038676, 0.0532350665, -0.1108342136]\begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2-6t+1\right) \end{bmatrix}$$

When the obtained results are matched against the exact solution when $\alpha = 1$ as demonstrated in Figure 6.1, we can clearly observe that when α approaches 1, our results approach the exact solution. We also solved this problem by using LPOMM and we compared the results with the LWOMM. The numerical computations for u(t) and v(t) when $\alpha = 0.9$ are also revealed in Table 6.1 and Table 6.2.

Table 6.1 Numerical solutions of $u(t)$ when $\alpha = 0.9$ attained by the introduced method and the LPOMM for Example 6.1

t	u_{LWOMM}	u_{LPOMM}	Absolute Error
0.0	0.3 10^{-9}	0.0000000000	0.3 10^{-9}
0.1	0.1483784330	0.1483784325	0.12 10^{-9}
0.2	0.3217645283	0.3217645277	0.633 10^{-9}
0.3	0.5201582862	0.5201582855	0.65 10^{-9}
0.4	0.7435597067	0.7435597059	0.83 10^{-9}
0.5	0.9919687898	0.9919687890	0.8 10^{-9}
0.6	1.2653855360	1.265385535	0.53 10^{-9}
0.7	1.563809944	1.563809943	0.95 10^{-9}
0.8	1.887242014	1.887242014	0.733 10^{-9}
0.9	2.235681748	2.235681748	0.62 10^{-9}
1.0	2.609129144	2.609129144	0.3 10^{-9}

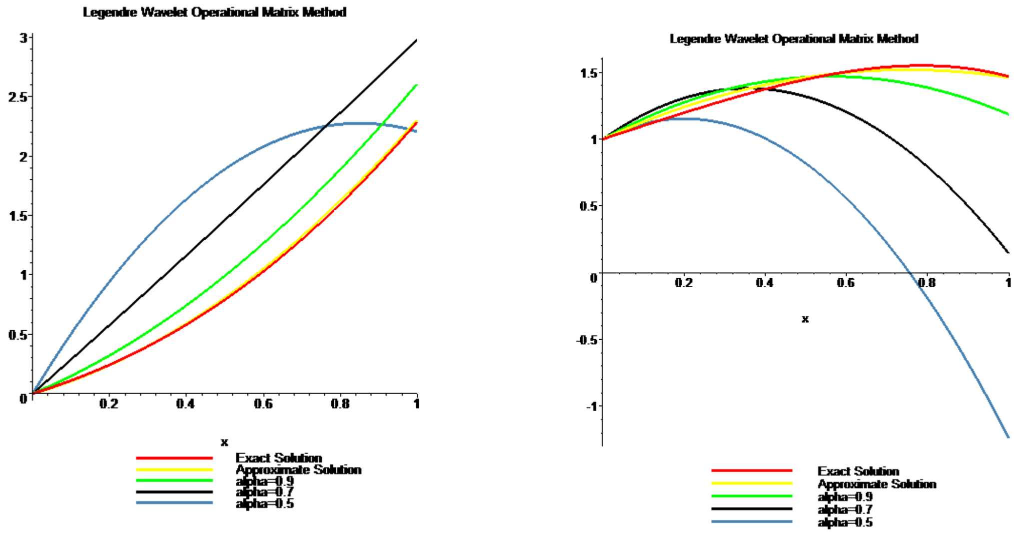

Figure 6.1 Comparison of our solutions and the exact solution when $\alpha = 0.9, 0.7, 0.5$ *and* 1 for Example 6.1

Table 6.2 Numerical solutions of $v(t)$ when $\alpha = 0.9$ attained by the introduced method and the LPOMM for Example 6.1

t	v_{LWOMM}	v_{LPOMM}	Absolute Error
0.0	1.000000000	1.000000000	$0.2\ 10^{-9}$
0.1	1.152270899	1.152270900	$0.5\ 10^{-9}$
0.2	1.274801858	1.274801858	$0.599\ 10^{-9}$
0.3	1.367592877	1.367592878	$0.67\ 10^{-9}$
0.4	1.430643956	1.430643957	$0.13\ 10^{-8}$
0.5	1.463955094	1.463955094	$0.1\ 10^{-8}$
0.6	1.467526291	1.467526293	$0.13\ 10^{-8}$
0.7	1.441357549	1.441357551	$0.171\ 10^{-8}$
0.8	1.385448866	1.385448868	$0.1461\ 10^{-8}$
0.9	1.299800244	1.299800245	$0.15\ 10^{-8}$
1.0	1.184411680	1.184411682	$0.18\ 10^{-8}$

Example 6.2 Consider the following non-linear system of FDEs [49]

$$D^{\frac{3}{2}}u(t) = -8u(t) + v^2(t) - 4t^6 + 4t^3 + \frac{8t^{\frac{3}{2}}}{\sqrt{\pi}} - 1$$
$$D^{\frac{1}{2}}v(t) = t^2 Du(t) + v(t) - 3t^4 - 2t^3 + \frac{32t^{\frac{5}{2}}}{5\sqrt{\pi}} - 1 \quad (6.10)$$

subject to

$$u(0)=0, \quad v(0)=1, \quad u(1)=1, \quad v(1)=3, \quad \frac{du(0)}{dt}=0, \quad \frac{du(1)}{dt}=3 \quad (6.11)$$

The exact solution of this system is known to be

$$u(t) = t^3, \quad v(t) = 2t^3 + 1$$

To solve the above system, we implemented the method presented in Section 6.2 with $M = 3, \ k = 0$. Approximating solution following as

$$u(t) \cong C^T\psi(t), \ v(t) \cong S^T\psi(t) \ Du(t) \cong C^T D\psi(t)$$

$$D^{\frac{3}{2}}u(t) \cong C^T D^{\left(\frac{3}{2}\right)}\psi(t), \quad D^{\frac{1}{2}}v(t) \cong S^T D^{\left(\frac{1}{2}\right)}\psi(t)$$

Approximating $g_0(t) = -4t^6 + 4t^3 + \frac{8t^{\frac{3}{2}}}{\sqrt{\pi}} - 1$, $\boldsymbol{h(t) = t^2}$, $g_1(t) = -3t^4 - 2t^3 + \frac{32t^{\frac{5}{2}}}{5\sqrt{\pi}} - 1$

following as $g_0(t) \cong G_0^T\psi(t)$, $g_1(t) \cong G_1^T\psi(t)$ and $h(t) = H^T\psi(t)$

where

$$\mathrm{G}_0(t) = \int_0^1 \left(-4t^6 + 4t^3 + \frac{8t^{\frac{3}{2}}}{\sqrt{\pi}} - 1 \right) \psi(t)dt$$

$$\mathrm{G}_1(t) = \int_0^1 \left(-3t^4 - 2t^3 + \frac{32t^{\frac{5}{2}}}{5\sqrt{\pi}} - 1 \right) \psi(t)dt$$

and

$$H(t) = \int_0^1 t^2\psi(t)dt$$

We get

$$D=\begin{pmatrix}0&0&0&0\\2\sqrt{3}&0&0&0\\0&2\sqrt{15}&0&0\\0&0&2\sqrt{35}&0\end{pmatrix},\quad D^{\left(\frac{3}{2}\right)}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\\frac{16\sqrt{5}}{\sqrt{\pi}}&\frac{16\sqrt{15}}{5\sqrt{\pi}}&-\frac{16}{7\sqrt{\pi}}&\frac{16\sqrt{35}}{105\sqrt{\pi}}\\-\frac{16\sqrt{7}}{\sqrt{\pi}}&\frac{80\sqrt{21}}{7\sqrt{\pi}}&\frac{16\sqrt{35}}{3\sqrt{\pi}}&-\frac{80}{11\sqrt{\pi}}\end{pmatrix}$$

$$D^{\left(\frac{1}{2}\right)}=\begin{pmatrix}0&0&0&0\\\frac{8\sqrt{3}}{3\sqrt{\pi}}&\frac{8}{5\sqrt{\pi}}&-\frac{8\sqrt{15}}{105\sqrt{\pi}}&\frac{8\sqrt{21}}{315\sqrt{\pi}}\\-\frac{8\sqrt{5}}{5\sqrt{\pi}}&\frac{8\sqrt{15}}{7\sqrt{\pi}}&\frac{8}{3\sqrt{\pi}}&-\frac{8\sqrt{35}}{77\sqrt{\pi}}\\\frac{16\sqrt{7}}{7\sqrt{\pi}}&-\frac{16\sqrt{21}}{45\sqrt{\pi}}&\frac{304\sqrt{35}}{385\sqrt{\pi}}&\frac{688}{195\sqrt{\pi}}\end{pmatrix}$$

If we consider (6.10) with (6.11), we have

$$\begin{aligned}R_1(t)&=C^TD^{\left(\frac{3}{2}\right)}\psi(t)+8C^T\psi(t)-\left(S^T\psi(t)\right)^2-G_0{}^T\psi(t)\\R_2(t)&=S^TD^{\left(\frac{1}{2}\right)}\psi(t)-\left(H^T\psi(t)\right)\left(C^TD\psi(t)\right)-S^T\psi(t)-G_1{}^T\psi(t)\end{aligned}\qquad(6.12)$$

Calculating Equations (6.12) at the first root of $P_4(t)$, i.e. $t_0=\frac{1}{2}+\frac{\sqrt{525-70\sqrt{30}}}{70}$

We have two non-linear equations and by utilising (6.11) we have

$$\begin{aligned}&c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}-\sqrt{7}c_{0,3}=0\\&c_{0,0}+\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}+\sqrt{7}c_{0,3}=1\\&s_{0,0}-\sqrt{3}s_{0,1}+\sqrt{5}s_{0,2}-\sqrt{7}s_{0,3}=1\\&s_{0,0}-\sqrt{3}s_{0,1}+\sqrt{5}s_{0,2}-\sqrt{7}s_{0,3}=3\\&2\sqrt{3}c_{0,1}-6\sqrt{5}c_{0,2}+10\sqrt{7}c_{0,3}=0\\&2\sqrt{3}c_{0,1}+6\sqrt{5}c_{0,2}+10\sqrt{7}c_{0,3}=3\end{aligned}$$

If we solve this system of nonlinear algebraic equations, we get

$$C^T=[0.2500000001,0.2525907427,0.1118033988,0.02362277957]$$

$$S^T=[1.490829745,0.4893097218,0.2277078606,0.05763606744]$$

Consequently,

$$u(t)=C^T\psi(t)=\begin{bmatrix}0.2500000001\\0.2525907427\\0.1118033988\\0.02362277957\end{bmatrix}^T\begin{bmatrix}1\\\sqrt{3}(-1+2t)\\\sqrt{5}\left(6t^2-6t+1\right)\\\sqrt{7}\left(20t^3-30t^2+12t-1\right)\end{bmatrix}$$

$$v(t)=S^T\psi(t)=\begin{bmatrix}1.490829745\\0.4893097218\\0.2277078606\\0.05763606744\end{bmatrix}^T\begin{bmatrix}1\\\sqrt{3}(-1+2t)\\\sqrt{5}\left(6t^2-6t+1\right)\\\sqrt{7}\left(20t^3-30t^2+12t-1\right)\end{bmatrix}$$

We applied both the proposed method and the LPOMM to solve this problem and show that our approach is more efficient and useful. Our numerical results support that our solution approaches the exact solution more than the approximate solution LPOMM. Comparison of the approximate and exact solutions are presented in Table 6.3 and Table 6.4.

Table 6.3 The numerical results attained by the introduced method in comparison with the LPOMM and the exact solution u(t) for Example 6.2.

t	Exact Solution $u(t)$	u_{LWOMM}	u_{LPOMM}
0.0	0.000	$-0.12\ 10^{-9}$	0.000000000000
0.1	0.001	0.01000000005	0.001000000000
0.2	0.008	0.02000000016	0.008000000000
0.3	0.027	0.03750000021	0.027000000000
0.4	0.064	0.07000000020	0.064000000000
0.5	0.125	0.12500000001	0.125000000000
0.6	0.216	0.21000000000	0.216000000000
0.7	0.343	0.33249999998	0.343000000000
0.8	0.512	0.49999999996	0.512000000000
0.9	0.729	0.71999999993	0.729000000000
1.0	1.000	0.99999999989	1.000000000000

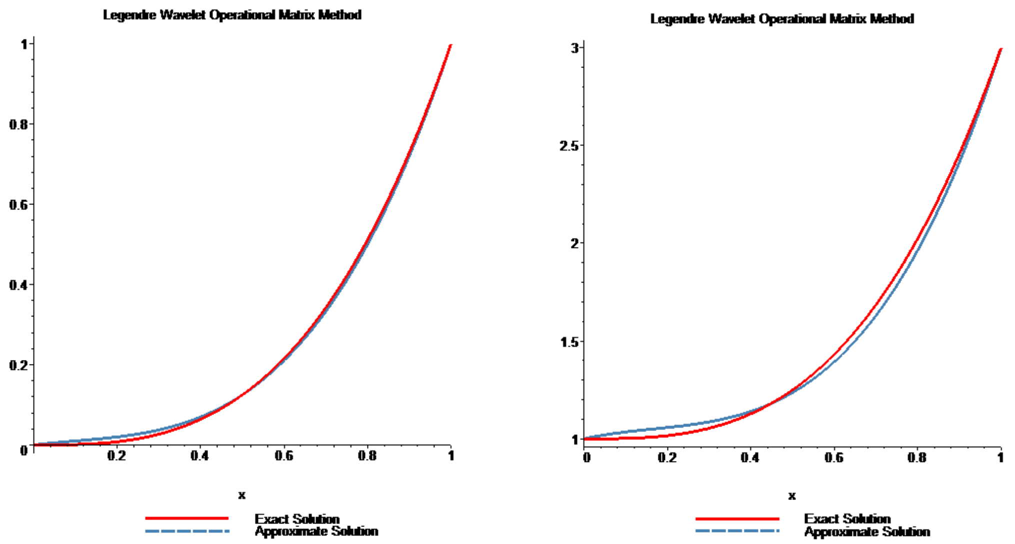

Figure 6.2 Comparison of our solutions $u(t), v(t)$ with the exact solution when $\alpha = 1.5, \quad \beta = 0.5$ for Example 6.2

Table 6.4 The numerical results attained by the introduced method in comparison with the LPOMM and the exact solution $v(t)$ for Example 6.2.

t	Exact Solution $v(t)$	v_{LWOMM}	v_{LPOMM}
0.0	1.000	1.000000000	0.9999999998
0.1	1.002	1.034841367	1.165803114
0.2	1.016	1.057587628	1.283854751
0.3	1.054	1.086537668	1.374911456
0.4	1.128	1.139990370	1.459729770
0.5	1.250	1.236244618	1.559066242
0.6	1.432	1.393599296	1.693677414
0.7	1.686	1.630353290	1.884319831
0.8	2.024	1.964805482	2.151750038
0.9	2.458	2.415254758	2.516724579
1.0	3.000	3.000000000	3.000000000

Example 6.3 Consider the following non-linear system of FDEs [41], [49]

$$\begin{aligned} D^{\alpha}u(t) &= \frac{u(t)}{2} \\ D^{\alpha}v(t) &= u^2(t) + v(t) \end{aligned} \tag{6.13}$$

subject to

$$u(0)=1, \quad v(0)=0 \tag{6.14}$$

The exact solution of this system when $\alpha = 1$ is known to be

$$u(t) = e^{\left(\frac{t}{2}\right)}, \quad v(t) = te^{t}$$

To solve the above system when $\alpha = 0.9$, we applied the method presented in Section 6.1 with $M = 2, \; k = 0$. Approximating solution following as

$$u(t) = C^T\psi(t), \;\; v(t) = S^T\psi(t)$$

$$D^{0.9}u(t) = C^T D^{(0.9)}\psi(t), \quad D^{0.9}v(t) = S^T D^{(0.9)}\psi(t)$$

We get

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}$$

$$D^{(0.9)} = \begin{pmatrix} 0 & 0 & 0 \\ 1.911059300\sqrt{3} & 0.2730084714 & -0.02642017466\sqrt{15} \\ -0.273008472\sqrt{5} & 1.664471004\sqrt{15} & 0.6325806046 \end{pmatrix}$$

If we consider (6.13) with (6.14), we have

$$\begin{aligned} R_1(t) &= C^T D^{(0.9)}\psi(t) - \frac{C^T\psi(t)}{2} \\ R_2(t) &= S^T D^{(0.9)}\psi(t) - \left(C^T\psi(t)\right)^2 - S^T\psi(t) \end{aligned} \tag{6.15}$$

Calculating Equations (6.15) at the first two roots of $P_3(t)$, i.e.

$$t_0 = \frac{1}{2}, \quad t_1 = \frac{1}{2} - \frac{\sqrt{15}}{10}$$

We obtain four non-linear equations following as

$$3.523070728c_{0,1} - 9.140726844c_{0,2} - 0.5c_{0,0} = 0$$
$$4.193891122s_{0,1} - 9.587940439s_{0,2} - \left(c_{0,0} - 1.341640787c_{0,1} + 0.8944271908\mathrm{c}_{0,2}\right)^2 - s_{0,0} = 0$$
$$3.424454517c_{0,1} - 0.7586951239c_{0,2} - 0.5c_{0,0} = 0$$
$$3.424454517s_{0,1} - 0.1996781297s_{0,2} - \left(c_{0,0} - 1.118033988\mathrm{c}_{0,2}\right)^2 - s_{0,0} = 0$$

and by utilising initial conditions we have

$$c_{0,0} - \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} = 1$$
$$s_{0,0} - \sqrt{3}s_{0,1} + \sqrt{5}s_{0,2} = 0$$

If we solve this system of non-linear algebraic equations, we get

$$C^T = \left[c_{0,0}, c_{0,1}, c_{0,2}\right] = \left[1.332807545,\ 0.1951100121,\ 0.002295506718\right]$$

$$S^T = \left[s_{0,0}, s_{0,1}, s_{0,2}\right] = \left[1.215158677,\ 0.8796261200,\ 0.1379199819\right]$$

Consequently,

$$u(t) = C^T\psi(t) = \left[1.332807545,\ 0.1951100121,\ 0.002295506718\right]\begin{bmatrix} 1 \\ \sqrt{3}\left(-1+2t\right) \\ \sqrt{5}\left(6t^2-6t+1\right) \end{bmatrix}$$

$$v(t) = S^T\psi(t) = \left[1.215158677,\ 0.8796261200,\ 0.1379199819\right]\begin{bmatrix} 1 \\ \sqrt{3}\left(-1+2t\right) \\ \sqrt{5}\left(6t^2-6t+1\right) \end{bmatrix}$$

The parameters $M = 2$, $k = 0$ and $\alpha = 0.9, 0.7, 0.5$ were utilised. Comparison of our results and the exact solution when $\alpha = 1$ were also displayed in Figure 6.3. The figures support that when α approximates 1, our results approximate the exact solution. We also solved this problem by using LPOMM and we compared the results with the LWOMM. Finally, we also presented the numerical computations for u(t) and v(t) when $\alpha = 0.9$ in Table 6.5 and Table 6.6.

Table 6.5 Our solutions $u(t)$ when $\alpha = 0.9$ attained by the presented method and the LPOMM for Example 6.3.

t	u_{LWOMM}	u_{LPOMM}	Absolute Error
0.0	1.000000000	1.000000000	$0.37\ 10^{-10}$
0.1	1.064816320	1.064816320	$0.132\ 10^{-9}$
0.2	1.130248589	1.130248589	$0.1375\ 10^{-9}$
0.3	1.196296807	1.196296807	$0.44\ 10^{-10}$
0.4	1.262960975	1.262960975	$0.808\ 10^{-9}$
0.5	1.330241091	1.330241091	$0.532\ 10^{-9}$
0.6	1.398137156	1.398137156	$0.168\ 10^{-9}$
0.7	1.466649171	1.466649171	$0.756\ 10^{-9}$
0.8	1.535777134	1.535777134	$0.1375\ 10^{-9}$
0.9	1.605521046	1.605521046	$0.468\ 10^{-9}$
1.0	1.675880908	1.675880908	$0.163\ 10^{-9}$

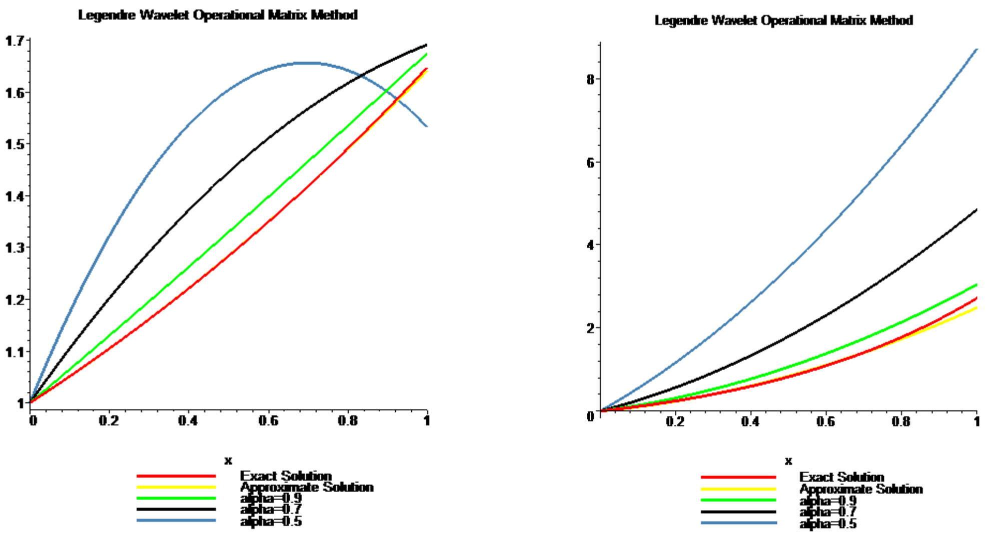

Figure 6.3 Comparison of our solutions $u(t), v(t)$ with the exact solution when $\alpha = 0.9, 0.7, 0.5$ *and* 1 for Example 6.3.

Table 6.6 Our solutions $v(t)$ when $\alpha = 0.9$ attained by the presented method and the LPOMM for Example 6.3.

t	v_{LWOMM}	v_{LPOMM}	Absolute Error
0.0	-0.1 10^{-9}	0.0000000000	0.1 10^{-9}
0.1	0.1381762607	0.1381762609	0.7 10^{-9}
0.2	0.3133603361	0.3133603363	0.2 10^{-9}
0.3	0.5255522260	0.5255522263	0.37 10^{-9}
0.4	0.7747519305	0.7747519309	0.5 10^{-9}
0.5	1.060959450	1.060959450	0.5 10^{-9}
0.6	1.384174783	1.384174784	0.8 10^{-9}
0.7	1.744397932	1.744397932	0.47 10^{-9}
0.8	2.141628895	2.141628895	0.7 10^{-9}
0.9	2.575867672	2.575867672	0.3 10^{-9}
1.0	3.047114264	3.047114264	0.1 10^{-9}

Example 6.4 Consider the following non-linear system of FDEs [49]

$$\begin{aligned} D^{\alpha}u(t) &= -1002u(t) + 1000v^2(t) \\ D^{\alpha}v(t) &= u(t) - v(t) - v^2(t) \end{aligned} \tag{6.16}$$

subject to

$$u(0) = 1, \quad v(0) = 1 \tag{6.17}$$

The exact solution of this system when $\alpha = 1$ is known to be

$$u(t) = e^{-2t}, \quad v(t) = e^{-t}$$

To solve the above system when $\alpha = 0.99$, we applied the method presented in Section 6.1 with $M = 4, \ k = 0$. Approximating solution following as

$$u(t) = C^T\psi(t), \quad v(t) = S^T\psi(t)$$

$$D^{0.99}u(t) = C^T D^{(0.99)}\psi(t), \quad D^{0.99}v(t) = S^T D^{(0.99)}\psi(t)$$

We get

$$D=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{63} & 0 \end{pmatrix}$$

$$D^{(0.99)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 3.449375852 & 0.02972385188 & -0.01262112773 & 0.007410898821 & -0.005015065172 \\ -0.06646455361 & 7.610540026 & 0.07375369729 & -0.03451765405 & 0.02160663680 \\ 5.225899340 & -0.06725669 & 11.54413503 & 0.1255710707 & -0.06260318730 \\ -0.1231327500 & 10.1398707 & -0.05433257 & 15.41452292 & 0.1827775379 \end{pmatrix}$$

If we consider (6.16) with (6.17), we have

$$\begin{aligned} R_1(t) &= C^T D^{(0.99)}\psi(t)+1002\,\mathrm{C}^T\,\psi(t)-1000\left(S^T\psi(t)\right)^2 \\ R_2(t) &= S^T D^{(0.99)}\psi(t)-C^T\psi(t)+S^T\psi(t)+\left(S^T\psi(t)\right)^2 \end{aligned} \tag{6.18}$$

Calculating Equations (6.18) at the first four roots of $P_5(t)$, i.e.

$$t_0=\frac{1}{2},\quad t_1=\frac{1}{2}-\frac{\sqrt{245-14\sqrt{70}}}{42},\quad t_2=\frac{1}{2}-\frac{\sqrt{245+14\sqrt{70}}}{42},\quad t_3=\frac{1}{2}+\frac{\sqrt{245-14\sqrt{70}}}{42}$$

We obtain eight non-linear equations and by utilising initial conditions we have

$$\begin{aligned} c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}-\sqrt{7}c_{0,3}+3c_{0,4} &= 1 \\ s_{0,0}-\sqrt{3}s_{0,1}+\sqrt{5}s_{0,2}-\sqrt{7}s_{0,3}+3s_{0,4} &= 1 \end{aligned}$$

If we solve this system of non-linear algebraic equations, we get

$$\begin{aligned} C^T &= \left[c_{0,0},c_{0,1},c_{0,2},c_{0,3},c_{0,4}\right] \\ &= \left[0.4415217604,\ -0.2476343090,\ 0.05158027301,\ -0.01297029961,\ -0.006696716433\right] \end{aligned}$$

$$\begin{aligned} S^T &= \left[s_{0,0},s_{0,1},s_{0,2},s_{0,3},s_{0,4}\right] \\ &= \left[0.6431515334,\ -0.1801527132,\ 0.02407737838,\ 0.0006364975451,\ -0.002446608841\right] \end{aligned}$$

Consequently,

$$u(t) = C^T\psi(t) = \begin{bmatrix} 0.4415217604 \\ -0.2476343090 \\ 0.05158027301 \\ -0.01297029961 \\ -0.006696716433 \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2-6t+1\right) \\ \sqrt{7}\left(20t^3-30t^2+12t-1\right) \\ 3\left(70t^4-140t^3+90t^2-20t+1\right) \end{bmatrix}$$

$$v(t) = S^T\psi(t) = \begin{bmatrix} 0.6431515334 \\ -0.1801527132 \\ 0.02407737838 \\ 0.0006364975451 \\ -0.002446608841 \end{bmatrix}^T \begin{bmatrix} 1 \\ \sqrt{3}(-1+2t) \\ \sqrt{5}\left(6t^2-6t+1\right) \\ \sqrt{7}\left(20t^3-30t^2+12t-1\right) \\ 3\left(70t^4-140t^3+90t^2-20t+1\right) \end{bmatrix}$$

We solved this problem by using LPOMM and we compared the results with the LWOMM for $M = 4, k = 0$ and $\alpha = 0.99, 0.9, 0.7$. The numerical computations for u(t) and v(t) when $\alpha = 0.99$ are also revealed in Table 6.7 and Table 6.8.

Table 6.7 Numerical solutions of $u(t)$ when $\alpha = 0.99$ obtained by the given method and the LPOMM for Example 6.4.

t	u_{LWOMM}	u_{LPOMM}	Absolute Error
0.0	1.000000000	1.000000000	0.4 10^{-10}
0.1	0.8144351529	0.8144351528	0.21 10^{-10}
0.2	0.6639425233	0.6639425233	0.15 10^{-9}
0.3	0.5429947229	0.5429947230	0.34 10^{-9}
0.4	0.4460643636	0.4460643636	0.42 10^{-9}
0.5	0.3676240568	0.3676240568	0
0.6	0.3021464142	0.3021464147	0.68 10^{-9}
0.7	0.2441040487	0.2441040481	0.24 10^{-9}
0.8	0.1879695699	0.1879695690	0.53 10^{-9}
0.9	0.1282155920	0.1282155905	0.461 10^{-9}
1.0	0.0593147248	0.0593147222	0.144 10^{-8}

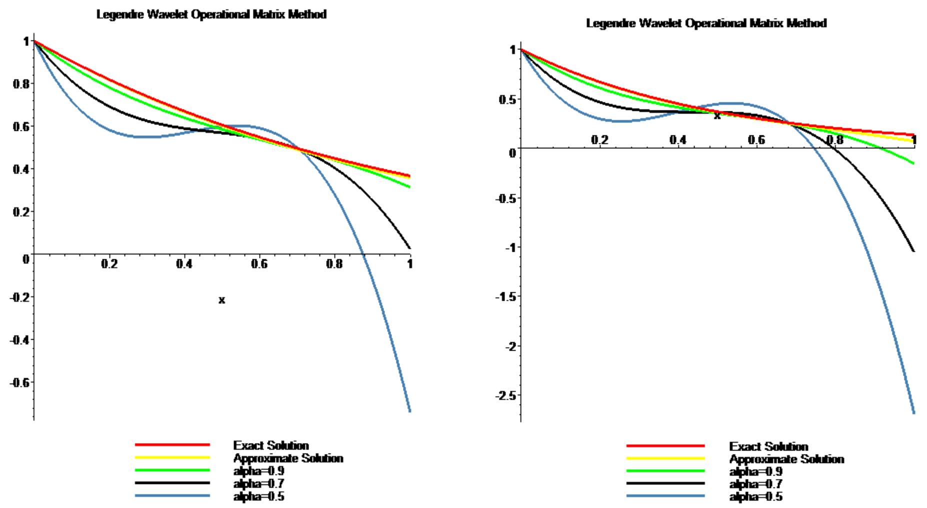

Figure 6.4 Comparison of our solutions $u(t), v(t)$ with the exact solution when $\alpha = 0.5, 0.7, 0.9$ *and* 1 for Example 6.4.

Table 6.8 Numerical solutions of $v(t)$ when $\alpha = 0.99$ obtained by the given method and the LPOMM for Example 6.4.

t	v_{LWOMM}	v_{LPOMM}	Absolute Error
0.0	1.000000000	0.9999999999	0.79 10^{-10}
0.1	0.9025601837	0.9025601837	0.1498 10^{-9}
0.2	0.8152646487	0.8152646488	0.119 10^{-9}
0.3	0.7371116577	0.7371116578	0.8 10^{-10}
0.4	0.6670994737	0.6670994739	0.11 10^{-9}
0.5	0.6042263597	0.6042263600	0.24 10^{-9}
0.6	0.5474905785	0.5474905788	0.19 10^{-9}
0.7	0.4958903932	0.4958903935	0.26 10^{-9}
0.8	0.4484240664	0.4484240670	0.443 10^{-9}
0.9	0.4040898614	0.4040898620	0.5898 10^{-9}
1.0	0.3618860408	0.3618860417	0.719 10^{-9}

Example 6.5 Consider the following fractional-order Brusselator system presented in [47] and [48]

$$\begin{aligned} D^{\alpha}u(t) &= -2u(t) + u^2(t)\,\mathrm{v(t)} \\ D^{\alpha}v(t) &= u(t) - u^2(t)\,\mathrm{v(t)} \end{aligned} \tag{6.19}$$

subject to

$$u(0) = 1, \quad v(0) = 1 \tag{6.20}$$

The approximate solution of this system when $\alpha = 1$ and $\alpha = 0.98$ was presented by Chang and Isah using Legendre wavelet operational matrix of fractional derivative through wavelet-polynomial transformation (LWPT) in [48] and by Bota and Caruntu using the polynomial least squares method (PLSM) in [47]. The solution of this system when $\alpha = 1$ was presented in [47] and [48] following as

$$u_{LWPT}(t) = 1 - 1.0120t + 0.1211t^2 + 0.1517t^3,$$
$$v_{LWPT}(t) = 1 + 0.0096t + 0.4069t^2 - 0.2461t^3$$

$$u_{PLSM}(t) = 1 - 1.02827t + 0.201028t^2 + 0.0750974t^3,$$
$$v(t)_{PLSM} = 1 + 0.0271107t + 0.334087t^2 - 180088t^3$$

The approximate solution of this system when $\alpha = 0.98$ was presented in [47] and [48] following as

$$u_{LWPT}(t) = 1 - 1.0791t + 0.2711t^2 - 0.0638t^3,$$
$$v_{LWPT}(t) = 1 + 0.0151t + 0.4185t^2 - 0.2624t^3$$

$$u_{PLSM}(t) = 1 - 1.08655t + 0.311138t^2 + 0.0243682t^3,$$
$$v(t)_{PLSM} = 1 + 0.0349127t + 0.333424t^2 - 0.184414t^3$$

To solve the above system when $\alpha = 0.98$, we applied the method presented in Section 6.1 with $M = 2, \; k = 0$. Approximating solution following as

$$u(t) = C^T\psi(t), \quad v(t) = S^T\psi(t)$$

$$D^{0.98}u(t) = C^T D^{(0.98)}\psi(t), \quad D^{0.98}v(t) = S^T D^{(0.98)}\psi(t)$$

We get

$$D=\begin{pmatrix}0&0&0\\2\sqrt{3}&0&0\\0&2\sqrt{15}&0\end{pmatrix}$$

$$D^{(0.98)}=\begin{pmatrix}0&0&0\\1.982905202\sqrt{3}&0.05889817430&-0.006370884195\sqrt{15}\\-0.1317003230&7.476321502&0.1450477428\end{pmatrix}$$

If we consider (6.19) with (6.20), we have

$$\begin{aligned}R_1(t)&=C^T D^{(0.98)}\psi(t)+2C^T\psi(t)-\left(C^T\psi(t)\right)^2\left(S^T\psi(t)\right)\\R_2(t)&=S^T D^{(0.98)}\psi(t)-C^T\psi(t)+\left(C^T\psi(t)\right)^2\left(S^T\psi(t)\right)\end{aligned}\tag{6.21}$$

Calculating Equations (6.21) at the first two roots of $P_3(t)$, i.e. $t_0=\dfrac{1}{2},\quad t_1=\dfrac{1}{2}-\dfrac{\sqrt{15}}{10}$

We obtain four non-linear equations and by utilising initial conditions we have

$$\begin{aligned}c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}&=1\\s_{0,0}-\sqrt{3}s_{0,1}+\sqrt{5}s_{0,2}&=1\end{aligned}$$

If we solve this system of non-linear algebraic equations, we get

$$C^T=\left[c_{0,0},c_{0,1},c_{0,2}\right]=\left[0.5654293689,\ -0.2188337915,\ 0.02483796840\right]$$

$$S^T=\left[s_{0,0},s_{0,1},s_{0,2}\right]=\left[1.087682270,\ 0.06669943989,\ 0.01245246086\right]$$

Consequently,

$$u(t)=C^T\psi(t)=\left[0.5654293689,\ -0.2188337915,\ 0.02483796840\right]\begin{bmatrix}1\\\sqrt{3}(-1+2t)\\\sqrt{5}\left(6t^2-6t+1\right)\end{bmatrix}$$

$$v(t)=S^T\psi(t)=\left[1.087682270,\ 0.06669943989,\ 0.01245246086\right]\begin{bmatrix}1\\\sqrt{3}(-1+2t)\\\sqrt{5}\left(6t^2-6t+1\right)\end{bmatrix}$$

The parameters $M=2$, $k=0$ with $\alpha=0.98$ were utilized. Comparison of our results and these approximate solutions introduced in [47] and [48] are also displayed in Figure 6.5. The figures support that our solution approaches the approximate solutions presented in [47] and [48]. Finally, we also presented the numerical computations for u(t) and v(t) when $\alpha = 0.98$ in Table 6.9 and Table 6.10.

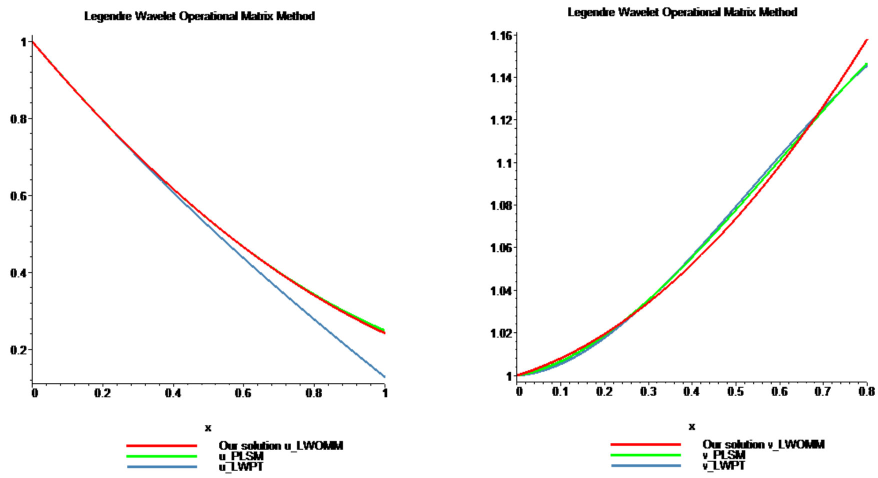

Figure 6.5 Comparison of our solutions u_{LOWMM} , v_{LOWMM} with the approximate solution u_{LPST} , v_{LPST} and the approximate solution u_{PLSM} , v_{PLSM} when $\alpha = 0.98$ for Example 6.5.

Table 6.9 Comparison between our approximate solution u_{LOWMM} and u_{LWPT} and u_{PLSM} when $\alpha = 0.98$ for Example 6.5.

t	u_{LWOMM}	u_{LWPT}	u_{PLSM}
0.0	1.000000000	1.0000000	1.0000000000
0.1	0.8942024826	0.8947372	0.8944807482
0.2	0.7950696916	0.7945136	0.7953304656
0.3	0.7026016268	0.6989464	0.7026953614
0.4	0.6167982883	0.6076528	0.6167216448
0.5	0.5376596761	0.5202500	0.5375555250
0.6	0.4651857902	0.4363552	0.4653432112
0.7	0.3993766306	0.3555856	0.4002309126
0.8	0.3402321973	0.2775584	0.3423648384
0.9	0.2877524902	0.2018908	0.2918911978
1.0	0.2419375095	0.1282000	0.2489562000

Table 6.10 Comparison between our approximate solution v_{LOWMM} with v_{LWPT} and v_{PLSM} when $\alpha = 0.98$ for Example 6.5.

t	v_{LOWMM}	v_{LWPT}	v_{PLSM}
0.0	1.000000000	1.0000000	1.000000000
0.1	1.008069307	1.0054326	1.006641096
0.2	1.019479961	1.0176608	1.018844188
0.3	1.034231959	1.0351102	1.035502792
0.4	1.052325304	1.0562064	1.055510424
0.5	1.073759995	1.0793750	1.077760600
0.6	1.098536032	1.1030416	1.101146836
0.7	1.126653415	1.1256318	1.124562648
0.8	1.158112143	1.1455712	1.146901552
0.9	1.192912217	1.1612854	1.167057064
1.0	1.231053638	1.1712000	1.183922700

CHAPTER 7

RESULTS AND DISCUSSION

Because variety of solution of higher order differential equations and the system of such equations can not be found analytically, numerical and approximate methods are needed. This situation is more difficult and complicated for the solution of fractional order differential equations and the system of such equations. There are a lot of tecniques that have been studied by many researchers to solve fractional differential equations numerically. In this thesis, high order differential equations and the system of such equations of the linear and non-linear form were solved by utilising operational matrix of derivative and by generalizing these matrices to these equations and systems. Also, fractional order differential equations and the system of such equations of the linear and non-linear form were examined by derivating a new operational matrix of the fractional derivative in some special conditions and by benefiting from charachteristics of these matrices.

So then the Legendre wavelet operational matrix method is introduced in related chapters of this thesis by using some significant features of shifted Legendre polynomials and Legendre wavelets. The most advantage of this method is that it gives a understandable procedure in reducing these equations and the system of such equations to a system of algebraic equations. Also, very effective algorithm have been also formulated to obtain the solution of equations and systems mentioned above on the Maple. We produced all numerical results and graphical representations via Maple. The results illustrate that the introduced procedure can solve such equations and systems very efficaciously and simply.

As the next step, the method introduced in this thesis can be applied to fractional partial differential equations and the system of such equations, fractional integral equations and the system of such equations, fractional integro-differential equations. These equations are at least as important as fractional differential equations and they are very significant in science, engineering and technology.

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1. Secer, A., Altun, S. and Bayram, M., (2018). "The Legendre wavelet operational matrix method: An efficient approximation for solving fractional order Brusselator system", ICAAMM 2018, 20-24 June, Istanbul.

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AWARDS

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